EVOLUTIONARY FINANCE AND DYNAMIC GAMES

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Evolutionary ideas in Economics and Finance

have a long history going back to Malthus, who played an inspirational role for Darwin.

A more recent stage of development of these ideas began in the 1950s with the publications of Alchian (1950) and others.

A powerful momentum to work in this area was given by the interdisciplinary research conducted in the 1980s and 1990s under the auspices of the Santa Fe Institute in New Mexico, USA, where researchers of different backgrounds—economists, mathematicians, physicists and biologists—combined their efforts to study evolutionary dynamics in biology, economics and finance.

Arthur, Farmer, Blume, Durlauf, Easley, Kirman, Holland, LeBaron, Dosi, Bottazzi, Brock, Hommes, Wagener, and others.

Objectives and approach

♦ While inspired by the above studies, especially by the pioneering work of Blume and Easley (1992), our approach to evolutionary finance is different from theirs both in the modeling frameworks and in the specific problems analyzed.

♦ We consider models based on *random dynamical systems* and *short-run (temporary) equilibrium*, rather than on the conventional general equilibrium settings (Radner equilibrium, perfect foresight), where agents maximize discounted expected utilities.

The models do not rely upon unobservable agents' characteristics, such as individual utilities or beliefs.

We aim at obtaining quantitative results, in particular, explicit formulas for surviving portfolio rules.

We consider the evolutionary approach as a means for developing a new class of dynamic equilibrium models, which would provide a plausible alternative to GE and make the theory closer to applications.

♦ We revive in the new context the *Marshallian* concept of temporary equilibrium. This notion is different from the one going back to Hicks and Lindahl (1930-40s), which has prevailed in the GE literature in the last decades (e.g. Grandmont 1988). The idea is to distinguish between two sets of economic variables changing with different speeds; "freeze" one of them, and equilibrate the other.

THE MODEL

Randomness

S space of ”states of the world” (a measurable space);

$s_t \in S$ ($t = 1, 2, \ldots$) state of the world at date t;

$s_1, s_2, \ldots$ an exogenous stochastic process.

Assets

There are K *assets* .

At each date t, assets $k = 1, \ldots, K$ pay *dividends* $D_{t,k}(s^t) \geq 0,\ k = 1, \ldots, K,$ depending on the history

$$s^t := (s_1, \ldots, s_t)$$

of the states of the world up to date t.

Basic assumptions

$$\sum_{k=1}^{K} D_{t,k}(s^t) > 0,$$

$$ED_{t,k}(s^t) > 0,\ k = 1, \ldots, K,$$

where E is the expectation with respect to the underlying probability P.

Asset supply

Total mass (the number of "physical units") of asset k available at date t is $V_{t,k} = V_{t,k}(s^t)$.

Investors and their portfolios. There are N *investors (traders)* $i \in \{1, \ldots, N\}$. Investor i at date $t = 0, 1, 2, \ldots$ selects a *portfolio*

$$x_t^i = (x_{t,1}^i, \ldots, x_{t,K}^i) \in \mathrm{R}_+^K,$$

where $x_{t,k}^i$ is the number of units of asset k in the portfolio x_t^i. The portfolio x_t^i for $t \geq 1$ depends, generally, on the current and previous states of the world:

$$x_t^i = x_t^i(s^t), \ \ s^t = (s_1, \ldots, s_t).$$

Asset prices. We denote by $p_t \in \mathrm{R}_+^K$ the vector of market prices of the assets. For each $k = 1, \ldots, K$, the coordinate $p_{t,k}$ of $p_t = (p_{t,1}, \ldots, p_{t,K})$ stands for the price of one unit of asset k at date t. The scalar product

$$\langle p_t, x_t^i \rangle := \sum_{k=1}^{K} p_{t,k} x_{t,k}^i$$

expresses the market value of the investor i's portfolio x_t^i at date t.

The state of the market at date t:

$$(p_t, x_t^1, \ldots, x_t^N),$$

where p_t is the vector of asset prices and $x_t^1, \ldots, x_t^N$ are the portfolios of the investors.

Investors' budgets. At date $t = 0$ investors have initial endowments $w_0^i > 0$ ($i = 1, 2, \dots, N$).

Trader i's budget at date $t \geq 1$ is

$$B_t^i(p_t, x_{t-1}^i) := \langle D_t(s^t) + p_t, x_{t-1}^i \rangle,$$

where

$$D_t(s^t) := (D_{t,1}(s^t), \dots, D_{t,K}(s^t)).$$

Two components:

the dividends $\langle D_t(s^t), x_{t-1}^i \rangle$ paid by the yesterday's portfolio x_{t-1}^i;

the market value $\langle p_t, x_{t-1}^i \rangle$ of the portfolio x_{t-1}^i in the today's prices p_t.

Investment rate. A fraction α of the budget is invested into assets. We will assume that the *investment rate* $\alpha \in (0, 1)$ is a fixed number, the same for all the traders. The number $1 - \alpha$ can represent the *tax rate* or the *consumption rate*. The assumption that $1 - \alpha$ is the same for all the investors is quite natural in the former case. In the latter case, it might seem restrictive, but in the present context it is indispensable since we focus in this work on the analysis of the comparative performance of trading strategies (portfolio rules) in the long run. Without this assumption, an analysis of this kind does not make sense: a seemingly worse performance of a portfolio rule might be simply due to a higher consumption rate of the investor.

Investment proportions. For each $t \geq 0$, each trader $i = 1, 2, \ldots, N$ selects a vector of *investment proportions*

$$\lambda_t^i = (\lambda_{t,1}^i, \ldots, \lambda_{t,K}^i) \in \Delta^K$$

in the unit simplex Δ^K, according to which the budget is distributed between assets.

In terms of the game under consideration, λ_t^i are the players' (investors') *actions* or *decisions*.

History of the game. Players' decisions might depend on the history $s^t := (s_1, \ldots, s_t)$ of states of the world and the *history of the game*

$$(p^{t-1}, x^{t-1}, \lambda^{t-1}),$$

where

$$p^{t-1} = (p_0, \ldots, p_{t-1}),$$

$$x^{t-1} := (x_0, x_1, \ldots, x_{t-1}), \; x_l = (x_l^1, \ldots, x_l^N),$$

$$\lambda^{t-1} = (\lambda_0, \lambda_1, \ldots, \lambda_{t-1}), \; \lambda_l = (\lambda_l^1, \ldots, \lambda_l^N),$$

The history of the game contains information about the *market history*

$$(p_0, x_0), \ldots, (p_{t-1}, x_{t-1})$$

and the decisions λ_l^i, $l = 0, \ldots, t-1$, of all the players $i = 1, \ldots, N$ at all the previous dates .

Investment strategies. A vector $\Lambda_0^i \in \Delta^K$ and a sequence of measurable functions with values in Δ^K

$$\Lambda_t^i(s^t, p^{t-1}, x^{t-1},\ \lambda^{t-1}),\ t = 1, 2, \ldots$$

form an *investment strategy* (*portfolio rule*) Λ^i of investor i.

Basic strategies. Among general portfolio rules, we distinguish those for which Λ_t^i depends only on s^t, and not on $(p^{t-1}, x^{t-1}, \lambda^{t-1})$. We will call such portfolio rules *basic*. They play an important role in the present work: the survival strategy we construct belongs to this class.

Investor i's demand function. Given a vector of investment proportions $\lambda_t^i = (\lambda_{t,1}^i, \ldots, \lambda_{t,K}^i)$ of investor i, the i's demand function is

$$X_{t,k}^i(p_t, x_{t-1}^i) = \frac{\alpha \lambda_{t,k}^i B_t^i(p_t, x_{t-1}^i)}{p_{t,k}}.$$

where α is the investment rate.

Equilibrium: for each t, aggregate demand for every asset is equal to supply:

$$\sum\nolimits_{i=1}^{N} X_{t,k}^i(p_t, x_{t-1}^i) = V_{t,k},\ k = 1, \ldots, K.$$

Equilibrium market dynamics.

Prices:

$$p_{t,k}V_{t,k} = \sum_{i=1}^{N} \alpha\lambda_{t,k}^{i}\langle D_t(s^t) + p_t, x_{t-1}^{i}\rangle,\ \ k = 1,\ldots,K;$$

Portfolios:

$$x_{t,k}^{i} = \frac{\alpha\lambda_{t,k}^{i}\langle D_t(s^t) + p_t, x_{t-1}^{i}\rangle}{p_{t,k}},\ \ k = 1,\ldots,K,\ \ i = 1,2,\ldots,N.$$

The vectors of *investment proportions* $\lambda_t^i = (\lambda_{t,k}^i)$:

$$\lambda_t^i(s^t) := \Lambda_t^i(s^t, p^{t-1}, x^{t-1},\ \lambda^{t-1}).$$

The pricing equation has a unique solution $p_{t,k} \geq 0$ if $V_{t,k} \geq V_{t-1,k}$ (growth), or under a weaker assumption: $\alpha V_{t-1,k}/V_{t,k} < 1$.

Admissible strategy profiles. We will consider only *admissible* strategy profiles – those for which aggregate demand for each asset is always strictly positive. This guarantees that $p_{t,k} > 0$ (only in this case the above formula for $x_{t,k}^i$ makes sense). If at least one of the portfolio rules has strictly positive investment proportions, then the strategy profile is admissible. This will be the case in all the situations we shall consider (in this sense, the focus on admissible strategy profiles does not restrict generality).

Market shares of the investors. Investor i's *wealth* at time t is

$$w_t^i = \langle D_t(s^t) + p_t, x_{t-1}^i \rangle$$

(dividends $+$ portfolio value). Investor i's *relative wealth*, or *market share,* is

$$r_t^i = \frac{w_t^i}{w_t^1 + \ldots + w_t^N}.$$

Survival strategies. Given an admissible strategy profile $(\Lambda^1, \ldots, \Lambda^N)$, we say that the portfolio rule Λ^1 (or the investor 1 using it) *survives* with probability one if

$$\inf_{t \geq 0} r_t^1 > 0 \text{ (a.s.)},$$

(the market share of investor 1 is bounded away from zero by a strictly positive random constant).

A portfolio rule is called a *survival strategy* if the investor using it survives with probability one *irrespective of what portfolio rules are used by the other investors*.

Our main goal is to identify survival strategies.

COMMENTS ON THE MODEL

Marshallian temporary equilibrium. In the model we deal with, the dynamics of the asset market is modeled in terms of a sequence of temporary equilibria. As it was noticed by Samuelson (1947), in order to study the process of market dynamics by using the Marshallian "moving equilibrium method," one needs to distinguish between at least two sets of economic variables changing with different speeds. Then the set of variables changing slower (in our case, the set of vectors of investment proportions) can be temporarily fixed, while the other (in our case, the asset prices) can be assumed to rapidly reach the unique state of partial equilibrium.

Asset allocation. Specification of portfolio rules in terms of proportions according to which wealth is allocated across assets is standard in financial practice. Typically, these proportions are held fixed during some period, which requires *portfolio rebalancing*. The question why portfolio rebalancing "adds value" has been considered from various angles. The role of volatility as a paradoxical endogenous source of growth, explaining the phenomenon of "volatility pumping", has been revealed in:

Dempster, Evstigneev and Schenk-Hoppé, Volatility-induced financial growth, Quantitative Finance (2007).

THE RESULTS

Assumption 1. Assume that the total mass of each asset grows (or decreases) at the same constant rate $\gamma > \alpha$:

$$V_{t,k} = \gamma^t V_k ,$$

where V_k ($k = 1, 2, \ldots, K$) are the initial amounts of the assets. In the case of real assets—involving long-term investments with dividends (e.g., real estate, transport, communications, IT, etc.)—the above assumption means that the system under consideration is on a *balanced growth path*.

Relative dividends. Define the *relative dividends* of the assets $k = 1, \ldots, K$ by

$$R_{t,k} = R_{t,k}(s^t) := \frac{D_{t,k}(s^t)V_k}{\sum_{m=1}^K D_{t,m}(s^t)V_m}, \; k = 1, \ldots, K, \; t \geq 1,$$

and put $R_t(s^t) = (R_{t,1}(s^t), \ldots, R_{t,K}(s^t))$.

Definition of the survival strategy Λ^*. Put

$$\rho := \alpha/\gamma, \; \rho_t := \rho^{t-1}(1 - \rho)$$

and consider the portfolio rule Λ^* with the vectors of investment proportions

$$\lambda_t^*(s^t) = (\lambda_{t,1}^*(s^t), \ldots, \lambda_{t,K}^*(s^t)),$$

$$\lambda_{t,k}^* = E_t \sum_{l=1}^{\infty} \rho_l R_{t+l,k} ,$$

where $E_t(\cdot) = E(\cdot|s^t)$ is the conditional expectation given s^t; $E_0(\cdot)$ is the unconditional expectation $E(\cdot)$.

The meaning of Λ^*.

♦ The portfolio rule Λ^* prescribes to distribute wealth across assets in accordance with the proportions of the expected flow of their discounted future relative dividends.

♦ The discount rate $\rho_{t+1}/\rho_t = \rho$ is equal to the investment rate α divided by the growth rate γ.

♦ Note that the portfolio rule Λ^* is *basic*: the investment proportions $\lambda^*_{t,k}(s^t)$ depend on the states of the world $s^t = (s_1, \ldots, s_t)$, but do not depend on the history of the game $(p^{t-1}, x^{t-1},\ \lambda^{t-1})$.

Λ^* **and the Kelly rule**. The strategy Λ^* is a generalization of the Kelly portfolio rule of “betting your beliefs” playing a central role in capital growth theory—Kelly (1956), Breiman (1961), Algoet and Cover (1988), Hakansson and Ziemba (1995), and others.

If $s_t \in S$ are i.i.d. and

$$R_{t,k}(s^t) = R_k(s_t),$$

then

$$\lambda^*_{t,k} = ER_k(s_t),$$

and so Λ^* is a *constant proportions strategy* (independent of ρ).

In the case of Arrow securities (or "horse race model"), the expectations $ER_k(s_t)$ are equal to the *probabilities* of the states of the world – hence “betting your beliefs”.

Assumption 2. There exists $\delta > 0$ such that

$$E_t R_{t+1,k}(s^{t+1}) > \delta \text{ (a.s.).}$$

This implies that $\lambda^*_{t,k} > 0$, and so any strategy profile containing Λ^* is admissible.

A central result is as follows.

Theorem 1: *The portfolio rule* Λ^* *is a survival strategy.*

Asymptotic uniqueness. The following theorem shows that in the class of basic strategies, the survival portfolio rule is essentially unique: any survival strategy is asymptotically similar to Λ^*.

Theorem 2. *If* $\Lambda = (\lambda_t)$ *is a basic survival strategy, then*

$$\sum_{t=0}^{\infty} \|\lambda_t^* - \lambda_t\|^2 < \infty \text{ (a.s.).}$$

Theorem 2 is akin to various *turnpike* results in the theory of economic dynamics, expressing the idea that all optimal or asymptotically optimal paths of an economic system follow in the long run essentially the same route — the turnpike (Samuelson, McKenzie, Radner, Nikaido, and others).

Theorem 2 is a direct analogue of Gale's turnpike theorem for "*good programs*":

D. Gale, On optimal development in a multi-sector economy, Rev. Econ. Stud. (1967).

Stochastic versions of this result:

V. Arkin and I. Evstigneev, Stochastic Models of Control and Economic Dynamics, Acad. Press, London (1987).

Game-theoretic content

Survival as a solution concept. In the model at hand, strategic interaction of portfolio rules of the players results in the outcome of the game for each player i – the random sequence of i's market shares $(r_t^i)_{t=0}^{\infty}$. The notion of a survival strategy is the *solution concept* we adopt in the analysis of this game. It is of course distinct from the Nash equilibrium concept: the players do not maximize any explicitly given objective functions or preference relations.

Asymptotic (**comparative**) **optimality**. Although the idea of survival does not initially involve optimization, we can reformulate the notion of a survival strategy so as to reveal its property of *asymptotic comparative optimality*.

♦ For two sequences of positive random numbers (w_t) and (w_t'), we write

$$(w_t) \preccurlyeq (w_t') \text{ iff } w_t \leq H w_t' \text{ (a.s.)}$$

for some random constant H, i.e. w_t does not grow asymptotically faster than w_t'.

♦ Let (w_t^i) denote the wealth process of investor i. A portfolio rule Λ^1 is a survival strategy if and only if the following condition holds. If investor 1 uses Λ^1, then

$$(w_t^i) \preccurlyeq (w_t^1) \text{ for all } i = 2, \ldots, N$$

and any strategies $\Lambda^2, \ldots, \Lambda^N$, i.e. *no other investor can outperform* 1 *in terms of the asymptotic growth rate of wealth*.

REFERENCES

The Kelly rule was first formulated in the pioneering study by Kelly (1956), followed by Breiman (1961), Algoet and Cover (1988) and others. An inspirational role for this line of work was played by ideas of Claude Shannon (unpublished lectures). For the history of these ideas see:

T. M. Cover, Shannon and investment, IEEE Information Theory Society Newsletter, Summer 1998.

A different notion of survival (similar to that in the classical ruin problem) was considered by

Milnor and Shapley, On games of survival, in: Contributions to the Theory of Games III, Princeton Univ. Press (1957).

The model under consideration is developed in:

E, Hens, S-H, Evolutionary stable stock markets, Economic Theory (2006);

E, Hens, S-H, Globally evolutionarily stable portfolio rules, J. Econ. Theory (2008).

As $\rho \to 0$, the model reduces to the one with "short-lived", one-period assets considered by

L. Blume and D. Easley, Evolution and market behavior, J. Econ. Theory (1992).

The results presented were obtained in two papers:

Amir, E. and S.-H., Asset market games of survival, Swiss Finance Institute, Working Paper (2008).

Amir, E., Hens and Xu, Evolutionary finance and dynamic games, Swiss Inst. of Banking, Univ. of Zurich (2009).