

# Resurgence of parabolic curves in $\mathbb{C}^2$

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## 1. Nondegenerate parabolic germs

Parabolic germ:  $F: \mathbb{C}^2, 0 \rightarrow$  holomorphic such that 1 is the only e.v. of  $DF(0)$   
but  $DF(0) \neq \text{Id}$

$$p=(x,y) \mapsto F(p)=(x_1, y_1) \quad \begin{cases} x_1 = x + y + a(x,y) \\ y_1 = y + b(x,y) \end{cases} \xrightarrow{\quad} O_2(x,y)$$

Nondegenerate if  $b(x,y) = b_{20}x^2 + b_{11}xy + b_{02}y^2$  with  $b_{20} \neq 0$

This condition is intrinsic, as observed in  
Levi-Civita, Annali di Matematica, 1901

Aim: Investigate local dynamics of  $F$  and related formal objects in  
the framework of Ecalle's theory of resurgent functions.

- Ref. for parabolic germs: M. Abate 2000, 2002 → existence of "parabolic curves"
- For a particular case (Hénon map): V. Gelfreich - D.S. 2001
- For tangent-to-identity germs ( $DF(0) = \text{Id}$ ): J. Ecalle 1985  
M. Hakim 1997, 1998  
N. Sibony 1999  
J. Ribón 2005

Asymptotic analysis of two invariant complex curves  $W^+, W^-$  s.t.

$$\forall p \in W^+, \quad F^n(p) \xrightarrow{n \rightarrow \infty} 0$$

"stable"

$$\forall p \in W^-, \quad F^n(p) \xrightarrow{n \rightarrow \infty} 0$$

"unstable"

Generically not analytic at the origin of  $\mathbb{C}^2$

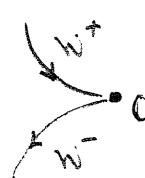
Wish to measure exp. small "splitting" between  $W^+$  and  $W^-$ .

Two Borel sums of the same formal series

Application: exponentially thin layer in  
parameter-space for the Bogdanov-Takens bifurcation for diffus... (V.G.)

Example: Hénon map  $\begin{cases} x_1 = x + y - x^2 \\ y_1 = y - x^2 \end{cases}$

[G-S. 2001]



Here, we want to consider any nondegenerate parabolic germ,  
possibly depending analytically  
on parameters.

## 2. Formal objects associated with $F$ : infinitesimal generator, separatrix, invariant foliation

Lemma

Given a parabolic germ  $F$ , there exists a unique formal v.f.

$$\tilde{X} = (y + \tilde{A}(x,y)) \frac{\partial}{\partial x} + \tilde{B}(x,y) \frac{\partial}{\partial y} \quad \text{with } \tilde{A}, \tilde{B} \in \mathcal{O}_2(x,y)$$

which has a formal flow  $\tilde{\Phi} = (\tilde{x}, \tilde{y})$   
satisfying  $\tilde{\Phi}|_{t=1} = F$ .

$$\begin{aligned} \tilde{x}, \tilde{y} &\in \mathbb{C}[t][[x,y]] \\ \tilde{\Phi}|_{t=0} &= \text{Id} \\ \dot{c}_t \tilde{\Phi} &= (\tilde{y} + \tilde{A} \circ \tilde{\Phi}, \tilde{B} \circ \tilde{\Phi}) \end{aligned}$$

$\tilde{X}$  is called the formal infinitesimal generator of  $F$ .

Notation:  $\tilde{\Phi}(t, \dots, \cdot) = \exp_t \tilde{X}, \quad F = \exp_1 \tilde{X}$

$$\sum_{k,l \geq 0} \frac{1}{k!l!} (y+a)^k b^l \frac{\partial^k}{\partial x^k} \frac{\partial^l}{\partial y^l} \varphi$$

Proof:  $\mathcal{F}$  = substitution operator:  $\varphi \in \mathbb{C}[[x,y]] \mapsto \varphi \circ \mathcal{F} \in \mathbb{C}[[x,y]]$   
algebra automorphism

We want to define  $\tilde{X}$  as  $\log \mathcal{F}$ :

$$\mathcal{F} = \text{Id} + G \rightarrow \tilde{X} \cdot \varphi = \sum_{r \geq 1} \frac{(-1)^{r-1}}{r} G^r \varphi$$

Argument to see that the r.h.s. converges in the Krull topology:

$$\text{valuations } \nu(G^r \varphi) \underset{r \rightarrow \infty}{\uparrow} \infty.$$

we  $\nu(\varphi) = \min \{ 2n+3m \mid [\varphi]_{n,m} \neq 0 \}$  if  $\varphi = \sum_{n,m \geq 0} [\varphi]_{n,m} x^n y^m$

associated with linear grading  $\delta(x^n y^m) = 2n+3m$   
(instead of usual  $\delta(x^n y^m) = n+m$ )

so that  $\nu(y+a) = 3, \nu(b) \geq 4, \nu(\partial_x^k \partial_y^l \varphi) \geq \nu(\varphi) - 2k - 3l$

$$\Rightarrow \nu(G \varphi) \geq \nu(\varphi) + 1.$$

Standard properties of the logarithm and exponential series imply  
that  $\tilde{X}$  is a derivation of  $\mathbb{C}[[x,y]]$  and

$$\exp \tilde{X} = \mathcal{F}, \quad \exp(t \tilde{X}) \varphi = \sum_{r \geq 0} \frac{t^r}{r!} \tilde{X}^r \varphi \in \mathbb{C}[t][[x,y]]$$

substitution operator  
for  $\mathcal{F}$

substitution operator for the formal flow.

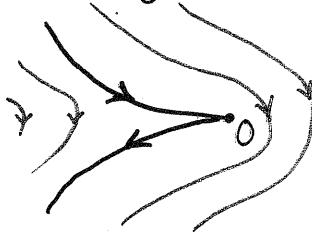
Note:  $[\tilde{B}]_{2,0} = b_{2,0}$

Now assume  $F$  non-degenerate:  $b_{2,0} \neq 0$ . Make  $b_{2,0} = -1$  by linear change of coord.

$$\tilde{X} = \underbrace{y \frac{\partial}{\partial x} - x^2 \frac{\partial}{\partial y}}_{X^{[1]}} + \tilde{X}^* \rightarrow \nu(\tilde{X}^* \varphi) \geq \nu(\varphi) + 2$$

Hamiltonian v.f.  
generated by

$$h(x,y) = \frac{1}{2} y^2 + \frac{1}{3} x^3$$



which has separatrix solution  $z \mapsto (-6z^2, 12z^3)$ .

It turns out that there is a "formal separatrix" for  $\tilde{X}$ : a formal curve of the form

formal Puiseux representation  
 $\{y = \tilde{Y}_0 \left( \left(\frac{x}{6}\right)^{-\frac{1}{2}} \right)\}$   
 $x^{\frac{1}{2}} \in \mathbb{C}[[x^{\frac{1}{2}}]]$

$$\{(x,y) \mid x = -6t^{-2}, y = \tilde{Y}_0(t)\}$$

$$\tilde{Y}_0 = 12t^{-3} + \dots \in \mathbb{C}[[t^{-1}]]$$

which is invariant by the flow of  $\tilde{X}$ ,

$\mathcal{C}((x^{\frac{1}{2}})[[t]])$  and also a "formal invariant foliation"

$\{y = \tilde{Y} \left( \left(\frac{x}{6}\right)^{-\frac{1}{2}}, t \right)\}$

$$\{(x,y) \mid x = -6t^{-2}, y = \tilde{Y}(t, b) = \sum_{n \geq 0} b^n \tilde{Y}_n(t)\}$$

$$\tilde{Y}_n \in \mathbb{C}[[t^{-1}]] [t]$$

Time-parametrisations:

The formal separatrix and the formal foliation are best studied through the corresponding solutions of  $\tilde{X}$ :  $z \mapsto \tilde{p}_0(z)$  and  $z \mapsto \tilde{p}(z, b)$

$$\partial_z \tilde{p} = \tilde{X}(\tilde{p})$$

$$\sum_{n \geq 0} b^n \tilde{p}_n(z)$$

which can themselves be found as solutions of the difference equation

$$\tilde{p}(z+1) = F(\tilde{p}(z))$$

in certain spaces of formal series.

### 3. Formal solutions of the difference equation

$E_0 = \mathbb{C}[[z^{-1}, z \log z]]$  differential algebra ( $\partial = \frac{d}{dz}$ )

$E_k = z^k E_0, k \in \mathbb{Z} \rightarrow$  increasing sequence of spaces

- For  $k < -1$ :  $E_{-1}, E_{-2}, \dots$  ideals of  $E_0$

- For fixed  $k \geq 0$ : monomials  $z^n (\log z)^m$  of  $E_k$  are ordered  
 $n \geq -k$        $z^{-p} (\log z)^m$  lexicographic order  
 $p-m \geq -k$  on  $(p, m) \in \mathbb{Z} \times \mathbb{N}$

$$z^k \gg$$

$$z^{k-1} \log z \gg z^{k-1} \gg$$

$$z^{k-2} (\log z)^2 \gg z^{k-2} \log z \gg z^{k-2} \gg$$

$$z^{k-3} (\log z)^3 \gg \dots$$

.....

.....

$$\begin{aligned} (\log z)^k \gg (\log z)^{k-1} \gg \dots \gg \log z \gg 1 \gg \\ \text{or} \quad \left[ \begin{array}{l} z^{-1} (\log z)^{k+1} \gg \dots \dots \dots \gg z^1 \\ \dots \dots \dots \end{array} \right] \end{aligned}$$

- $E = \bigcup_{k \geq 0} E_k = E_0[z]$  differential algebra

- Largest differential algebra we shall deal with:  $(E \cap \sigma(1)) + b E[[b]]$

Lemma

For  $\tilde{p}(z, b) \in ((E \cap \sigma(1)) + b E[[b]])^2$ ,

$$\tilde{p}(z+1, b) = F(\tilde{p}(z, b)) \iff \partial_z \tilde{p}(z, b) = \tilde{X}(\tilde{p}(z, b))$$

(Proof in the same vein as above. Use the property)

$$p(z) \text{ solution of } \tilde{X} \iff \forall \varphi \in \mathbb{C}[[x, y]], \partial_z (\varphi \circ p) = (\tilde{X} \varphi) \circ p.$$

Prop • The difference equation  $\tilde{p}(z+1) = F(\tilde{p}(z))$  admits nonzero solutions in  $(\mathbb{E}_{-2})^2$ . Let  $\tilde{p}_0(z)$  be one of them:

All nonzero solutions in  $(\mathbb{E} \cap \sigma(1))^2$  are given by  $\tilde{p}_0(z+a)$ ,  $a \in \mathbb{C}$ .

• In adapted coordinates, there is a unique nonzero solution

$\tilde{p}_0(z) = (\tilde{x}_0(z), \tilde{y}_0(z))$  in  $(\mathbb{E}_{-2})^2$  for which  $\tilde{x}_0$  has no  $z^3$ -term.

$$\tilde{x}_0(z) = -6z^{-2}(1 + 2cz^{-1}\log z + 3c^2z^{-2}\log^2 z - 2c^2z^{-2}\log z + O(z^{-2}))$$

$$\tilde{y}_0(z) = 12z^{-3}(1 + 3cz^{-1}\log z + (\frac{3}{2} - c)z^{-1} + 6c^2z^{-2}\log^2 z + c(6-7c)z^{-2}\log z + O(z^{-2}))$$

By a suitable analytic change of coord. we can always put  $F$

in the form  $F(x, y) = (x_1, y_1) \quad \begin{cases} x_1 = x + y + f(x, y) \\ y_1 = y + g(x, y) \end{cases}$

with no  $y^2$ -term in  $f$ :

$$f(x, y) = -x^2 - \gamma xy + O_3(x, y). \quad (c = \frac{\epsilon}{\gamma})$$

Idea of pf:  $(*) \Leftrightarrow \begin{cases} y(z+1) - y(z) = f(x(z), y(z)) \\ x(z+1) - x(z) = y(z+1) \end{cases}$

$$\Leftrightarrow \begin{cases} y(z) = x(z) - x(z-1) := (\mathcal{D}x)(z) \\ \boxed{Px = f(x, \mathcal{D}x)} \end{cases} (**)$$

$$(Px)(z) := x(z+1) - 2x(z) + x(z-1).$$

Substitute a series with undetermined coeff. in  $(**)$  ...  $\square$

Formal machinery

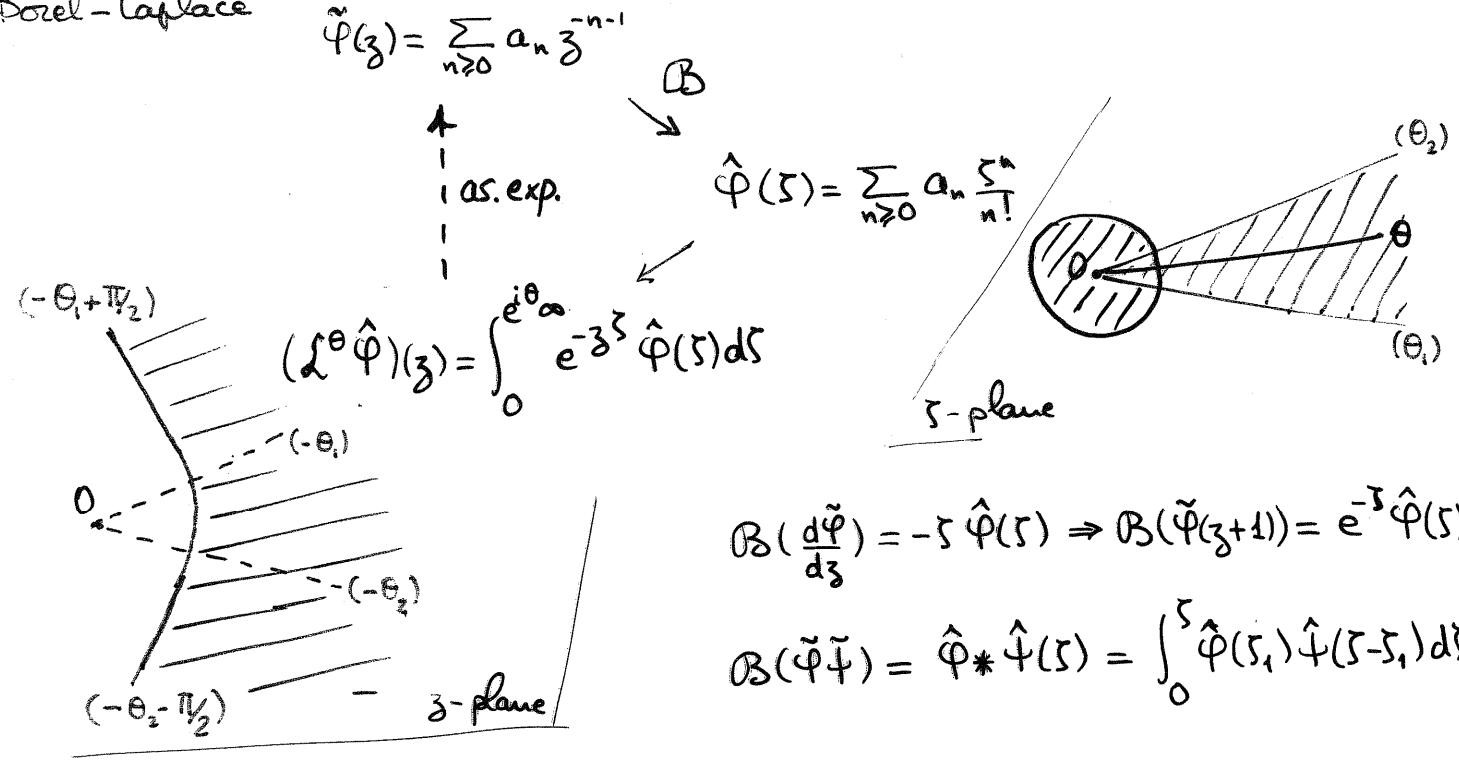
$$\boxed{\tilde{x}_0(z e^{2\pi i}) = \tilde{x}_0(z - 2\pi i c)} \Rightarrow \tilde{x}_0(z) = \tilde{X}_0(z_*) \in \mathbb{C}[[z_*^{-1}]]$$

$$z_* = z(1 + \sigma(1)) \in z\mathbb{C}\{\bar{z}, \bar{z}\log z\}$$

implicitly determined by  $z = z_* + c \log z_*$ .

# Borel summability and resurgence of $\tilde{p}_0 = (\tilde{x}_0, \tilde{y}_0)$

Borel-Laplace



$$\text{For non-integer powers: } B(\tilde{z}^{-\sigma}) = \hat{I}_{\sigma}(s) = \frac{s^{\sigma-1}}{\Gamma(\sigma)}, \quad \operatorname{Re} \sigma > 0$$

$$\text{Consistent since } \tilde{z}^{-\sigma} = \int_0^{e^{i\theta} \infty} \hat{I}_{\sigma}(s) e^{-s\tilde{z}} ds$$

$$\tilde{\varphi}(z) = \sum_{n \geq 0} (-1)^n \frac{(a)_n (b)_n}{n!} z^{-n-1} \text{ s.t. of } \varphi'' - (1 + Az^{-1})\varphi' - z^{-1}(1+Bz^{-1})\varphi$$

$$A = a+b-3 \quad B = -(a-1)(b-1)$$

$\hat{\varphi}(s)$  has radius of CV = 1 and extends holomorphically to  $\overline{\mathbb{C} \setminus \{0, -1\}}$

$$\text{In fact } \hat{\varphi}(s) = \frac{s^{-a}}{\Gamma(-a+1)} * \underbrace{\left[ \frac{s^{a-1}}{\Gamma(a)} (1+s)^{-b} \right]}_{\hat{U}(s) \in s^{a-1} \mathbb{C}\{s\}} \quad \text{for } 0 < \operatorname{Re} a < 1$$

$$\tilde{\varphi}(z) = z^{a-1} \tilde{U}(z) \text{ with } \tilde{U}(z) \in z^{-a} \mathbb{C}[[z^{-1}]]$$

$$U(a, 1+a-b, z) = \int_0^\infty e^{-s^a} \hat{U}(s) ds$$

Logarithms Differentiate w.r.t.  $\sigma$

$$J_{\sigma,m}(z) = \left(\frac{\partial}{\partial \sigma}\right)^m z^{-\sigma} = (-1)^m z^{-\sigma} (\log z)^m$$

$\operatorname{Re} \sigma > 0, m \in \mathbb{N}^*$

B ↴

$$\hat{J}_{\sigma,m}(s) = \sum_{k=0}^m \binom{m}{k} \left(\frac{1}{\Gamma}\right)^{(k)}(s) s^{s-1} (\log s)^{m-k}$$

Extension to  $\operatorname{Re} \sigma < 0$  Use "majors"  $\check{\varphi}(s)$  instead of "minors"  $\hat{\varphi}(s)$   
defined modulo  $\mathbb{C}\{s\}$

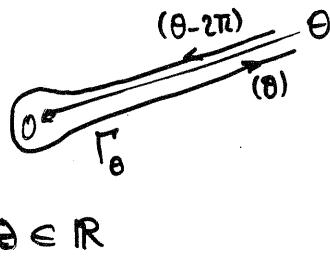
$$\check{\varphi}(s) - \hat{\varphi}(s e^{-2\pi i})$$

$$\check{I}_\sigma(s) = \frac{s^{\sigma-1}}{(1-e^{-2\pi i \sigma}) \Gamma(\sigma)} = \frac{e^{i\pi \sigma}}{2\pi i} \Gamma(1-\sigma) s^{\sigma-1}$$

Replace trsf of majors

$$(\check{L}^\theta \check{\varphi})(z) = \int_{\Gamma_\theta} e^{-sz} \check{\varphi}(s) ds$$

$\arg z \in \mathbb{R}$



$\theta \in \mathbb{R}$

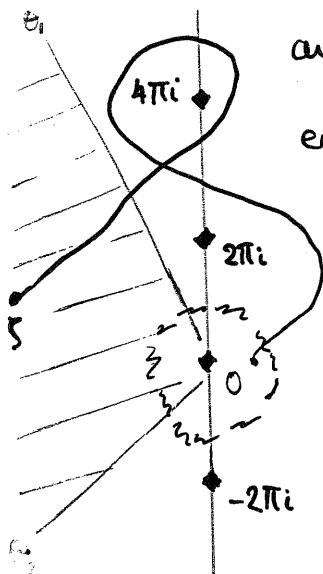
$$(\check{L}^\theta \check{I}_\sigma)(z) = z^{-\sigma}$$

for all  $\sigma \in \mathbb{C} \setminus \mathbb{N}^*$

(Extension of convolution → unit  $\delta$  has major  $\check{I}_0 = \frac{1}{2\pi i s}$ )

Thm Let  $R$  be the universal cover of  $\mathbb{C} \setminus 2\pi i \mathbb{Z}$  with base-point at 1.

$\check{x}_0 = \hat{x}_0(s) \in \mathbb{C}[[s, s \log s]]$  is convergent for  $|s|$  small  
and defines a holomorphic function of  $R$  which has at most  
exponential growth in any closed sector  $\bar{\delta}_{\theta_1, \theta_2} \subset R$  ..



In the case of an analytic family  $F_\lambda$ ,

$\hat{x}_0^{(\lambda)}(s)$  depends analytically on  $\lambda$ .

Note:  $\hat{x}_0(s e^{2\pi i}) = e^{-2\pi i \sigma} \hat{x}_0(s) \Rightarrow \hat{x}_0(s) = \hat{X}(s) e^{-c s \log s}$   
with  $\hat{X}(s) \in \mathbb{C}\{s\}$ .

Origin of singularities:

$$(*) \quad P \tilde{x}_0 = -\tilde{x}_0^2 - Y \tilde{x}_0 D \tilde{x}_0 + L_{3,0} \tilde{x}_0^3 + \dots$$

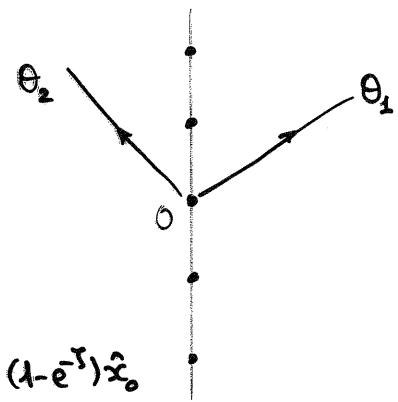
$$\Leftrightarrow (4 \pi i h^2 \frac{s}{2}) \hat{x}_0 = -\hat{x}_0 * \hat{x}_0 + Y \hat{x}_0 * [(e^s - 1) \hat{x}_0] + \dots$$

Apply  $\mathcal{L}^\theta$  with  $\theta \in \mathbb{R} \setminus (\frac{\pi}{2} + \pi\mathbb{Z})$

$\theta \in ]-\frac{\pi}{2}, \frac{\pi}{2}[$  or  $]\frac{\pi}{2}, \frac{3\pi}{2}[$  sufficient in view of monodromy of  $\hat{x}_0$  around 0

Cor There exist two analytic solutions of (\*):

$$\begin{cases} p^+(z) \sim \tilde{p}_0(z), & z \in \mathcal{D}^+ \\ p^-(z) \sim \tilde{p}_0(z), & z \in \mathcal{D}^- \end{cases}$$



$$x = e^{\theta_2} \hat{x}_0$$

$$y = D x = e^{\theta_2} (1 - e^{-\lambda}) \hat{x}_0$$

$$x^+ = e^{\theta_1} \hat{x}_0$$

$$y^+ = D x^+ = e^{\theta_1} (1 - e^{-\lambda}) \hat{x}_0$$

(Unicity:  $p^+$  and  $p^-$  can be characterized by a weaker asymptotic property.)

These are "time-parametrisations" for stable and unstable curves

$$W^\pm = \{ p^\pm(z) : z \in \mathcal{D}^\pm \} \quad F^n(p^\pm(z)) = p^\pm(z+n) \xrightarrow[n \rightarrow \pm\infty]{} 0$$

- Then
- There exists a sequence  $\tilde{x}_1 \in E_4, \tilde{x}_2 \in E_{10}, \dots$  such that
- $$\tilde{x}(z, b) = \sum_{n \geq 0} b^n \tilde{x}_n(z) \in (E_n \alpha(1)) + b E[[b]]$$
- solves (\*\*).  $\tilde{x}_1(z) = \frac{1}{84} z^4 (1 - 4c z^{-1} \log z + O(z))$  without  $z^3$ -term
- $n \geq 2$ :  $\tilde{x}_n(z)$  without  $z^4$  and  $z^2$  terms.
- All the solutions are of the form  $\tilde{x}(z + A(b), B(b)), A \in \mathbb{C}[[b]], B \in \mathbb{C}[[b]]$
  - Let  $\tilde{p}(z, b) = (\tilde{x}(z, b), \underbrace{D \tilde{x}(z, b)}_{\tilde{y}(z, b)})$ : solution of (\*).
- All the  $\tilde{x}_n(z), \tilde{y}_n(z)$  are Borel summable and resurgent like  $\tilde{x}_0$ .

Formal monodromy:  $\tilde{x}(ze^{2\pi i}, b) = \tilde{x}(z - 2\pi i \underline{\alpha(b)}, \underline{\beta(b)})$

$\alpha_0 = c, \alpha_1$ : formal invariants.

#### 4. Resurgence relations and Bridge Equation.

Thm

For each  $\omega = 2\pi m e^{\pm i\pi/2}$ ,  $m \in \mathbb{N}^*$ , there exist

$$\Theta_\omega(b) = \sum_{n \geq 0} b^n \Theta_{\omega,n} \text{ and } \mu_\omega(b) = \sum_{n \geq 0} b^n \mu_{\omega,n} \in \mathbb{C}[[b]]$$

such that

$$\Delta_\omega \tilde{P} = \Theta_\omega(b) \partial_b \tilde{P} + \mu_\omega(b) \partial_3 \tilde{P}$$

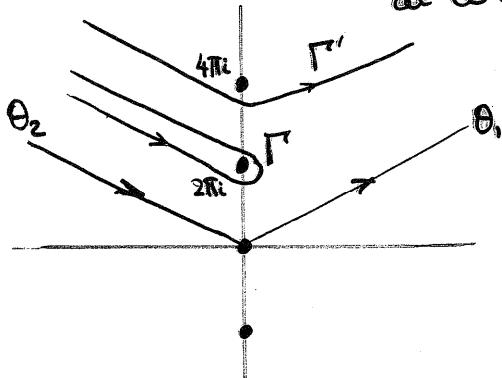
$$\Delta_\omega \tilde{x}_0 = \Theta_{\omega,0} \tilde{x}_1 + \mu_{\omega,0} \partial_3 \tilde{x}_0$$

$$\Delta_\omega \tilde{x}_1 = 2 \Theta_{\omega,0} \tilde{x}_2 + \Theta_{\omega,1} \tilde{x}_1 + \mu_{\omega,0} \partial_3 \tilde{x}_1 + \mu_{\omega,1} \partial_3 \tilde{x}_0$$

⋮

Rough explanation:  $\Delta_\omega$  is the "alien derivation of index  $\omega$ ",

e.g.  $\Delta_\omega \tilde{x}_0$  measures singularity of (certain branches of)  $\hat{x}_0$  at  $\omega$ : its major is defined by translating the minor  $\hat{x}_0$ .



Stokes phenomenon

As a consequence, e.g. for  $\omega = +2\pi i$

and  $z \in \mathcal{D}^+ \cap \mathcal{D}^-$  with  $\operatorname{Im} z < 0$ ,

$$x^+(z) - x^-(z) = \int_{e^{i\theta_2} \infty}^{e^{i\theta_1} \infty} e^{-3\tau} \hat{x}_0(\tau) d\tau = \left( \int_{\Gamma} + \int_{\Gamma'} \right) \dots$$

$$= \underbrace{e^{-2\pi i z}}_{\text{exp. small}} \left( \Theta_{\omega,0} \partial_z^{\theta_2} \tilde{x}_1 + \mu_{\omega,0} \partial_3 \partial_z^{\theta_2} \hat{x}_0 \right) + \underbrace{\dots}_{\text{exp. smaller}}$$

$$\sim e^{-2\pi i z} (\Theta_{\omega,0} \tilde{x}_1 + \mu_{\omega,0} \partial_3 \tilde{x}_0).$$

Leading term:

$$x^+ - x^- \sim \frac{1}{84} \Theta_{\omega,0} z^4 e^{-2\pi i z}$$

provided  $\Theta_{\omega,0} \neq 0$ , which is generically true  
( $\omega = 2\pi i$ )

because this number doesn't vanish for certain maps  
and depends analytically on parameters when F does.

Note:  $\Theta_{2\pi i, 0}$  and  $\Theta_{-2\pi i, 0}$  are almost analytic invariants for  $F$ .

Namely: if  $f$  and  $g \in \mathbb{C}\{x, y\}$  define two nondegenerate parabolic germs which analytically conjugate, then there exist

$$A(b) \in \mathbb{C}[[b]] \text{ and } B(b) = b + O(b^2) \in \mathbb{C}[[b]]$$

such that

$$\Theta_\omega^g(b) = \frac{e^{-\omega A(b)}}{B'(b)} \Theta_\omega^f(B(b)), \quad \mu_\omega^g(b) = e^{-\omega A(b)} \left( \mu_\omega^f(B(b)) - \frac{A'(b)}{B'(b)} \Theta_\omega^f(B(b)) \right).$$

$$\text{In particular } \Theta_{\omega, 0}^g = e^{-\omega A_0} \Theta_{\omega, 0}^f.$$

Formal invariant foliation

$$\text{Invert } -6t^{-2} = \tilde{x}(z, b) = -6z^2(1 + \tilde{\alpha}(z, b)).$$

$$t = z(1 + \tilde{\alpha}(z, b))^{-\frac{1}{2}} \Leftrightarrow z = Z(t, b) \in \mathbb{E}[[b]]$$

Substitute  $\tilde{Y}(t, b) := \tilde{y}(Z(t, b), b) \in \mathbb{E}[[b]]$

$\leftarrow$  no log  $t$  because of formal monodromy.

Alien calculus yields

$$\boxed{\Delta_\omega \tilde{Y} = \Theta_\omega(b) e^{-\omega(Z-t)} \partial_b \tilde{Y}}.$$

Nontriviality

Suppose  $f(x, y) = \sum_{n \geq 1} f_n x^{2n}$  with  $f_1 = -1$  and all  $f_n \leq 0$ .

(e.g.  $f(x, y) = -x^2$ : Hénon map)

Then  $\hat{x}_0(\zeta) \in \mathbb{R}\{\zeta\}$  is odd

and  $\hat{x}_0(\zeta) > 0$  on  $[0, 2\pi i]$ , not bounded because of  $(*)^*$

$\Rightarrow \Theta_{2\pi i, 0} \neq 0$  because of resurgence relation.

In fact  $\Theta_{2\pi i, 0} = \Theta_{-2\pi i, 0} \in i\mathbb{R}^{*-}$ .

## Case of a convergent vector field

Start with  $X = \underbrace{(y + A(x,y))}_{\Omega_2} \frac{\partial}{\partial x} + \underbrace{B(x,y)}_{\Omega_2} \frac{\partial}{\partial y}$      $A, B \in C\{x,y\}$   
 $B_{2,0} \neq 0.$

Then  $F = \exp_1 X$  is a non deg. parabolic germ.

In this case the separatrix is convergent:  $\tilde{x}_0, \tilde{y}_0 \in C\{\tilde{z}^1, \tilde{z}^{\log z}\}$   
and the numbers  $\Omega_{\omega,0}$  and  $\mu_{\omega,0}$  vanish.

Examples coming from 1-dimensional germs

Let  $\Phi \in C\{x,y\}$  with  $\Phi(0)=0$  and  $X^{(1)} = y \frac{\partial}{\partial x} - x^2 \frac{\partial}{\partial y}:$

$F(p) = (\exp_{1+\Phi(p)} X^{(1)})(p)$  defines a non deg. parabolic germ.

In this case:

- same foliation as for  $X^{(1)}$ , thus  $\Omega_{\omega}(b) = 0$
- $\tilde{p}_0(z) = (-6 \tilde{u}(z)^2, 12 \tilde{u}(z)^3)$  with

$\tilde{u}(z) \in z + C[[z^{-1}]]$  solution of

$$\tilde{u}(z+1) = \tilde{u}(z) + 1 + a(\tilde{u}(z))$$

$$a(z) = \Phi(-6z^2, 12z^3) \in C[z]$$

i.e.  $\tilde{u}$  conjugates  $z \mapsto z+1$  and  $\boxed{z \mapsto z+1+a(z)}$   
tangent-to-identity germ  
at  $\infty$

This is the first example of resurgence!

Ecalle (1981)

$$\Delta_{\omega} \tilde{u} = A_{\omega} \partial_z \tilde{u}$$

$$\tilde{x}_0 = -6 \tilde{u}^{-2} \Rightarrow \Delta_{\omega} \tilde{x}_0 = A_{\omega} \partial_z \tilde{x}_0 \quad \text{i.e. } \underline{\underline{\mu_{\omega,0} = A_{\omega}}}$$

