# Global Existence for Rate-Independent Gradient Plasticity

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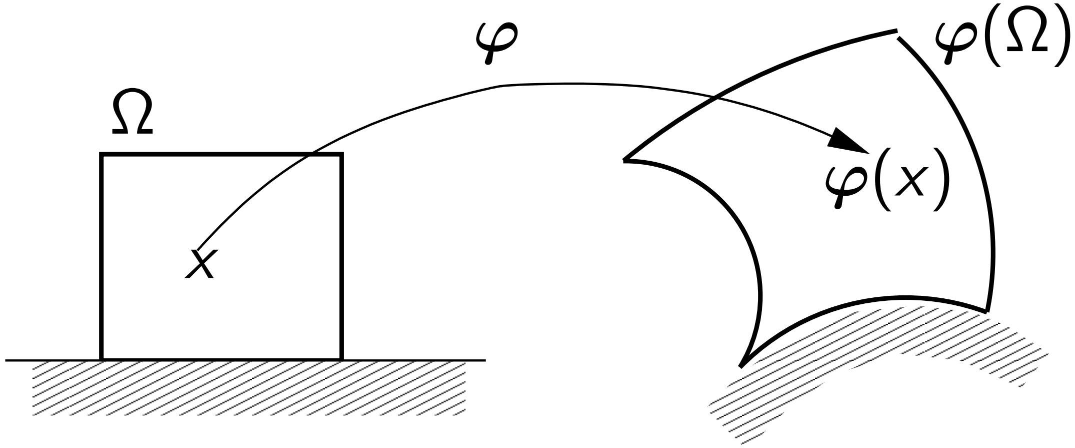
Rate-Independence, Homogenization and Multiscaling

*Joint work with Andreas Mainik*
*with credits to Stefan Müller and Gilles Francfort*

## Overview

- **Finite-strain** relates to **geometric nonlinearities**
  multiplicative decomposition $\nabla\varphi = \boldsymbol{F} = \boldsymbol{F}_{\text{elast}}\boldsymbol{F}_{\text{plast}}$
- plastic yielding gives **nonsmoothness**
  partially hysteretic behavior in loading cycling




Develop **sufficient conditions** for **existence** such as

- strong hardening
- gradient regularization

to understand the mechanisms for failure

- localization, shearbands
- microstructure formation

**Finite-strain** relates to **geometric nonlinearities**

- invariance under rigid-body transformations (frame indifference)
  $\boldsymbol{C} = \boldsymbol{F}^{\mathsf{T}}\boldsymbol{F}$ symmetric Cauchy strain tensor
- no self-penetration, i.e., $\det \boldsymbol{F} > 0$
- multiplicative decomposition $\boldsymbol{F} = \boldsymbol{F}_{\mathrm{elast}}\boldsymbol{P}$ or $\boldsymbol{F}_{\mathrm{elast}} = \nabla\varphi\, \boldsymbol{P}^{-1}$
- $\boldsymbol{F} \in \mathrm{GL}^{+}(d) := \{\, \boldsymbol{A} \in \mathbb{R}^{d\times d} \mid \det \boldsymbol{A} > 0 \,\}$ and
  $\boldsymbol{P} \in \mathrm{SL}(d) := \{\, \boldsymbol{A} \in \mathbb{R}^{d\times d} \mid \det \boldsymbol{A} = 1 \,\}$

Use (sub) Riemannian and Finslerian geometry
on the Lie groups $\mathrm{GL}^{+}(d)$, $\mathrm{SL}(d)$, ...

**Nonsmoothness** is treated by a weak solution concept:

- energetic solutions do not need derivatives

## Literature

Ortiz et al 1999, . . . , Miehe et al 2002, . . .

M/Theil/Levitas 1999, NoDEA 2001/04, ARMA 2002

Carstensen/Hackl/M [PRSL 2002]
*Non-convex potentials and microstructure in finite-strain plasticity*

M. [CMT 2003] *Energetic formulation of multiplicative elastoplasticity*

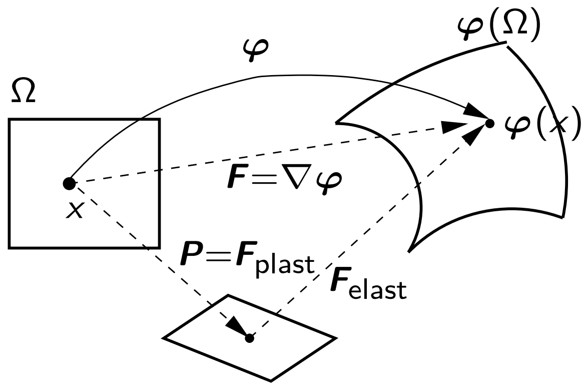
M. [SIMA 2004] *Existence of minimizers in incremental elastoplasticity*

Francfort/M. [JRAM 2006] *Existence results for rate-independent material models.* (based on Dal Maso et al)

M/Müller [ZAMM 2006] *Lower semicontinuity and existence of minimizers for a functional in elastoplasticity*

Mainik/M. 2007 *Global existence for rate-independent strain-gradient plasticity at finite strains.* In preparation.

# Overview




$\boldsymbol{P}$ plastic tensor

$p$ hardening variables

### Multiplicative decomposition

$$\boxed{\nabla\varphi = \boldsymbol{F} = \boldsymbol{F}_{\text{elast}}\boldsymbol{F}_{\text{plast}}}$$

$$\boldsymbol{P} = \boldsymbol{F}_{\text{plast}} \in \mathrm{SL}(d), \quad \boldsymbol{F}_{\text{elast}} = \nabla\varphi\, \boldsymbol{P}^{-1}$$

For elastic properties only $\boldsymbol{F}_{\text{elast}}$ matters.

Plastic invariance (except for hardening)

$$R(\boldsymbol{P}, p, \dot{\boldsymbol{P}}, \dot{p}) = \widehat{R}(\dot{\boldsymbol{P}}\boldsymbol{P}^{-1}, p, \dot{p})$$

**Energy-storage functional** $\mathcal{E}(t, \varphi, \boldsymbol{P}, p) =$

$$\int_\Omega \underbrace{W(\nabla\varphi \boldsymbol{P}^{-1})}_{\text{elastic energy}} + \underbrace{H(\boldsymbol{P}, p)}_{\text{hardening}} + \underbrace{U(\boldsymbol{P}, p, \nabla\boldsymbol{P}, \nabla p)}_{\text{regularization}} \,\mathrm{d}x - \langle \ell(t), \varphi \rangle$$

**Elastic equilibrium:** "$0 = \mathrm{D}_\varphi \mathcal{E}(t, \varphi, \boldsymbol{P}, p)$"

**Plastic flow law:**

$$0 \in \partial_{\dot{\boldsymbol{P}}} R(\boldsymbol{P}, p, \dot{\boldsymbol{P}}, \dot{p}) + \mathrm{D}_{\boldsymbol{P}} \mathcal{E}(t, \varphi, \boldsymbol{P}, p)$$
$$0 \in \partial_{\dot{p}} R(\boldsymbol{P}, p, \dot{\boldsymbol{P}}, \dot{p}) + \mathrm{D}_{p} \mathcal{E}(t, \varphi, \boldsymbol{P}, p)$$

# 2. Finite-strain elastoplasticity

Rayleigh's dissipation potential $R(\boldsymbol{P}, p, \dot{\boldsymbol{P}}, \dot{p})$

Rate independence $R(\boldsymbol{P}, p, \gamma\dot{\boldsymbol{P}}, \gamma\dot{p}) = \gamma^1 R(\boldsymbol{P}, p, \dot{\boldsymbol{P}}, \dot{p})$ for $\gamma \geq 0$.

$\rightsquigarrow$ (sub) Riemannian/Finslerian metric on $\mathrm{SL}(d) \times \mathbb{R}^m$.

To avoid rates we introduce a dissipation distance

$$D(\boldsymbol{P}_0, p_0, \boldsymbol{P}_1, p_1) = \min \Big\{ \int_0^1 R(\widetilde{\boldsymbol{P}}, \widetilde{p}, \dot{\widetilde{\boldsymbol{P}}}, \dot{\widetilde{p}})\,\mathrm{d}s \;\Big|\; (\widetilde{\boldsymbol{P}}, \widetilde{p})(0)) = (\boldsymbol{P}_0, p_0), \\ (\widetilde{\boldsymbol{P}}, \widetilde{p})(1)) = (\boldsymbol{P}_1, p_1),\ (\widetilde{\boldsymbol{P}}, \widetilde{p}) \in \mathrm{C}^1([0,1]; \mathrm{SL}(d) \times \mathbb{R}^m) \Big\}$$

Triangle inequality
$D(\boldsymbol{P}_0, p_0, \boldsymbol{P}_2, p_2) \leq D(\boldsymbol{P}_0, p_0, \boldsymbol{P}_1, p_1) + D(\boldsymbol{P}_1, p_1, \boldsymbol{P}_2, p_2)$

No symmetry, in general $D(\boldsymbol{P}_0, p_0, \boldsymbol{P}_1, p_1) \neq D(\boldsymbol{P}_1, p_1, \boldsymbol{P}_0, p_0)$

Plastic invariance
$D(\boldsymbol{P}_0\boldsymbol{P}_*, p_0, \boldsymbol{P}_1\boldsymbol{P}_*, p_1) = D(\boldsymbol{P}_0, p_0, \boldsymbol{P}_1, p_1)$ for all $\boldsymbol{P}_* \in \mathrm{SL}(d)$

$D(\boldsymbol{P}_0\boldsymbol{P}_*, p_0, \boldsymbol{P}_1\boldsymbol{P}_*, p_1) = D(\boldsymbol{P}_0, p_0, \boldsymbol{P}_1, p_1)$ for all $\boldsymbol{P}_* \in \mathrm{SL}(d)$

Plastic invariance implies

- $D(\boldsymbol{P}_0, p_0, \boldsymbol{P}_1, p_1) = D(\boldsymbol{I}, p_0, \boldsymbol{P}_1\boldsymbol{P}_0^{-1}, p_1)$
- $D(\boldsymbol{I}, p_0, \boldsymbol{P}, p_1) \approx |\log \boldsymbol{P}|$ $\rightsquigarrow$ nonconvex, bad coercivity

**von Mises plasticity:** isotropic hardening $p \in [0, \infty)$

$$R(\boldsymbol{P}, p, \dot{\boldsymbol{P}}, \dot{p}) = \begin{cases} A'(p)\dot{p} & \text{if } \|\dot{\boldsymbol{P}}\boldsymbol{P}^{-1}\|_{\mathrm{vM}} \leq \dot{p}, \\ \infty & \text{else,} \end{cases}$$

$$\text{with } \|\boldsymbol{\xi}\|_{\mathrm{vM}} = \big(\alpha|\boldsymbol{\xi}+\boldsymbol{\xi}^{\mathsf{T}}|^2 + \beta|\boldsymbol{\xi}-\boldsymbol{\xi}^{\mathsf{T}}|^2\big)^{1/2}$$

$\beta = 0$: $D(\boldsymbol{I}, p_0, \boldsymbol{P}, p_1) = A(p_1) - A(p_0)$ for $p_1 - p_0 \geq \frac{\alpha}{2}\big\|\log(\boldsymbol{P}^{\mathsf{T}}\boldsymbol{P})\big\|_{\mathrm{vM}}$

$\beta > 0$: No formula is known!

Under what conditions is $\boldsymbol{P} \mapsto D(\boldsymbol{I}, p_0, \boldsymbol{P}, p_1)$ polyconvex ?

## Overview

Admissible deformations $\Gamma_{\mathsf{Dir}} \subset \partial\Omega$

$\mathcal{F} = \{ \boldsymbol{\varphi} \in \mathrm{W}^{1,q_\varphi}(\Omega;\mathbb{R}^d) \mid \boldsymbol{\varphi} = \boldsymbol{\varphi}_{\mathsf{Dir}} \text{ on } \Gamma_{\mathsf{Dir}} \}$

(for the moment, we assume time-independent Dirichlet data)

Set of internal states $\boldsymbol{z} = (\boldsymbol{P}, p)$

$\mathcal{Z} = \{ (\boldsymbol{P},p) \in \mathrm{W}^{1,r_P}(\Omega;\mathbb{R}^d) \times \mathrm{W}^{1,r_p}(\Omega;\mathbb{R}^m) \mid \boldsymbol{P} \in \mathrm{SL}(d) \text{ a.e. in } \Omega \}$

Energy-storage functional (as above)

$\mathcal{E} : [0,T] \times \mathcal{F} \times \mathcal{Z} \to \mathbb{R}_\infty := \mathbb{R} \cup \{\infty\}$

$(t, \boldsymbol{\varphi}, \boldsymbol{z}) \mapsto \int_\Omega W(\nabla\boldsymbol{\varphi}\boldsymbol{P}^{-1}) + H(\boldsymbol{z}) + U(\boldsymbol{z}, \nabla\boldsymbol{z})\,\mathrm{d}x - \langle \ell(t), \boldsymbol{\varphi} \rangle$

Dissipation distance on $\mathcal{Z}$:

$\mathcal{D} : \mathcal{Z} \times \mathcal{Z} \to [0,\infty]$

$(\boldsymbol{z}_0, \boldsymbol{z}_1) = (\boldsymbol{P}_0, p_0, \boldsymbol{P}_1, p_1) \mapsto \int_\Omega D(\boldsymbol{P}_0(x), p_0(x), \boldsymbol{P}_1(x), p_1(x))\,\mathrm{d}x.$

**Definition [M/Theil'99, M.'03]:**
$(\varphi, \boldsymbol{z}) : [0, T] \to \mathcal{F} \times \mathcal{Z}$ is called an **energetic solution** associated with $\mathcal{E}$ and $\mathcal{D}$, if for all $t \in [0, T]$ the **stability condition (S)** and the **energy balance (E)** hold:

**(S)** $\forall\, (\widetilde{\varphi}, \widetilde{\boldsymbol{z}}) \in \mathcal{F}{\times}\mathcal{Z}$: $\mathcal{E}(t, \varphi(t), \boldsymbol{z}(t)) \leq \mathcal{E}(t, \widetilde{\varphi}, \widetilde{\boldsymbol{z}}) + \mathcal{D}(\boldsymbol{z}(t), \widetilde{\boldsymbol{z}})$

**(E)** $\mathcal{E}(t, \varphi(t), \boldsymbol{z}(t)) + \mathrm{Diss}_{\mathcal{D}}(\boldsymbol{z}, [0, t])$

$$= \mathcal{E}(0, \varphi(0), \boldsymbol{z}(0)) + \int_0^t \underbrace{\partial_\tau \mathcal{E}(\tau, \varphi(\tau), \boldsymbol{z}(\tau))}_{\text{power ext. forces}} \,\mathrm{d}\tau$$

Any sufficiently smooth energetic solution solves

**Elastic equilibrium:** $0 = \mathrm{D}_\varphi \mathcal{E}(t, \varphi, \boldsymbol{z})$

**Plastic flow law:** $0 \in \partial_{\dot{\boldsymbol{z}}} R(\boldsymbol{z}, \dot{\boldsymbol{z}}) + \mathrm{D}_{\boldsymbol{z}} \mathcal{E}(t, \varphi, \boldsymbol{z})$

However, in general energetic solutions have jumps w.r.t. to time.

WIAS

# Overview

W I A S

Main assumptions on $\mathcal{E}$ and $\mathcal{D}$:

$\mathcal{E}(t,\varphi,\boldsymbol{P},p) \mapsto \int_\Omega W(\nabla\varphi\boldsymbol{P}^{-1})+H(\boldsymbol{P},p)+U(\boldsymbol{P},p,\nabla\boldsymbol{P},\nabla p)\,\mathrm{d}x-\langle \ell(t),\varphi\rangle$

$\mathcal{D}(\boldsymbol{P}_0,p_0,\boldsymbol{P}_1,p_1) = \int_\Omega D(\boldsymbol{P}_0,p_0,\boldsymbol{P}_1,p_1)\,\mathrm{d}x$

## (A1) Lower semicontinuity:

$W : \mathbb{R}^{d\times d} \to \mathbb{R}_\infty$ is polyconvex and lsc

$H : \mathrm{SL}(d)\times\mathbb{R}^m \to \mathbb{R}_\infty$ is lsc

$U$ continuous and $U(\boldsymbol{P},p,\cdot,\cdot)$ is convex

$D : (\mathrm{SL}(d)\times\mathbb{R}^m)\times(\mathrm{SL}(d)\times\mathbb{R}^m) \to [0,\infty]$ lsc

## (A2) Coercivity:

$W(\boldsymbol{F}_\mathrm{el}) \geq c|\boldsymbol{F}_\mathrm{el}|^{q_F} - C$

$H(\boldsymbol{P},p) + D(\boldsymbol{I},p_*,\boldsymbol{P},p) \geq c\big(|\boldsymbol{P}|^{q_P}+|\boldsymbol{P}^{-1}|^{q_P}+|p|^{r_p}\big) - C$

$U(\boldsymbol{P},p,\nabla\boldsymbol{P},\nabla p) \geq c\big(|\nabla\boldsymbol{P}|^{r_P}+|\nabla p|^{r_p}\big) - C$

**Proposition.** Let (A1) and (A2) hold with
$$\frac{d}{q_{\boldsymbol{P}}} + \frac{1}{q_{\boldsymbol{F}}} < 1 \text{ and } r_p, r_{\boldsymbol{P}} > 1.$$
Define $q_\varphi$ via $\frac{1}{q_\varphi} = \frac{1}{q_{\boldsymbol{P}}} + \frac{1}{q_{\boldsymbol{F}}}$ and assume $q_\varphi > d$ and $\ell \in \mathrm{C}^1\big([0,T], (\mathrm{W}^{1,q_\varphi}(\Omega;\mathbb{R}^d))^*\big)$. Consider

$\mathcal{F} \subset \mathrm{W}^{1,q_\varphi}(\Omega;\mathbb{R}^d)$ and

$\mathcal{Z} \subset \big(\mathrm{W}^{1,r_{\boldsymbol{P}}}(\Omega;\mathbb{R}^{d\times d}) \cap \mathrm{L}^{q_{\boldsymbol{P}}}(\Omega;\mathrm{SL}(d))\big) \times \mathrm{W}^{1,r_p}(\Omega;\mathbb{R}^m)$

equipped with the weak topologies, respectively. Then,

- $\mathcal{D}$ is weakly lower semicontinuous on $\mathcal{Z} \times \mathcal{Z}$
- $\mathcal{E}(t,\cdot)$ has weakly compact sublevels in $[0,T]\times\mathcal{F}\times\mathcal{Z}$ ($\Longleftrightarrow$ coercivity and lower semicontinuity)

### Lower semicontinuity

$\nabla\varphi_k \rightharpoonup \nabla\varphi$ in $\mathrm{L}^{q_\varphi}$ and $\boldsymbol{P}_k \to \boldsymbol{P}$ in $\mathrm{L}^{q_{\boldsymbol{P}}}$

$\Longrightarrow$ minors converge: $\mathbb{M}_s(\nabla\varphi_k\boldsymbol{P}_k^{-1}) \rightharpoonup \mathbb{M}_s(\nabla\varphi\boldsymbol{P}^{-1})$ in $\mathrm{L}^{q_s}$

# 4. Existence results

Existence via the **Incremental Minimization Problem (IP):**

Partitions $0 = t_0^N < t_1^N < \cdots < t_{K_N-1}^N < t_{K_N}^N = T$

$(\varphi_k^N, \boldsymbol{z}_k^N) \in \operatorname{Arg\,Min}\{\, \mathcal{E}(t_k^N, \widetilde{\varphi}, \widetilde{\boldsymbol{z}}) + \mathcal{D}(\boldsymbol{z}_{k-1}^N, \widetilde{\boldsymbol{z}}) \mid \widetilde{\varphi} \in \mathcal{F},\, \widetilde{\boldsymbol{z}} \in \mathcal{Z} \,\}$.

Piecewise constant interpolants $(\overline{\varphi}^N, \overline{\boldsymbol{z}}^N) : [0, T] \to \mathcal{F} \times \mathcal{Z}$.

Limit passage needs the **Joint Recovery Condition**

If $(\varphi_n, \boldsymbol{z}_n)$ stable at $t_n$, $(t_n, \varphi_n, \boldsymbol{z}_n) \rightharpoonup (t, \varphi, \boldsymbol{z})$, $\mathcal{E}(t_n, \varphi_n, \boldsymbol{z}_n) \le C$
and $(\widehat{\varphi}, \widehat{\boldsymbol{z}}) \in \mathcal{F} \times \mathcal{Z}$,
then there exists $(\widehat{\varphi}_n, \widehat{\boldsymbol{z}}_n)$ such that

$$\limsup_{n\to\infty} \mathcal{E}(t_n, \widehat{\varphi}_n, \widehat{\boldsymbol{z}}_n) + \mathcal{D}(\boldsymbol{z}_n, \widehat{\boldsymbol{z}}_n) - \mathcal{E}(t_n, \varphi_n, \boldsymbol{z}_n) \le \mathcal{E}(t, \widehat{\varphi}, \widehat{\boldsymbol{z}}) + \mathcal{D}(\boldsymbol{z}, \widehat{\boldsymbol{z}}) - \mathcal{E}(t, \varphi, \boldsymbol{z})$$

Easy to satisfy, if $\mathcal{D}$ is weakly continuous: $\widehat{\boldsymbol{z}}_n = \widehat{\boldsymbol{z}}$

**Existence Result [Mainik/M.'07]**
Assume (A1), (A2) and that either (i) or (ii) holds:

(i) $D : (\mathrm{SL}(d)\times\mathbb{R}^m) \times (\mathrm{SL}(d)\times\mathbb{R}^m) \to [0,\infty)$ is continuous and

$$D(\boldsymbol{P}_0, p_0, \boldsymbol{P}_1, z_1) \leq C\big(|\boldsymbol{P}_0|^{q_{\boldsymbol{P}}} + |\boldsymbol{P}_1|^{q_{\boldsymbol{P}}} + |p_0|^{r_p} + |p_1|^{r_p}\big)$$

(ii) $r_{\boldsymbol{P}}, r_p > d$ and for each $(\boldsymbol{P}_0, p_0)$ the set
$\{\,(\boldsymbol{P}_1, p_1) \mid D(\boldsymbol{P}_0, p_0, \boldsymbol{P}_1, p_1) < \infty\,\}$ has nonempty interior.

Then, for each stable initial state $(\varphi_0, \boldsymbol{z}_0)$ with $\mathcal{D}(\boldsymbol{I}, p_*, \boldsymbol{z}_0) < \infty$ there exists an energetic solution $(\varphi, \boldsymbol{z}) : [0, T] \to \mathcal{F} \times \mathcal{Z}$.

Case (i) applies for "kinematic hardening" (no $p$ needed)

Case (ii) is needed for isotropic hardening, i.e., $p \in \mathbb{R}$:

$$H(\boldsymbol{P}, p) = \exp(\kappa p), \quad D(\boldsymbol{I}, p_0, \boldsymbol{P}, p_1) = \begin{cases} p_1 - p_0 & \text{for } p_1 - p_0 \geq |\log(\boldsymbol{P}^{\mathsf{T}}\boldsymbol{P})| \\ \infty & \text{else} \end{cases}$$

# Overview

Essential idea in FRANCFORT/M. 2006

using FRANCFORT, DAL MASO, TOADER 2005 and BALL 2002

Dirchlet data $\varphi(t,x) = \varphi_{\mathrm{Dir}}(t,x)$ for $(t,x) \in [0,T] \times \Gamma_{\mathrm{Dir}}$.

Usual additive ansatz:

$\varphi(t,\cdot) = \varphi_{\mathrm{Dir}}(t,\cdot) + \psi(t,\cdot)$ with

$\psi(t) \in \overline{\mathcal{F}} = \{\, \psi \in \mathrm{W}^{1,q_\varphi}(\Omega;\mathbb{R}^d) \mid \psi|_{\Gamma_{\mathrm{Dir}}} = 0 \,\}$

$\overline{\mathcal{E}}(t,\psi,\boldsymbol{z}) = \mathcal{E}(t,\varphi_{\mathrm{Dir}}(t){+}\psi,\boldsymbol{z})$

**Power of external loading** (now including power of $\varphi_{\mathrm{Dir}}$)

$$\partial_t\overline{\mathcal{E}}(t,\psi,\boldsymbol{z}) = \int_\Omega \underbrace{\partial_{\boldsymbol{F}} W((\nabla\varphi_{\mathrm{Dir}}(t){+}\nabla\psi)\boldsymbol{P}^{-1})}_{1^{\mathrm{st}}\text{ Piola-Kirchhoff tensor. }\boldsymbol{T}} : \underbrace{(\nabla\dot\varphi_{\mathrm{Dir}}\boldsymbol{P}^{-1})}_{\in \mathrm{L}^q(\Omega)}\,\mathrm{d}x - \langle \dot\ell(t),\psi\rangle$$

In general we cannot expect $\boldsymbol{T} \in \mathrm{L}^{q'}(\Omega)$, not even $\mathrm{L}^{1}(\Omega)$

# 5. Time-dependent boundary data

***Way out?*** $\rightsquigarrow$ Matrices are elements of Lie groups: Multiply !!

Extend $\varphi_{\mathrm{Dir}}$ to all of $\mathbb{R}^d$, namely $\boldsymbol{g}_{\mathrm{Dir}} \in \mathrm{C}^2([0,T] \times \mathbb{R}^d; \mathbb{R}^d)$ such that $\nabla \boldsymbol{g}_{\mathrm{Dir}}, (\nabla \boldsymbol{g}_{\mathrm{Dir}})^{-1} \in \mathrm{L}^\infty([0,T] \times \mathbb{R}^d; \mathbb{R}^{d\times d})$

$\varphi(t,x) = \boldsymbol{g}_{\mathrm{Dir}}(t, \psi(t,x))$ with
$\psi(t) \in \widetilde{\mathcal{F}} = \{\, \psi \in \mathrm{W}^{1,q_\varphi}(\Omega; \mathbb{R}^d) \mid \psi|_{\Gamma_{\mathrm{Dir}}} = \mathrm{id}_{\Gamma_{\mathrm{Dir}}} \,\}$

**Chain rule** replaces linearity:

$$\nabla_x \varphi(t,x) = \nabla_y \boldsymbol{g}_{\mathrm{Dir}}(t, \psi(t,x)) \nabla_x \psi(t,x)$$

$$\widetilde{\mathcal{E}}(t, \psi, \boldsymbol{z}) = \mathcal{E}(t, \boldsymbol{g}_{\mathrm{Dir}}(t,\cdot)\circ\psi, \boldsymbol{z})$$

**Power of external loading**

$$\partial_t \widetilde{\mathcal{E}}(t,\psi,\boldsymbol{z}) = \int_\Omega \partial_{\boldsymbol{F}} W(\nabla \boldsymbol{g}_{\mathrm{Dir}}(t,\psi)\nabla\psi \boldsymbol{P}^{-1}) : (\nabla \dot{\boldsymbol{g}}_{\mathrm{Dir}}(t,\psi)\nabla\psi \boldsymbol{P}^{-1})\,\mathrm{d}x$$

$$\partial_t \widetilde{\mathcal{E}}(t,\boldsymbol{\psi},\boldsymbol{z}) = \int_\Omega \partial_{\boldsymbol{F}} W(\underbrace{\nabla \boldsymbol{g}_{\text{Dir}}}_{\boldsymbol{G}} \underbrace{\nabla \psi}_{\boldsymbol{F}} \boldsymbol{P}^{-1}) : (\underbrace{\nabla \dot{\boldsymbol{g}}_{\text{Dir}}}_{\boldsymbol{H}} \nabla \psi \boldsymbol{P}^{-1})\, \mathrm{d}x$$

$$= \int_\Omega \underbrace{\partial_{\boldsymbol{F}} W(\boldsymbol{GFP}^{-1})(\boldsymbol{GFP}^{-1})^{\mathsf{T}}}_{\text{Kirchhoff } \boldsymbol{K}(\boldsymbol{GFP}^{-1})} : \underbrace{\left(\boldsymbol{HFP}^{-1}(\boldsymbol{GFP}^{-1})^{-1}\right)}_{\boldsymbol{V} := \boldsymbol{HG}^{-1}} \mathrm{d}x$$

$\boldsymbol{V} \in \mathrm{L}^\infty([0,T]\times\Omega;\mathbb{R}^{d\times d})$ independently of $\psi : [0,T] \to \widetilde{\mathcal{F}}$

## Multiplicative stress control

(Baumann, Phillips, Owen'91, Ball '02)

Kirchhoff tensor $\boldsymbol{K}(\boldsymbol{F}) := \partial_{\boldsymbol{F}} W(\boldsymbol{F})\boldsymbol{F}^{\mathsf{T}}$

$$\exists\, c_0, c_1\ \forall\, \boldsymbol{F} \in \mathrm{GL}^+(d)_+(\mathbb{R}^d): \quad |\boldsymbol{K}(\boldsymbol{F})| \leq c_1\big(W(\boldsymbol{F}) + c_0\big)$$

# 5. Time-dependent boundary data

Multiplicative stress control is
***compatible with finite-strain polyconvexity:***

$W(\boldsymbol{F}) = \alpha|\boldsymbol{F}|^q + \frac{\beta}{(\det \boldsymbol{F})^\gamma}$

$\boldsymbol{T}(\boldsymbol{F}) = \alpha q|\boldsymbol{F}|^{q-2}\boldsymbol{F} - \frac{\beta\gamma}{(\det \boldsymbol{F})^{\gamma+1}} \operatorname{cof} \boldsymbol{F}$ (not bounded by $W$)

$\boldsymbol{K} = \boldsymbol{T}\boldsymbol{F}^\mathsf{T} = \alpha q|\boldsymbol{F}|^{q-2}\boldsymbol{F}\boldsymbol{F}^\mathsf{T} - \frac{\beta\gamma}{(\det \boldsymbol{F})^\gamma}\boldsymbol{I}$ (as $(\operatorname{cof} \boldsymbol{F})\,\boldsymbol{F}^\mathsf{T} = (\det \boldsymbol{F})\boldsymbol{I}$)

Hence, $|\boldsymbol{K}(\boldsymbol{F})| \leq \max \alpha, \beta\sqrt{d}\,W(\boldsymbol{F})$.

**Corollary:** There exist $c_0^E, c_1^E$ such that
$|\partial_t\widetilde{\mathcal{E}}(t, \psi, \boldsymbol{z})| \leq c_1^E\big(\widetilde{\mathcal{E}}(t, \psi, \boldsymbol{z}) + c_0^E\big)$
whenever the right-hande side is finite.

- Provides useful a priori **energetic** estimates
- needs of course sufficient hardening
- ALBER's group: Self-controling property

# Conclusion

Using **full regularization** and **strong coercivity** the existence of energetic solutions can be shown for many plasticity models.

Future tasks:

- Study formation of microstructure for vanishing regularization: $U(\boldsymbol{z},\nabla\boldsymbol{z}) = \varepsilon|\nabla\boldsymbol{z}|^r$ with $\varepsilon \to 0$

- Study failure via localization for vanishing strong coercivity

  $H(\boldsymbol{P},p) = |p|^2 + \varepsilon\exp(\kappa|p|)$ for $\varepsilon \to 0$
  - deformation gradients $\nabla\boldsymbol{\varphi}^\varepsilon$ behaves badly,
  - but energy densities and stresses have suitable limits (cf. [Bouchitté/M/Roubíček'07])

- Replace global minimization by local minimization in (IP) viscous approximations

# Thank you for your attention!

Papers available under http://www.wias-berlin.de/people/mielke

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