On the complicated price dynamics of simple one- and two-dimensional discontinuous financial markets with heterogeneous interacting traders

Fabio Tramontana – University of Ancona and University of Urbino

Laura Gardini – University of Urbino (Italy)

Tony He – University of Technology (Sidney, Australia)

Frank Westerhoff – University of Bamberg (Germany)

Financial markets

✓ *Bubbles* and *crashes* have been frequently observed in the past. In some cases, these events had an impact on the real economy, triggering, for instance, deeper recessions;

✓ Moreover, the volatility in financial markets may be regarded as excessively high in the sense that *prices fluctuate more strongly than warrented by the underlying fundamentals*;

✓ Also extreme price changes, which make up a large part of financial market risk, occur rather frequently;

✓ Empirical accounts on thse phenomena are provided by Sornette (2003), Shiller (2005, 2008) and Lux (2009).

What is driving the dynamics of financial makets?

A fundamental economic principle says that prices are put into motion if demand is unequal to supply.

In the case of financial markets, a markets' order imbalance is caused by the trading behavior of its market partecipants.

Some empirical evidence helps us to understand how agents determine their speculative orders (see Menkoff and Taylor, 2007 and Murphy, 1999).

In particular, market partecipants rely on both technical and fundamental trading rules.

- Technical analysis is a trading method that seeks to identify trading signals out of past price movements. As a result, technicians (or chartists) may have a destibilizing impact on the dynamics of financial markets;

- Fundamental analysis presumes that prices will mean revert toward fundamental values, inducing, in general, some kind of market stability.

Models with heterogeneous agents

Models with heterogeneous agents take into account how exactly do markets with a diverse ecology of interacting technical and fundamental traders functions.

We can sketch three of the main frameworks:

I. Day and Huang (1990) show that endogenous price dynamics may be trigerred by nonlinear trading rules. In their model chartists apply a linear trading rule, while the trading behavior of fundamentalists is nonlinear. The model displays an intricate and unpredictable alternance of fundamental and chartist dominance;

II. Agents may switch between technical and fundamental behaviors. Depending on the popularity of fundamental or technical analysis the price may fluctuate close to the fundamental value and far from it. See Brock and Hommes (1998), Kirman (1991) and Lux (1998);

III. If traders can switch among several markets, they can be stabilized (destabilized) by the relative number of fundamentalists (chartists). See Westerhoff (2004), Chiarella et al. (2005) or Dieci and Westerhoff (2008).

<u>Our first contribution</u>

We develop a financial market with five different types of agents:

1) Fundamentalist traders whose demand (or supply) depends (asymmetricaly) on the difference between the fundamental and the actual price and on its absolute value;

2) Fundamentalist traders whose <u>fixed</u> demand (or supply) depends (asymmetricaly) only on the sign of the difference between the fundamental and the actual price;

3) Chartists traders whose demand (or supply) depends (asymmetricaly) on the difference between the fundamental and the actual price and on its absolute value;

4) Chartists traders whose <u>fixed</u> demand (or supply) depends (asymmetricaly) only on the sign of the difference between the fundamental and the actual price;

5) A market maker which adjusts prices with respect to excess demand.

The setup

The market maker mediates transactions out of equilibrium and adjusts prices according to

$$P_{t+1} = P_t + a(D_t^{C,1} + D_t^{C,12} + D_t^{F,1} + D_t^{F,2})$$

where P is the log of the price, a a positive price adjustment parameter (that can be normalized to 1 without loss of generality) and the orders of the two types of technical traders are indicated by $D_t^{C,1}$ and $D_t^{C,2}$ while the orders of the two types of fundamental traders are denoted by $D_t^{F,1}$ and $D_t^{F,2}$, respectively.

Following Day and Huang (1990) chartists believe in the persistence of bull and bear markets. The orders of the first type of chartists are expressed by:

$$D_t^{C,1} := \begin{cases} +c^{1,a}(P_t - F) & \text{if} \quad P_t - F \geq 0 \\ +c^{1,b}(P_t - F) & \text{if} \quad P_t - F < 0 \end{cases}$$

where $c^{1,a}$ and $c^{1,b}$ are positive reaction parameters and F is the log of the fundamental price.

The setup

Type 2 chartists submit orders according to

$$D_t^{C,2} := \begin{cases} +c^{2,a} & \text{if} \quad P_t - F \geq 0 \\ -c^{2,b} & \text{if} \quad P_t - F < 0 \end{cases}$$

where $c^{2,a}$ and $c^{2,b}$ are positive reaction parameters .
The size of their orders does not depend on the deviation from the fundamental value (see Lux (1998)).

Fundamentalists believe that prices return towards their fundamental values in the long run. The orders of type 1 and type 2 fundamentalists are formalized as

$$D_t^{F,1} := \begin{cases} +f^{1,a}(F - P_t) & \text{if} \quad F - P_t > 0 \\ +f^{1,b}(F - P_t) & \text{if} \quad F - P_t \leq 0 \end{cases} \qquad D_t^{F,2} := \begin{cases} +f^{2,a} & \text{if} \quad F - P_t > 0 \\ -f^{2,b} & \text{if} \quad F - P_t \leq 0 \end{cases}$$

where $f^{1,a}$, $f^{1,b}$, $f^{2,a}$ and $f^{2,b}$ are positive reaction parameters .

The dynamical system

Introducing $\tilde{P}_t = P_t - F$, the orders imply the following dynamical system:

$$\tilde{P}_{t+1} = \begin{cases} (1+c^{1,a} - f^{1,b})\tilde{P}_t + c^{2,a} - f^{2,b} & \text{if} \quad \tilde{P}_t \geq 0 \\ (1+c^{1,b} - f^{1,a})\tilde{P}_t - c^{2,b} + f^{2,a} & \text{if} \quad \tilde{P}_t < 0 \end{cases}$$

To simplify the notation, let us defines the slopes as

$$s_R = 1 + c^{1,a} - f^{1,b} \qquad \text{and} \qquad s_L = 1 + c^{1,b} - f^{1,a}$$

and the intercepts are

$$m_R = c^{2,a} - f^{2,b} \qquad \text{and} \qquad m_L = f^{2,a} - c^{2,b}$$

We then obtain the one-dimensional, in general discontinuous, dynamical system

$$\boxed{\tilde{P}_{t+1} = T(\tilde{P}_t) \begin{cases} f_R(\tilde{P}_t) = s_R \tilde{P}_t + m_R & \text{if} \quad \tilde{P}_t \geq 0 \\ f_L(\tilde{P}_t) = s_L \tilde{P}_t + m_L & \text{if} \quad \tilde{P}_t < 0 \end{cases}}$$

Piecewise-linear systems

In this case, the local bifurcations associated with the eigenvalues are degenerate.

Only two kinds of bifurcations can occur:

- Contact bifurcations (see Fournier-Prunaret et al. (1994) and Mira et al. (1996)) occurs when two invariant sets of different nature have a contact in one or more points;

- Border-Collision bifurcation (BCB) are contacts between an invariant set of the map with the border of its region of definition that in some cases produce a bifurcation.

The one dimensional piecewise linear case, continuous and discontinuous, was considered by Banerjee et al. (2000), Jain and Banerjee (2003) Avrutin and Schanz (2006, 2008), Avrutin et al. (2006), Di Bernardo et al. (2008).

This case is so rich that it is still not completely studied.

Economic scenarios

$$\tilde{P}_{t+1} = T(\tilde{P}_t)\begin{cases} f_R(\tilde{P}_t) = s_R\tilde{P}_t + m_R & \text{if} \quad \tilde{P}_t \geq 0 \\ f_L(\tilde{P}_t) = s_L\tilde{P}_t + m_L & \text{if} \quad \tilde{P}_t < 0 \end{cases}$$

Given that the reaction parameters are all positive, **slopes and intercepts of the map *T* can take any values**.

We investigate four economic scenarios:

I. Only type 1 traders are present (special case);

II. Only type 2 traders are present (special case);

III. General case with $s_R>1$, $s_L>1$, $m_R<0$ and $m_L>0$ (type 1 chartists dominate type 1 fundamentalists, but type 2 fundamentalists dominate type 2 chartists);

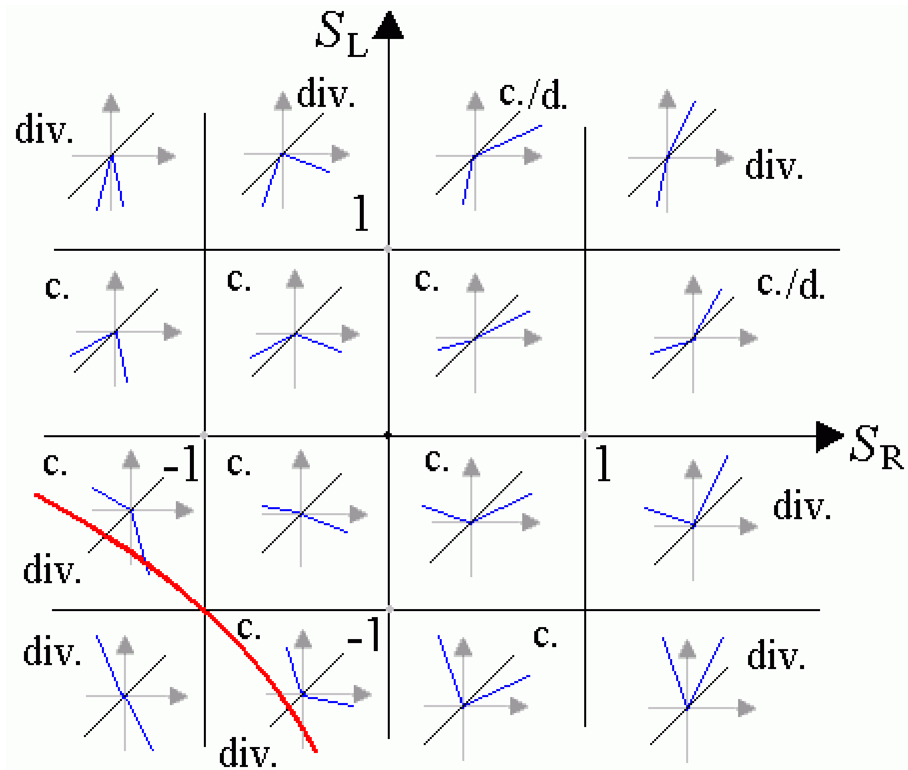
IV. General case with $s_R<0$, $s_L>0$, $m_R>0$ and $m_L<0$ (type 1 fundamentalists trade much more forcefully than type 1 chartists and the opposite is true for type 2 traders.

Case I

If only type 1 traders are present we have that $m_L = m_R = 0$, so the map becomes

$$\tilde{P}_{t+1} = T(\tilde{P}_t)\begin{cases} f_R(\tilde{P}_t) = s_R \tilde{P}_t & \text{if} \quad \tilde{P}_t \geq 0 \\ f_L(\tilde{P}_t) = s_L \tilde{P}_t & \text{if} \quad \tilde{P}_t < 0 \end{cases}$$

and the trajectories are either diverging or converging to the unique fixed point $\tilde{P} = 0$ depending on the sign and modulus of the slopes.

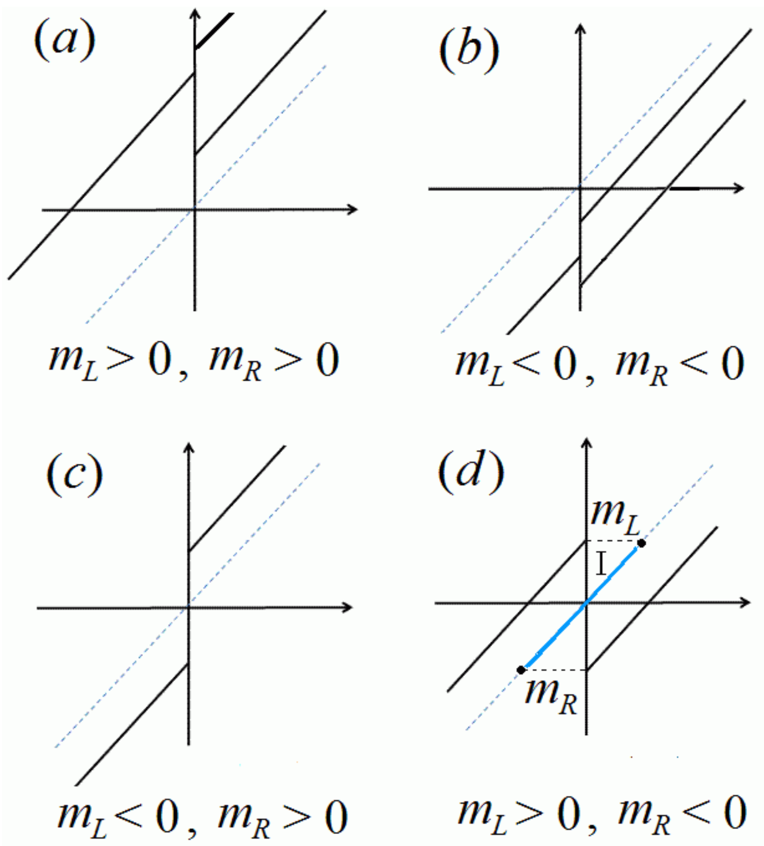


Case II

If only type 2 traders are present we have that $s_L = s_R = 1$, so the map becomes

$$\tilde{P}_{t+1} = T(\tilde{P}_t) \begin{cases} f_R(\tilde{P}_t) = \tilde{P}_t + m_R & \text{if} \quad \tilde{P}_t \geq 0 \\ f_L(\tilde{P}_t) = \tilde{P}_t + m_L & \text{if} \quad \tilde{P}_t < 0 \end{cases}$$

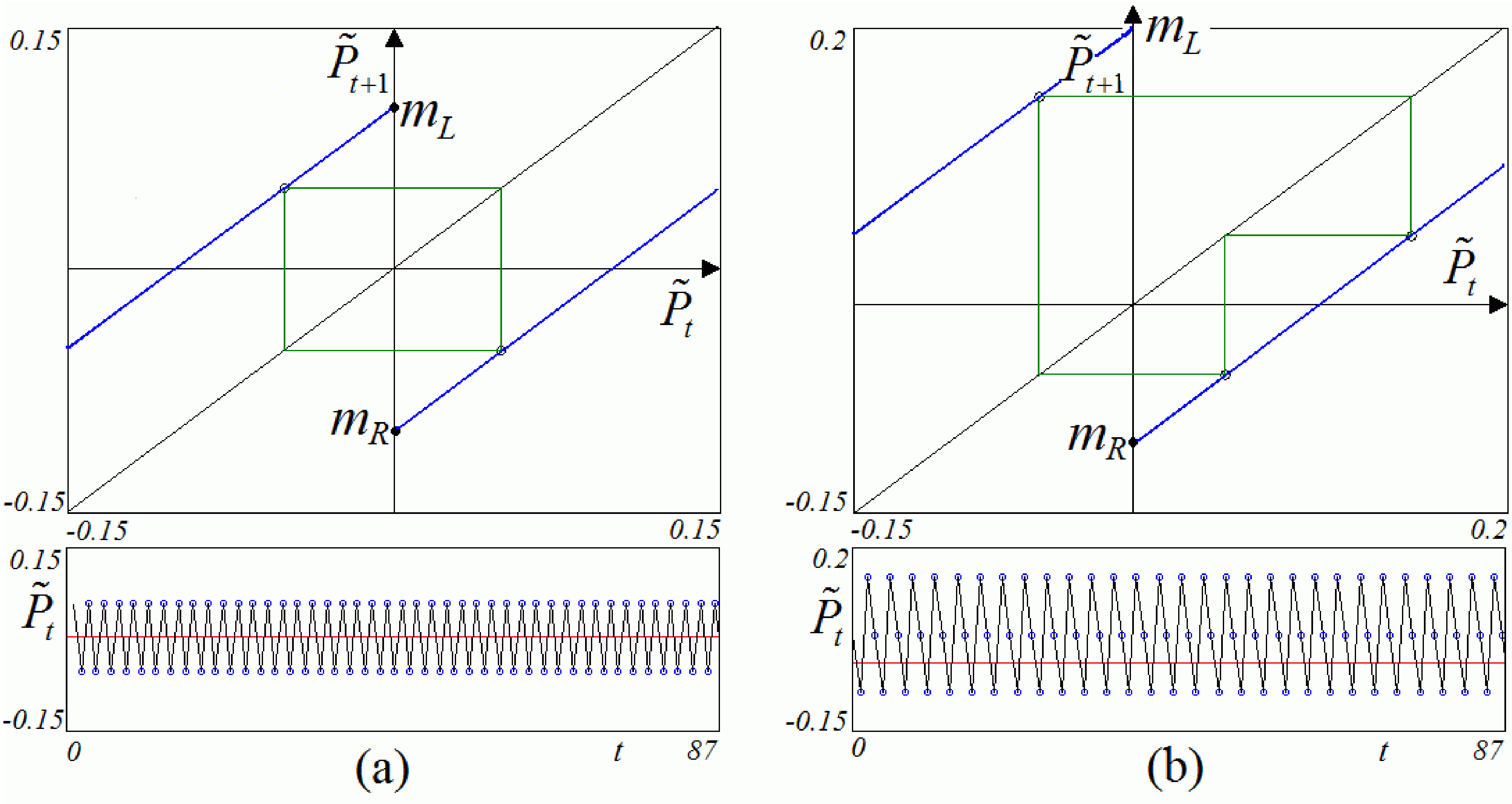
Which is discontinuos (without fixed points) and the trajectories are diverging, except for the case (d):



SUBCASE (*d*)

- The interval $I=[m_R,m_L]$ is an invariant absorbing interval
- We cannot have neither attractive nor repelling states or diverging orbits, because the eigenvalues are equal to 1
- Orbits can only be periodic or quasiperiodic inside I
- It can be proved that the period of the orbit is related to the ratio $s = m_L / -m_R$. If s is rational the orbit is periodic, if not it is quasiperiodic

Case II: periodic and quasiperiodic orbits



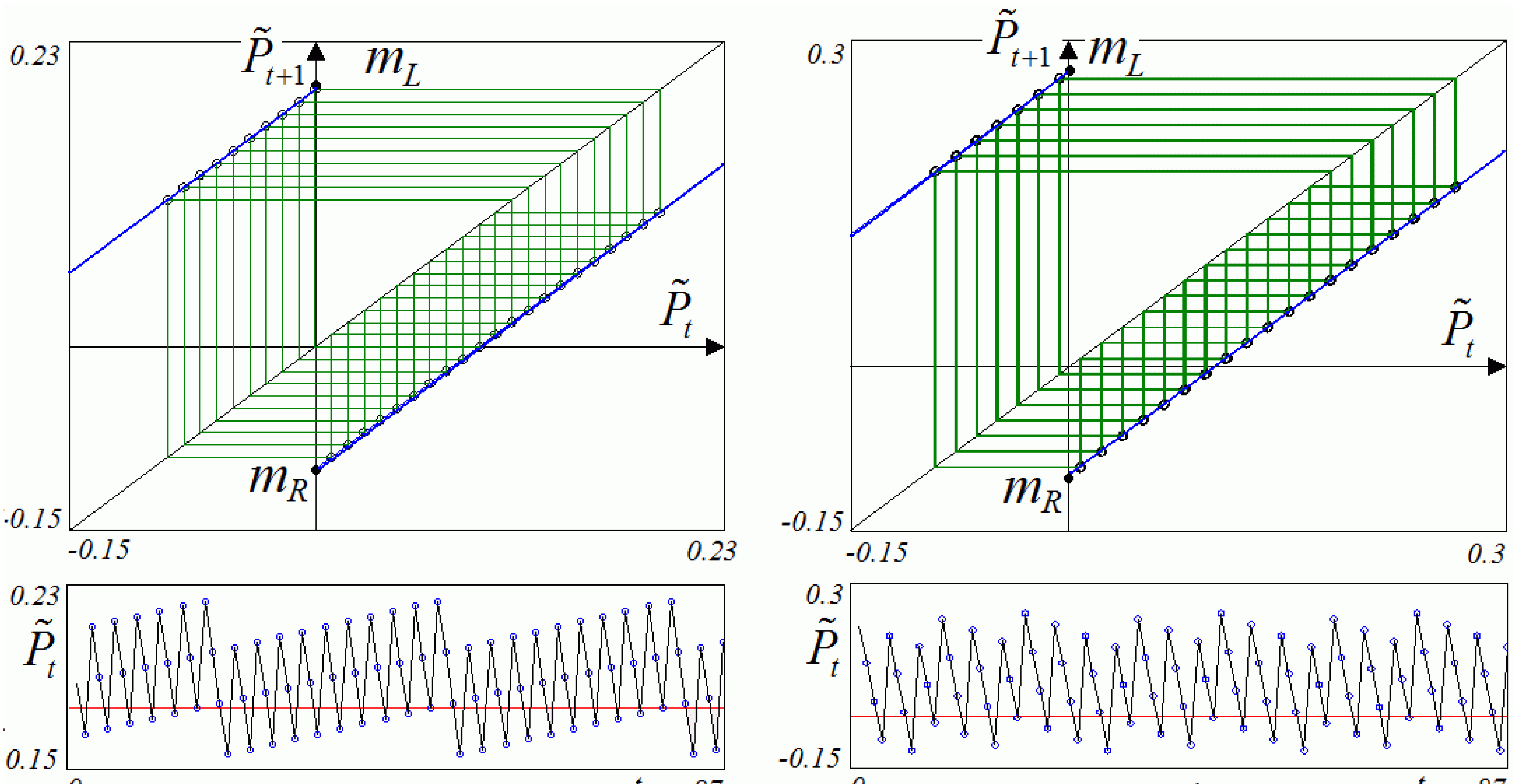
$$\frac{m_L}{-m_R} = 1$$

$$\frac{m_L}{-m_R} = \frac{2}{1}$$

2 → p.ts on the *R* side

1 → p.ts on the *L* side

Case II: periodic and quasiperiodic orbits



(c) (d)

$$\frac{m_L}{-m_R} = \frac{21}{10}$$

$\xrightarrow{\quad}$ p.ts on the R side

$\xrightarrow{\quad}$ p.ts on the L side

$$m_L = \frac{\sqrt{2}}{3} - 0.2$$

$$m_R = 0.1$$

quasiperiodic motion

Case III

In this case type 1 chartists dominate stype 1 fundamentalists:

$$c^{1,a} - f^{1,b} > 0 \quad \Rightarrow \quad s_R > 1 \qquad \text{and} \qquad c^{1,b} - f^{1,a} > 0 \quad \Rightarrow \quad s_L > 1$$

Type 2 fundamentalists, instead, dominates type 2 chartists:

$$c^{2,a} - f^{2,b} < 0 \quad \Rightarrow \quad m_R < 0 \qquad \text{and} \qquad f^{2,a} - c^{2,b} > 0 \quad \Rightarrow \quad m_L > 0$$

The map T contains the four parameters:

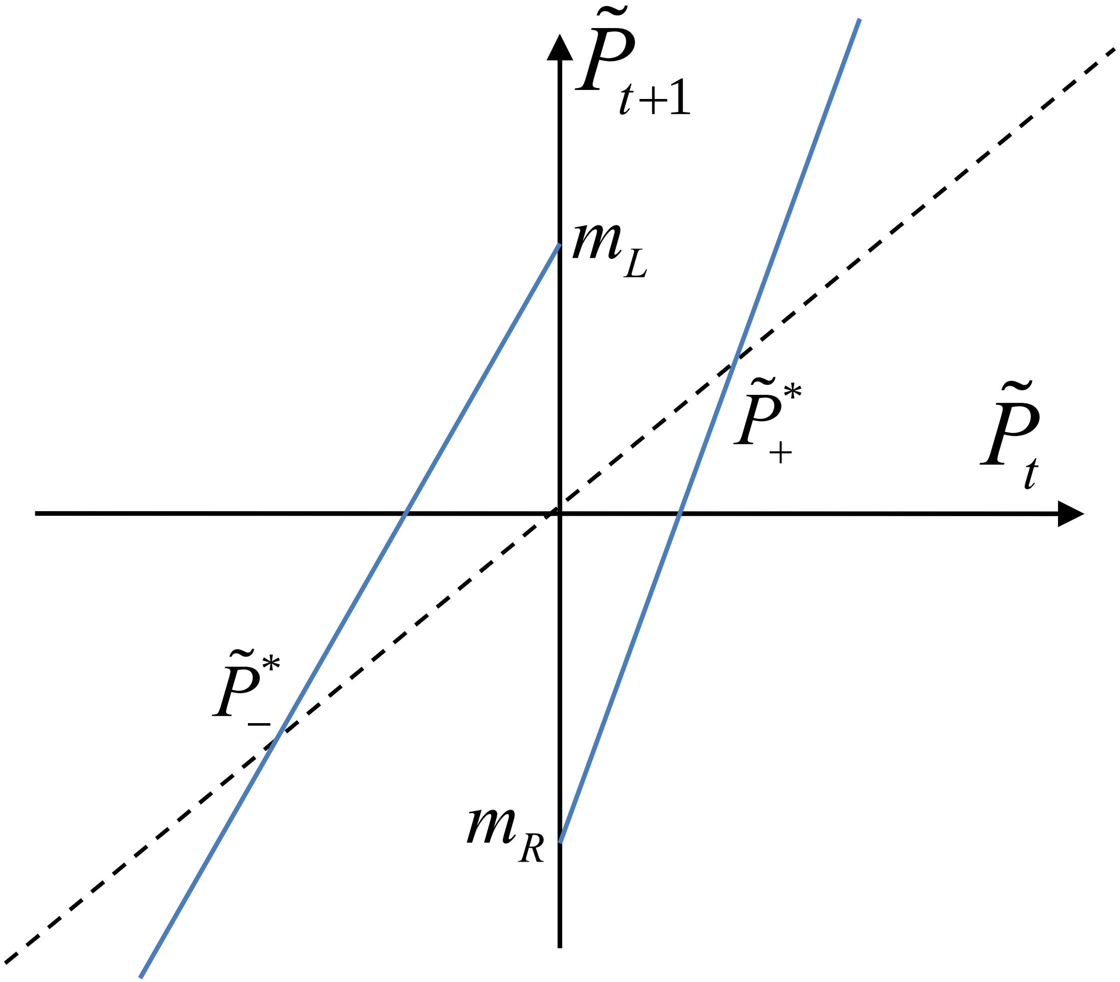
$$\tilde{P}_{t+1} = T(\tilde{P}_t) \begin{cases} f_R(\tilde{P}_t) = s_R \tilde{P}_t + m_R & \text{if} \quad \tilde{P}_t \geq 0 \\ f_L(\tilde{P}_t) = s_L \tilde{P}_t + m_L & \text{if} \quad \tilde{P}_t < 0 \end{cases}$$

with slopes higher than 1.

So each exisitng *k*-cycle must be unstable because its eigenvalue is $s_L^n s_R^{k-n} > 1$

Case III

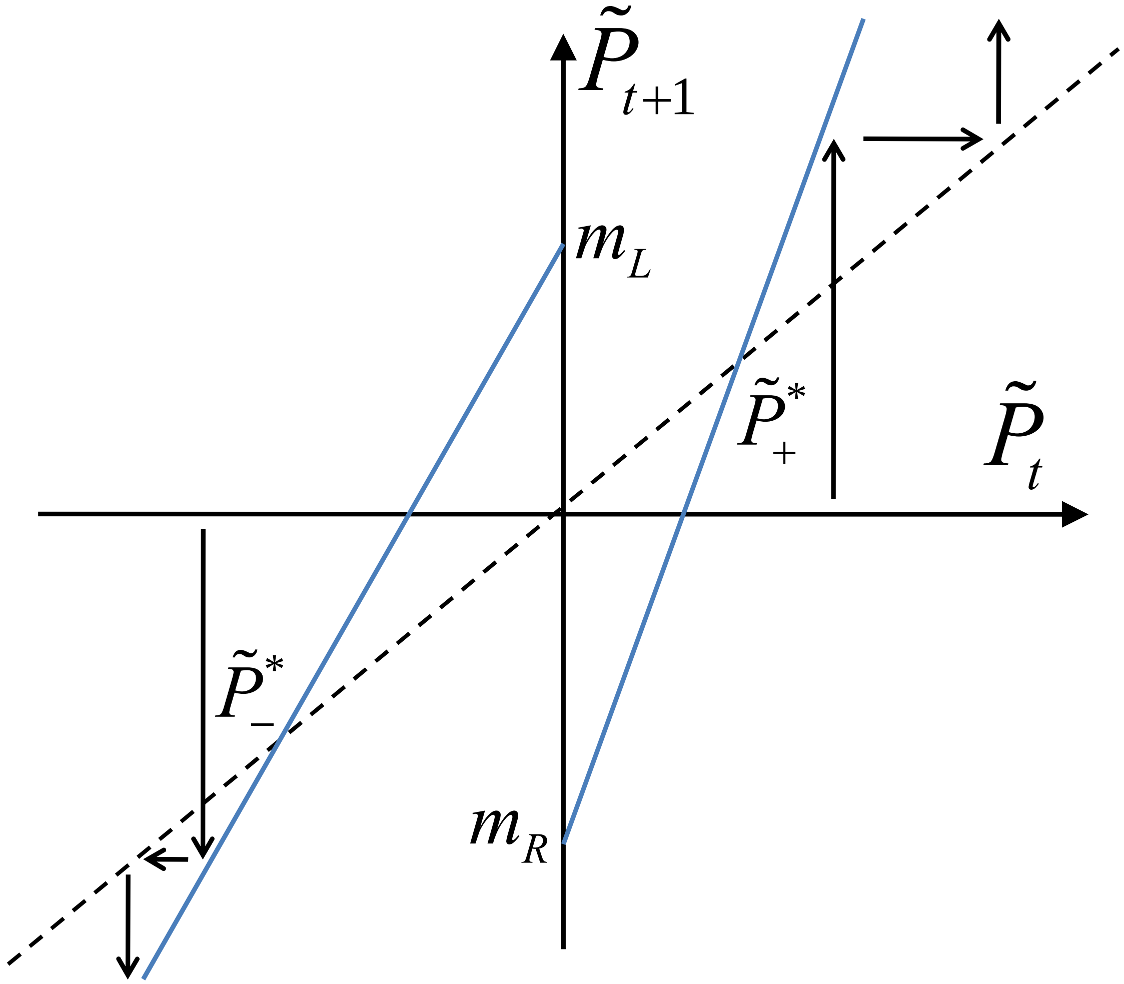
Qualitatively the map T appears as follows:



Two unstable steady states ($\tilde{P}_-^*$ and $\tilde{P}_+^*$) always exist.

Case III

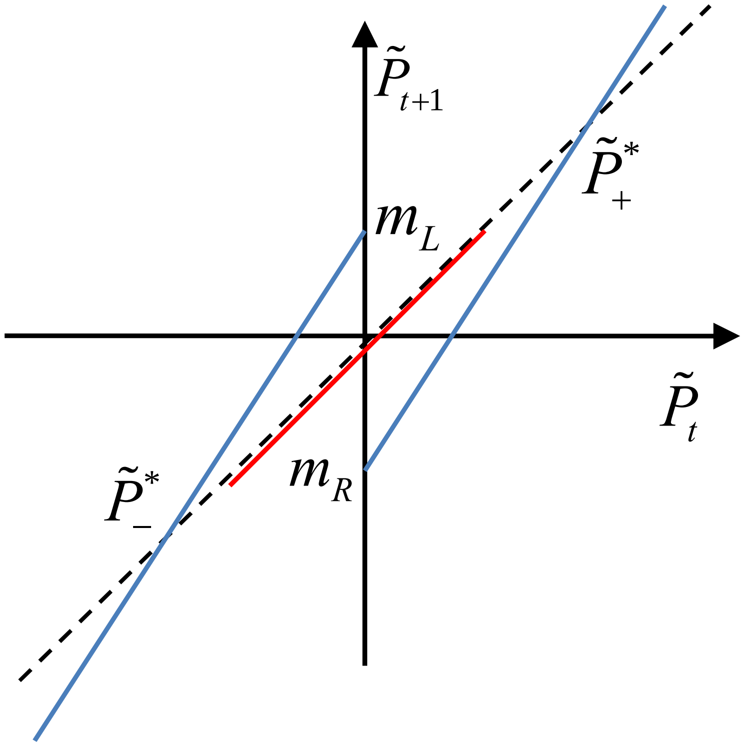
Initial conditions higher than $\tilde{P}_+^*$ or lower than $\tilde{P}_-^*$ lead to divergence:



What happens to the orbits starting inside the interval $]P_-^*, P_+^*[$?

Case III: scenario 1

The first scenario is the case in which $P_{+}^{*} > m_L$ and $P_{-}^{*} < m_R$:



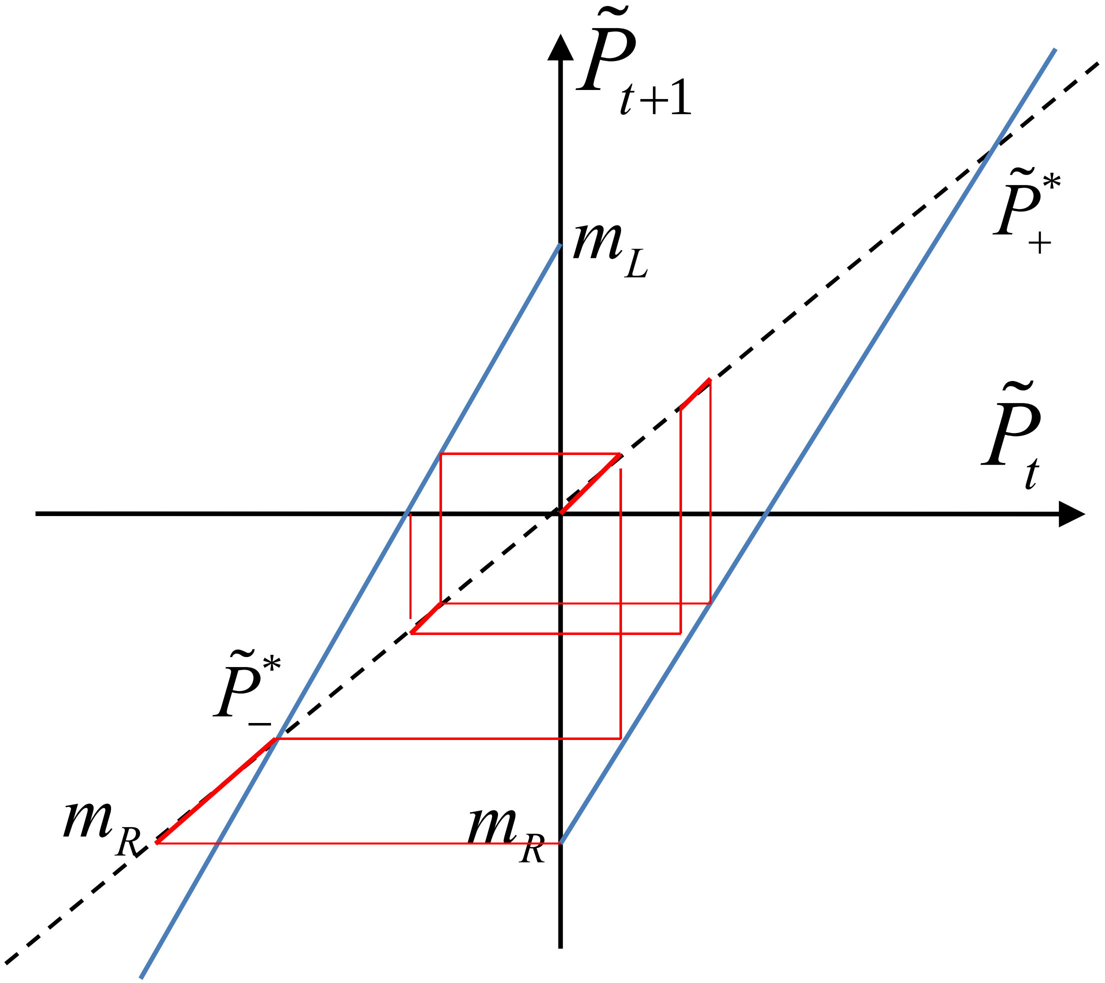
- ❖The interval I (in red) is still invariant and absorbs the orbits starting from the interval $\left]P_{-}^{*}, P_{+}^{*}\right[$

- ❖ The dynamics inside I are bounded and can only be *chaotic*, in fact the eigenvalues of any existing cycles must be higher than 1 becasue the slopes are both higher than 1.

Starting from scenario 1 and incresing one of the slopes (or reducing in absolute value one of the intercepts), a fixed point becomes closer and closer to a border of the interval I. At the contact a contact bifurcation occurs and a different scenario is verified…

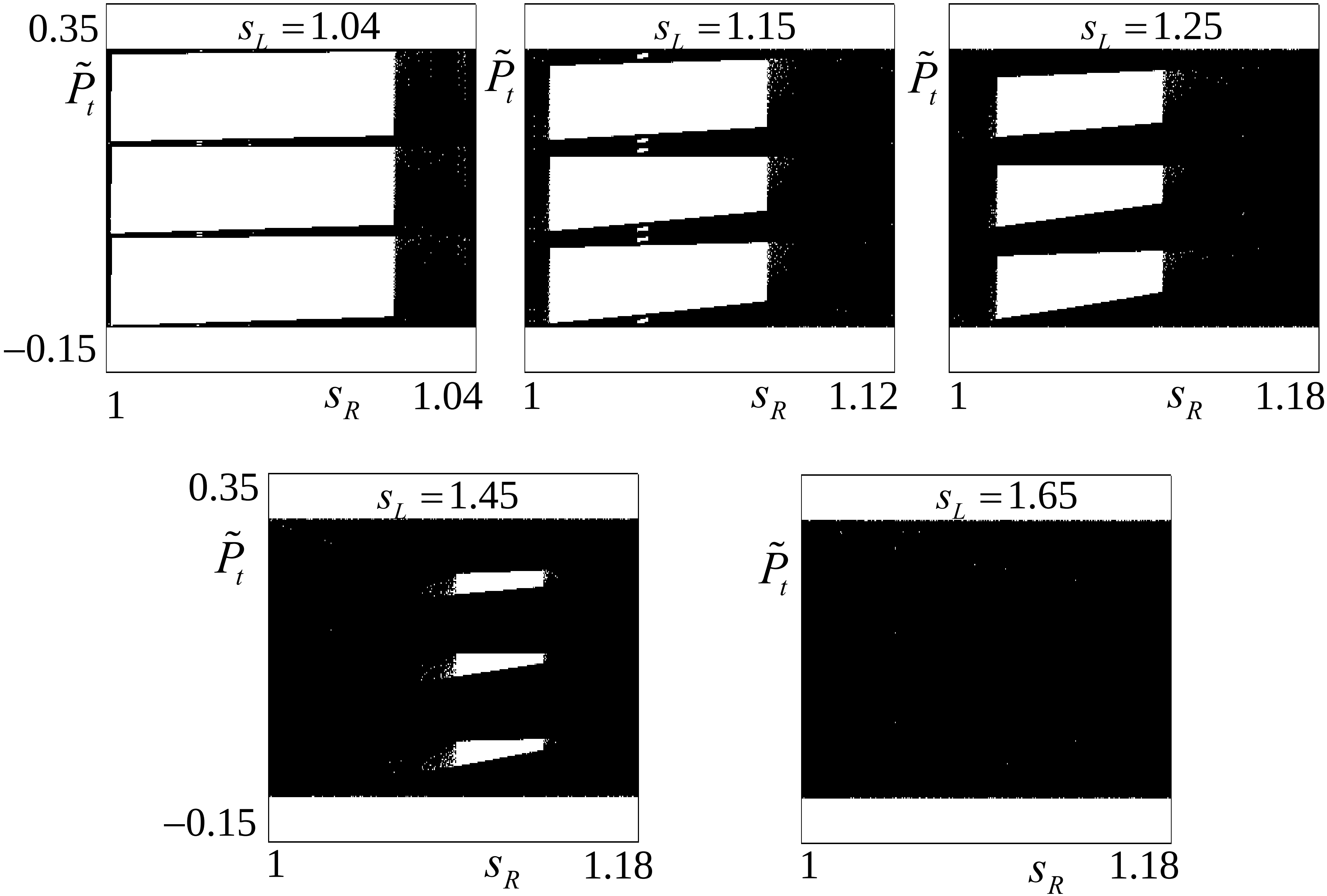
<u>Case III: scenario 2</u>

The second scenario is the case in which $P_+^* < m_L$ or $P_-^* > m_R$ (or both):



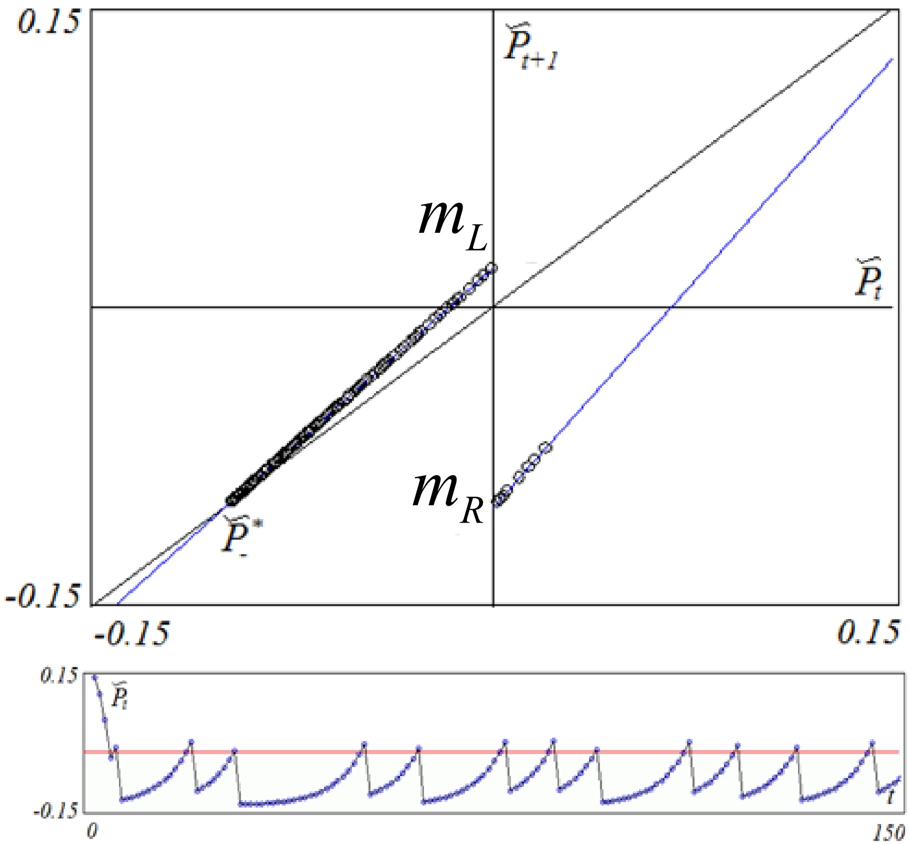
The preimages of the interval $]m_R, P_-^*[$ cover the whole interval I so <u>each orbit is now diverging</u> (except for the unstable cycles in I).

Case III: scenario 1, the role of the slopes



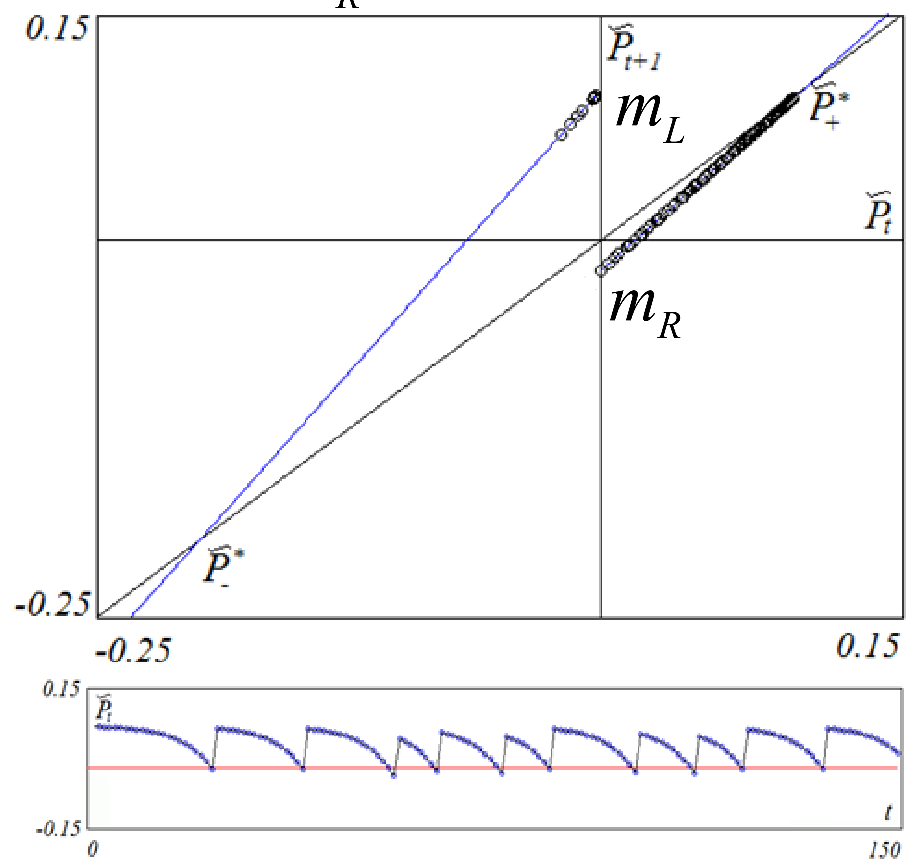
Case III: scenario 1, the role of the intercepts

m_L close to 0



(a)

m_R close to 0

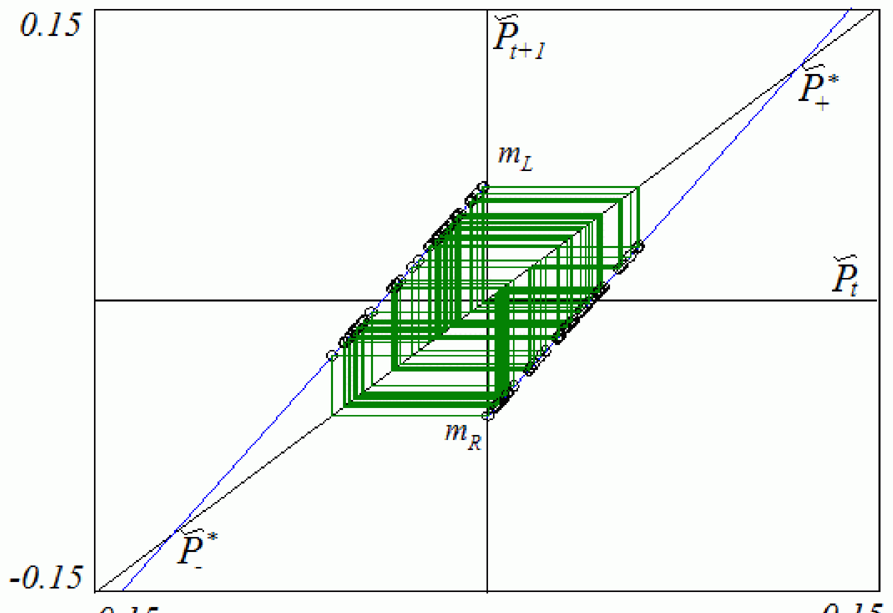


(b)

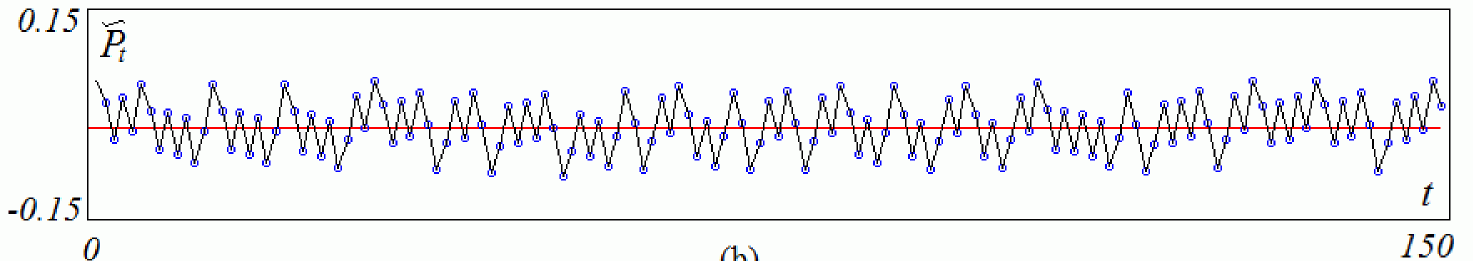
If one intercept is (much) more close to the origin than the other one, the number of iterations and that side is (much) more than the number of interations on the other side.

Case III: scenario 1, unpredictable switching

If the slopes are large enough and there is not an intercept (much) closer to the origin than the other, the orbits switch in an unpredictable way between the bull and bear region.



(a)



(b)

Case IV

In this case type 1 fundamentalists (strongly) dominates type 1 chartists:

$$f^{1,b} - c^{1,a} > 1 \quad \Rightarrow \quad s_R < 0 \qquad \text{and} \qquad f^{1,a} - c^{1,b} > 1 \quad \Rightarrow \quad s_L < 0$$

Type 2 chartists, instead, dominates type 2 fundamentalists:

$$c^{2,a} - f^{2,b} > 0 \quad \Rightarrow \quad m_R > 0 \qquad \text{and} \qquad f^{2,a} - c^{2,b} < 0 \quad \Rightarrow \quad m_L < 0$$

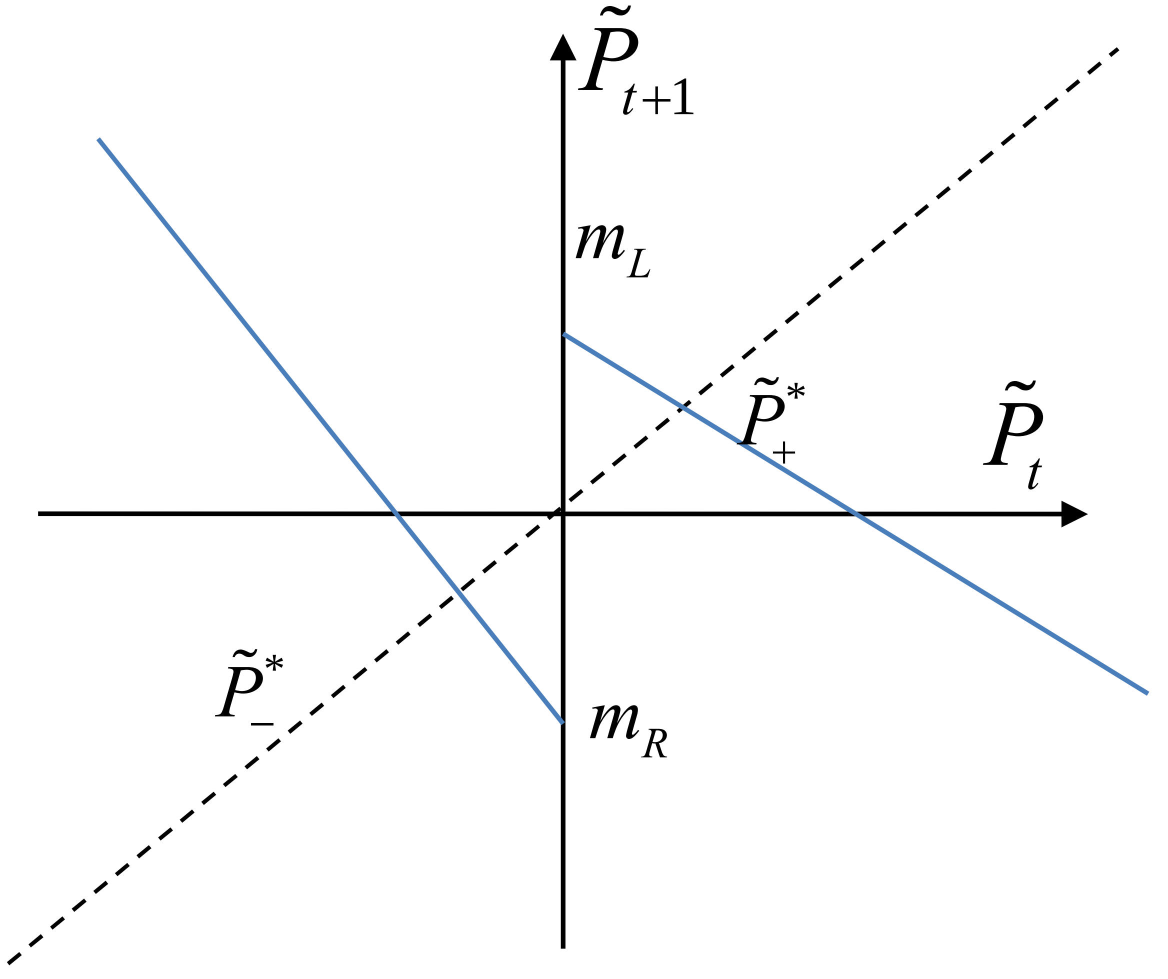
The map T contains the four parameters:

$$\boxed{\tilde{P}_{t+1} = T(\tilde{P}_t) \begin{cases} f_R(\tilde{P}_t) = s_R \tilde{P}_t + m_R & \text{if} \quad \tilde{P}_t \geq 0 \\ f_L(\tilde{P}_t) = s_L \tilde{P}_t + m_L & \text{if} \quad \tilde{P}_t < 0 \end{cases}}$$

with negative slopes.

Case IV

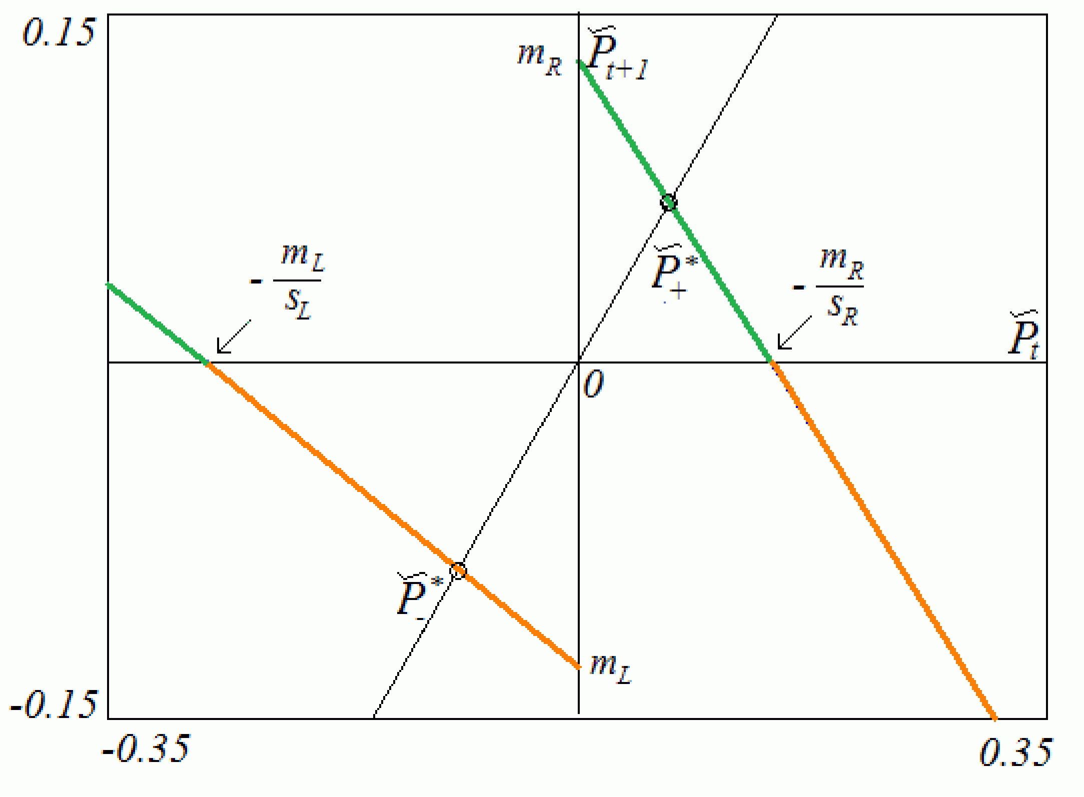
Qualitatively the map T appears as follows:



Two steady states ($\tilde{P}_-^*$ and $\tilde{P}_+^*$) always exist and can be stable and unstable.

Case IV: subcase A

The fixed points are both locally stable, i.e. $-1 < s_R < 0$ and $-1 < s_L < 0$.



✓ All the orbits are converging towards one of the fixed points;

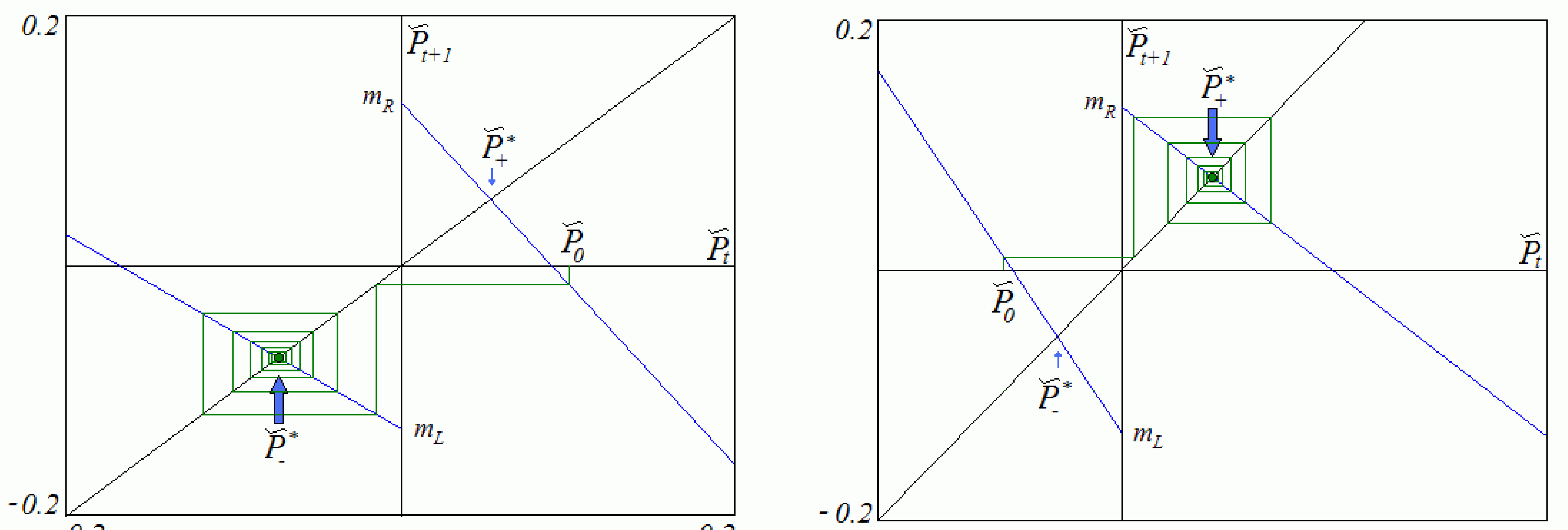
✓The basins of attraction of the fixed points alternate on the real line.

Case IV: from subcase A to subcase B

Decresing the value of one of the slopes (f.i. s_R) a bifurcation occurs when s_R=-1.

At the bifurcation value the basin of the fixed point which is no more stable is filled with stable (but not attracting) cycles of period 2. The bifurcation is a *degenerate flip*.

When s_R(or s_L)<-1 and $s_R s_L < 1$ (almost) all the orbits converge towards the stable fixed points. Exceptions are given by the unstable fixed point and its preimages.



(a) (b)

<u>Case IV: subcase B</u>

When $s_R s_L > 1$ an unstable 2-cycle is created.
Let us indicate the periodic points by $\{\overline{x}_1, \overline{x}_2\}$.

The basin of the stable fixed point is now given by:

- The interval $]\overline{x}_1, \overline{x}_2[$ if $m_R < \overline{x}_2$ and $m_L > \overline{x}_1$;
- If $m_R > \overline{x}_2$ <u>or</u> $m_L < \overline{x}_1$ a chaotic frontier sperates the basin of the fixed point by the basin of divergence, inside the interval $]\overline{x}_1, \overline{x}_2[$;

We still have to analyze tha case in which both the fixed points are unstable, that is:

$$s_{R,L} < -1$$

In this subcase (subcase C) we have that the unstable 2-cycle must exist because $s_R s_L > 1$.

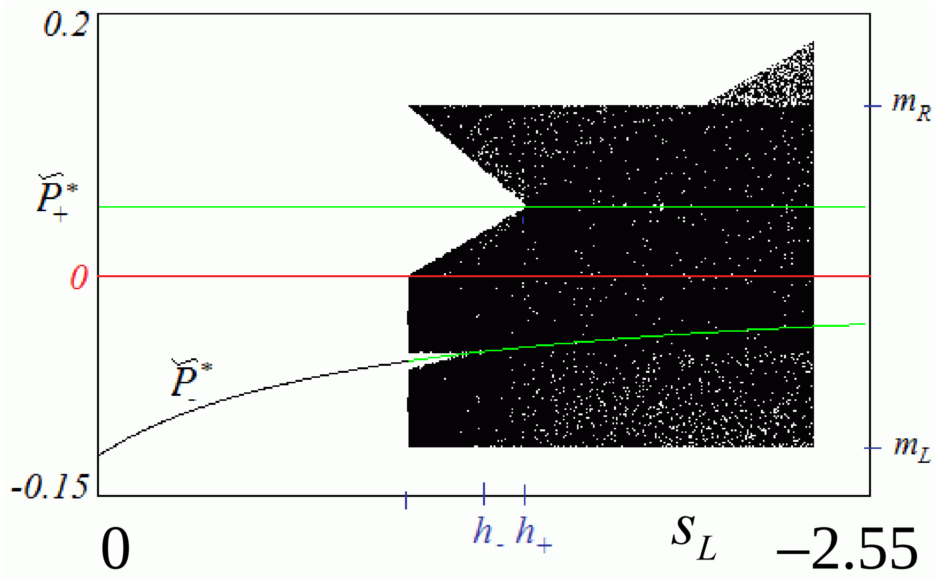
Case IV: subcase C

When the slopes are both lower than -1 the generic trajectories can only be chaotic or diverging.

Again, the orbits are bounded (that is generically chaotic) if are fulfilled the conditions:

$$m_R < \overline{x}_2 \qquad \text{and} \qquad m_L > \overline{x}_1$$

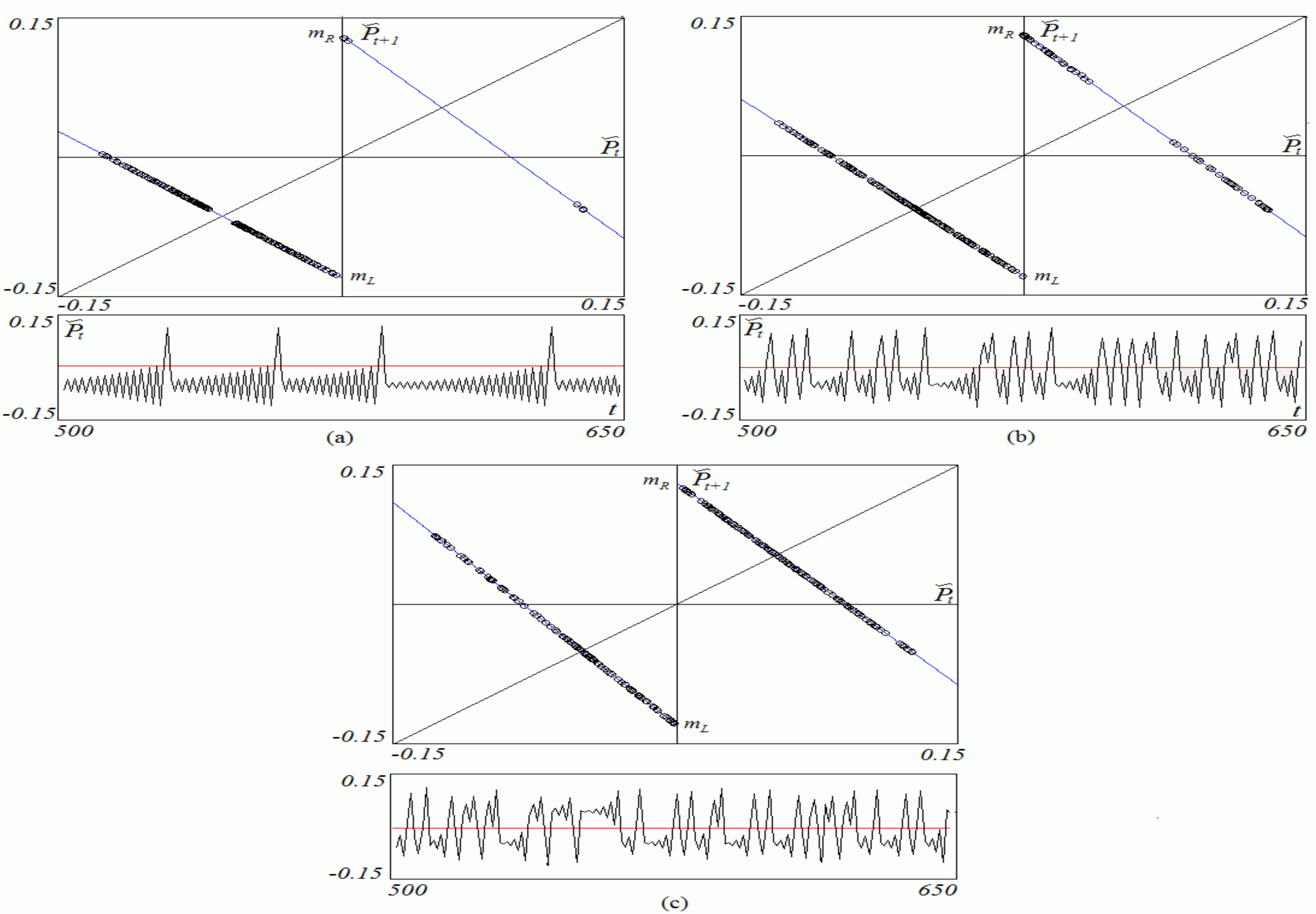
and the attractor consists in *k*-chaotic pieces, where *k* becomes lower and lower (until it reach 1) by decreasing slopes:



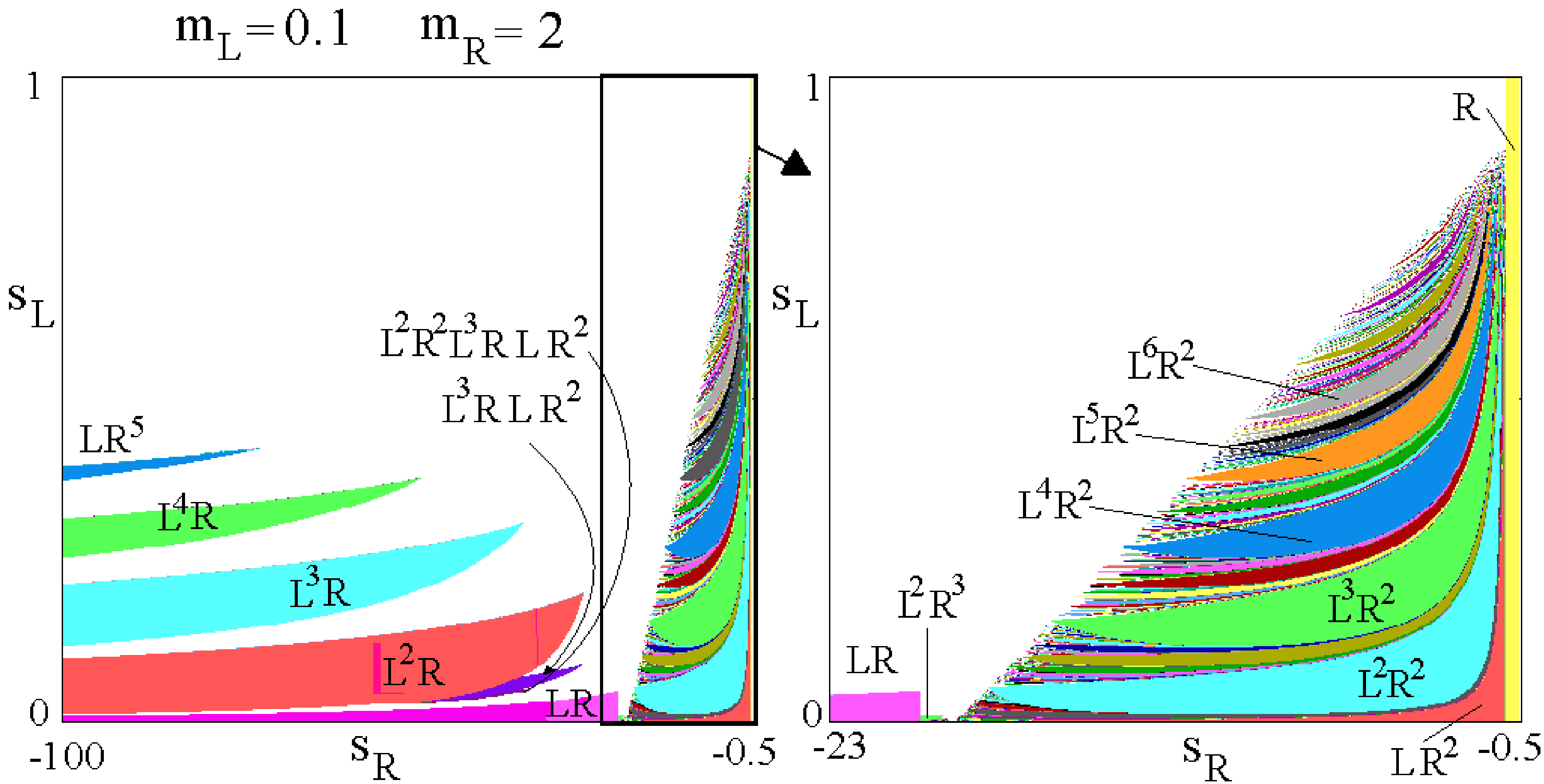
If the condition is not fulfilled the generical orbit diverges, except for a chaotic repellor.

Case IV: subcase C

To lower numbers of chaotic bands is associated an higher level of unpredictability of the switching between bull and bear regions:

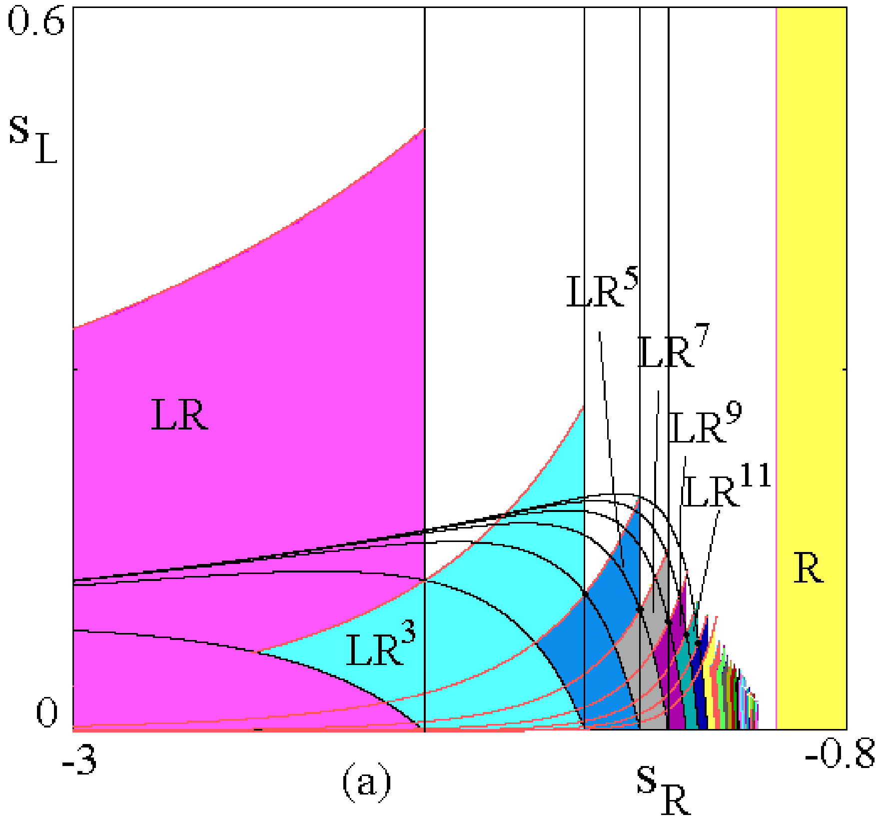


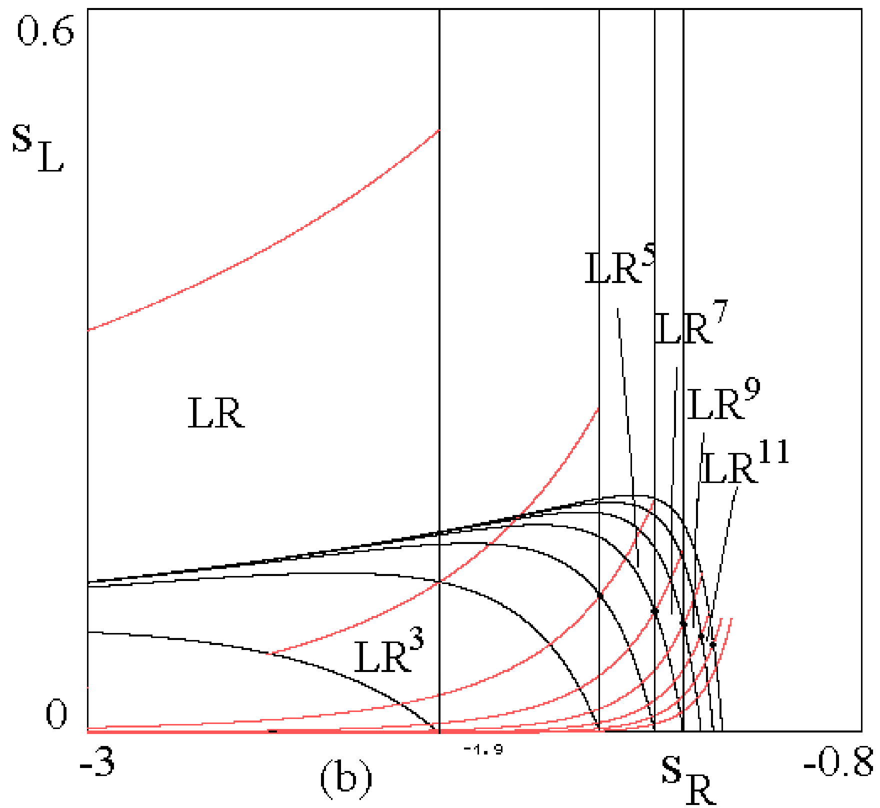
BCB curves



$m_L = 0.1$ $m_R = 2$
1
s_L
0
-100
s_R
-0.5
LR^5
L^4R
L^3R
L^2R
LR
$L^2R^2L^3RLR^2$
L^3RLR^2
1
s_L
0
-23
s_R
-0.5
R
L^6R^2
L^5R^2
L^4R^2
L^3R^2
L^2R^2
L^2R^3
LR
LR^2

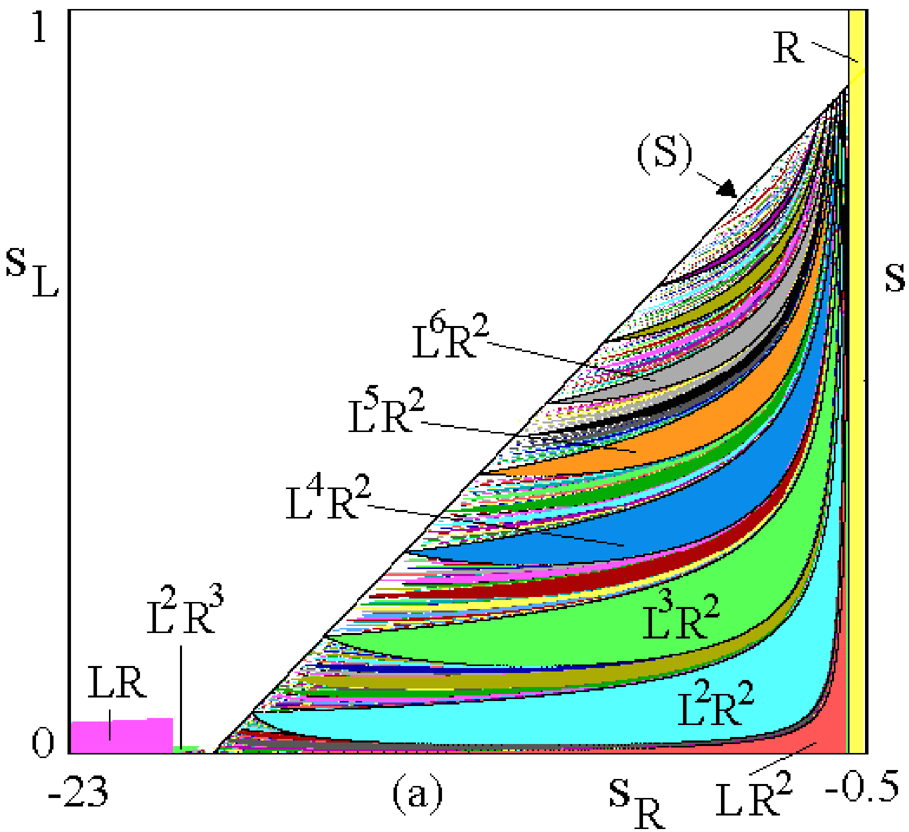
Numerical and analytical BCB curves

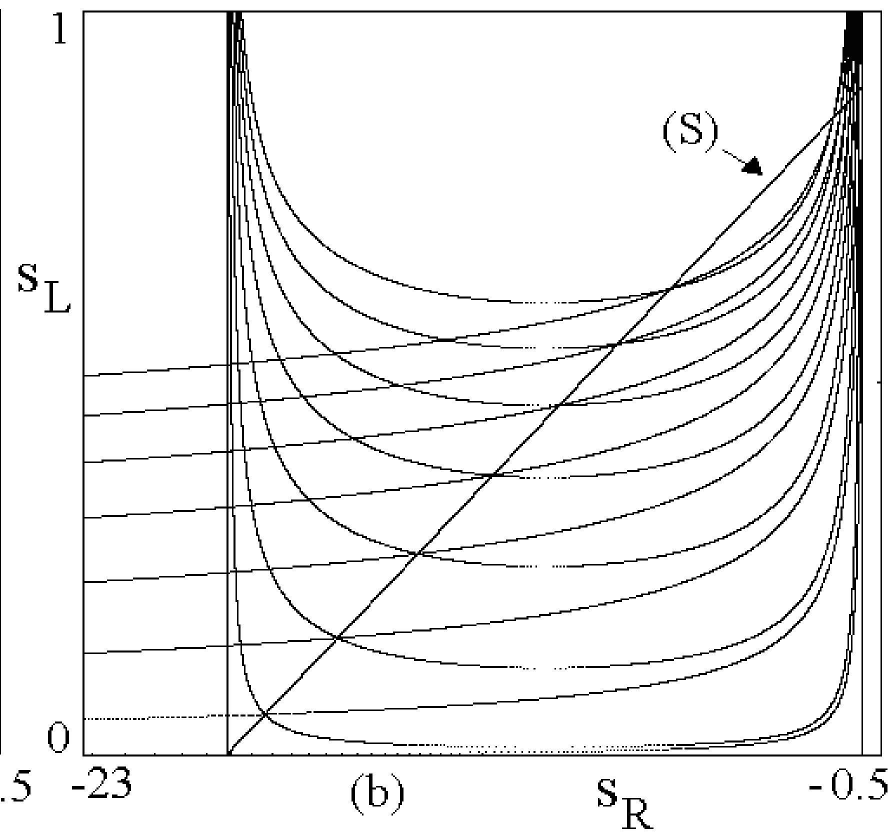




Numerical and analytical BCB curves

$m_L = 0.1 \quad m_R = 2$





Our second contribution

The second model we propose includes the following assumptions:

1) We consider a market of one risky asset and two types of traders: fundamentalists and chartists;

2) While the excess demand of the fundamentalist is formulated in the usual way, chartists do not have knowledge about the fundamental price and they only use the price information. In general their excess demand is given by:

$$D_t^c = h(\xi_t), \qquad \xi_t = \phi_t - \psi_t$$

where ϕ_t and ψ_t are short and long term price trend, respectively.

3) In particular we introduce the hyphotesis that chartists are reluctant when the difference between short and long term price term is small:

$$h(x) = \begin{cases} h(x+\delta) & \text{for} \quad x \le -\delta \\ 0 & \text{for} \quad |x| < \delta \\ h(x-\delta) & \text{for} \quad x \ge \delta \end{cases}$$

Excess demand functions

- The excess demand function of fundamentalists is given by:

$$D_t^f = \alpha(P_t^* - P_t), \qquad \alpha > 0$$

- For the chartists, usually $h(x)$ is an S-shaped function, but we would expect the chartists are cautious when the difference between short and long time price trend is far away from zero, meaning that their demand would reduce when the difference is beyond certain threshold value. A possible function is:

$$h(x) := \frac{ax}{1+cx^2}, \qquad a > 0, c > 0$$

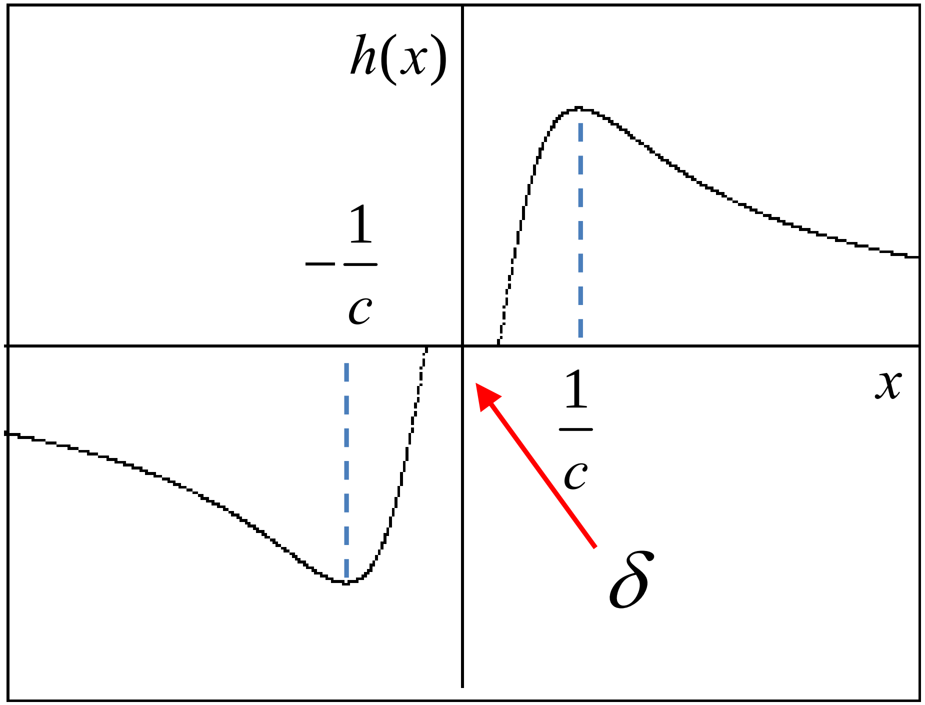
which has the following features:

$$h(0) = 0,\ h'(0) = a,\ h'(x) > 0 \ \text{ for } |x| < 1/c \ \text{ and } \ h'(x) < 0 \ \text{ for } |x| > 1/c,$$
$$\lim_{x \to \pm\infty} h(x) = 0$$

parameter a measures chartists' extrapolation when x is small while parameter c measures the price difference limit for agents to extrapolate.

The behavior of chartists

The shape of $h(x)$ is the following:



Price trends

- The short term trend can be taken as $\phi_t = P_t$ for simplicity;

- As a first approximation, the long term tren can be taken as $\psi_t = P_{t-1}$, so

$$x_t = \phi_t - \psi_t = P_t - P_{t-1} \qquad (1)$$

In a second moment we can complicate the model considering a standard moving average or a geometric moving average with infinite memory (see He (2003), Chiarella and He (2003) and He and Li (2008)).

- The market maker equation is assumed as:

$$P_{t+1} = P_t + \frac{\mu}{2}\left[(1+m)D_t^f + (1-m)D_t^c\right] \qquad (2)$$

where $m \in [-1,1]$ is constant but it could be considered endogenous whenever we consider the possibility of switching

The 2D map

By making the change of variable $X = P_{t-1}$ and $Y = P_t$ we have the two dimensional system:

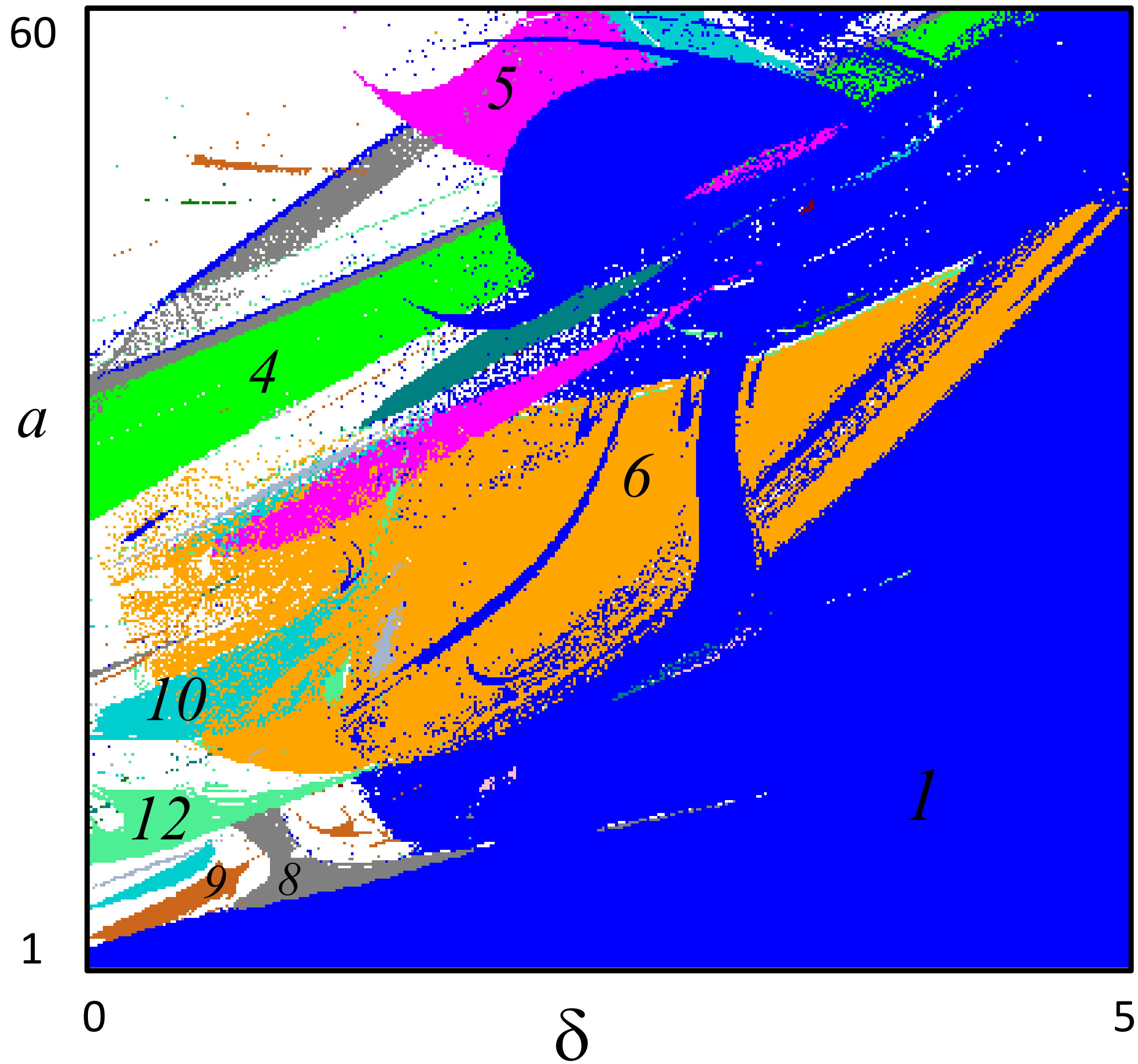
$$T = \begin{cases} X' = Y \\ Y' = Y + \dfrac{\mu}{2}\left[(1+m)\alpha(P^* - Y) + (1-m)h(Y - X)\right] \end{cases}$$

where the discontinuity is due to $h(x)$ which is discontinuous for $\delta > 0$.

Two-dimensional discontinuous systems are still not much studied, so it is not easy to obtain analytical results.

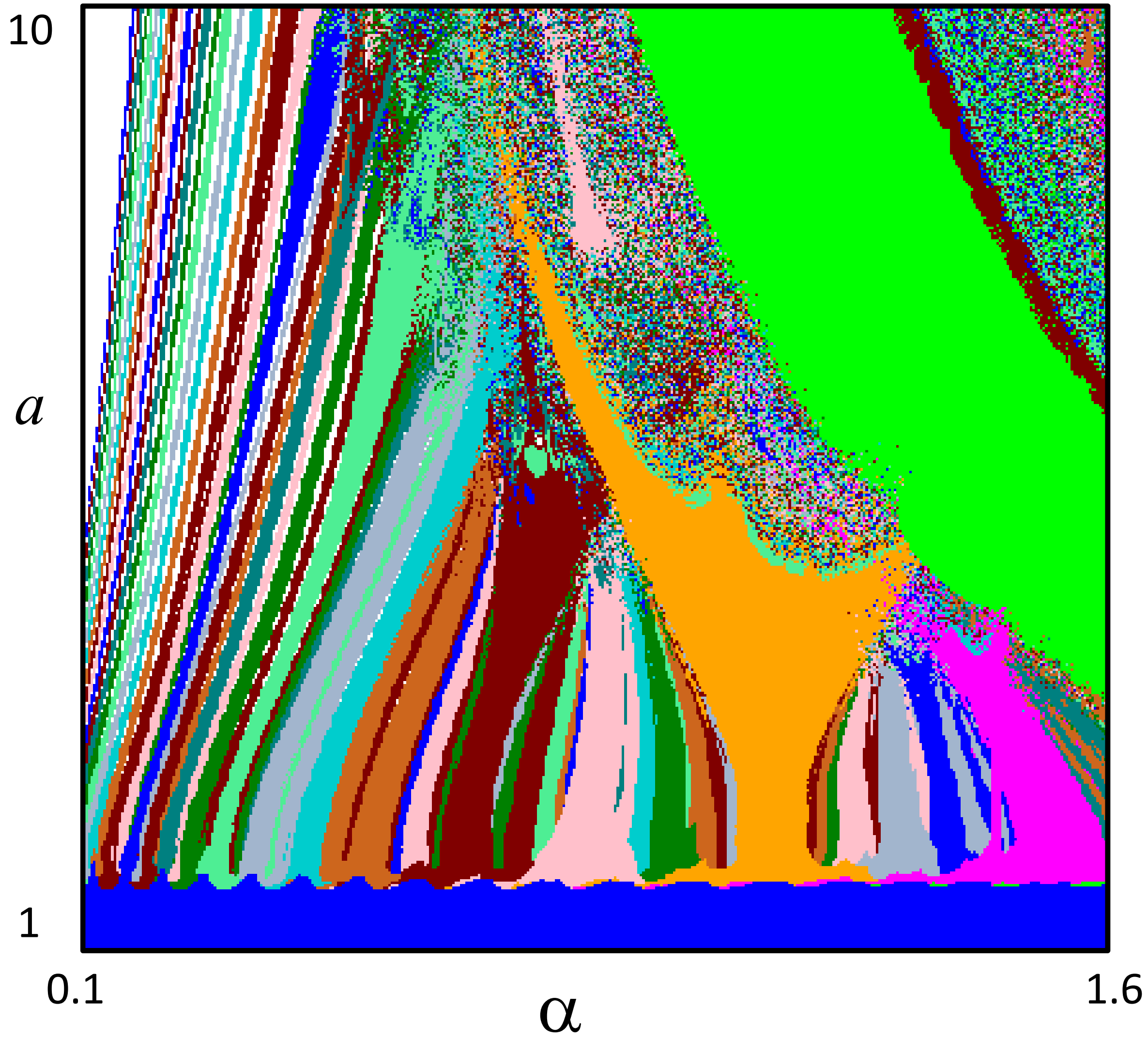
We have just some preliminar numerical result.

$\alpha = 0.514$; μ=1.4488; c=0.325; m=0.2872; F=100



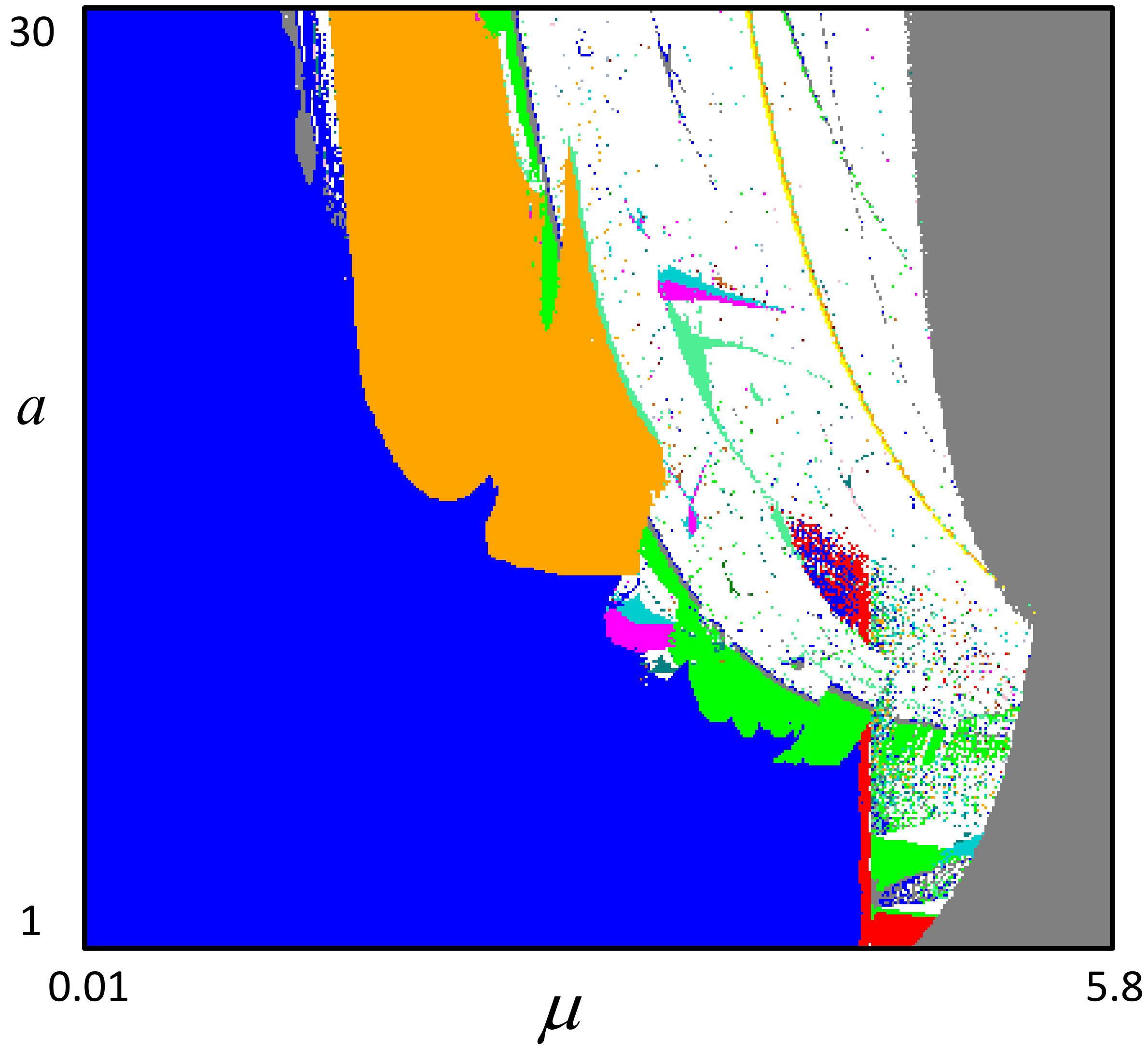
Bifurcation Diagram

$\delta = 0.5$; $\mu=0$; $c=0.3$; $m=0.2$; $F=100$



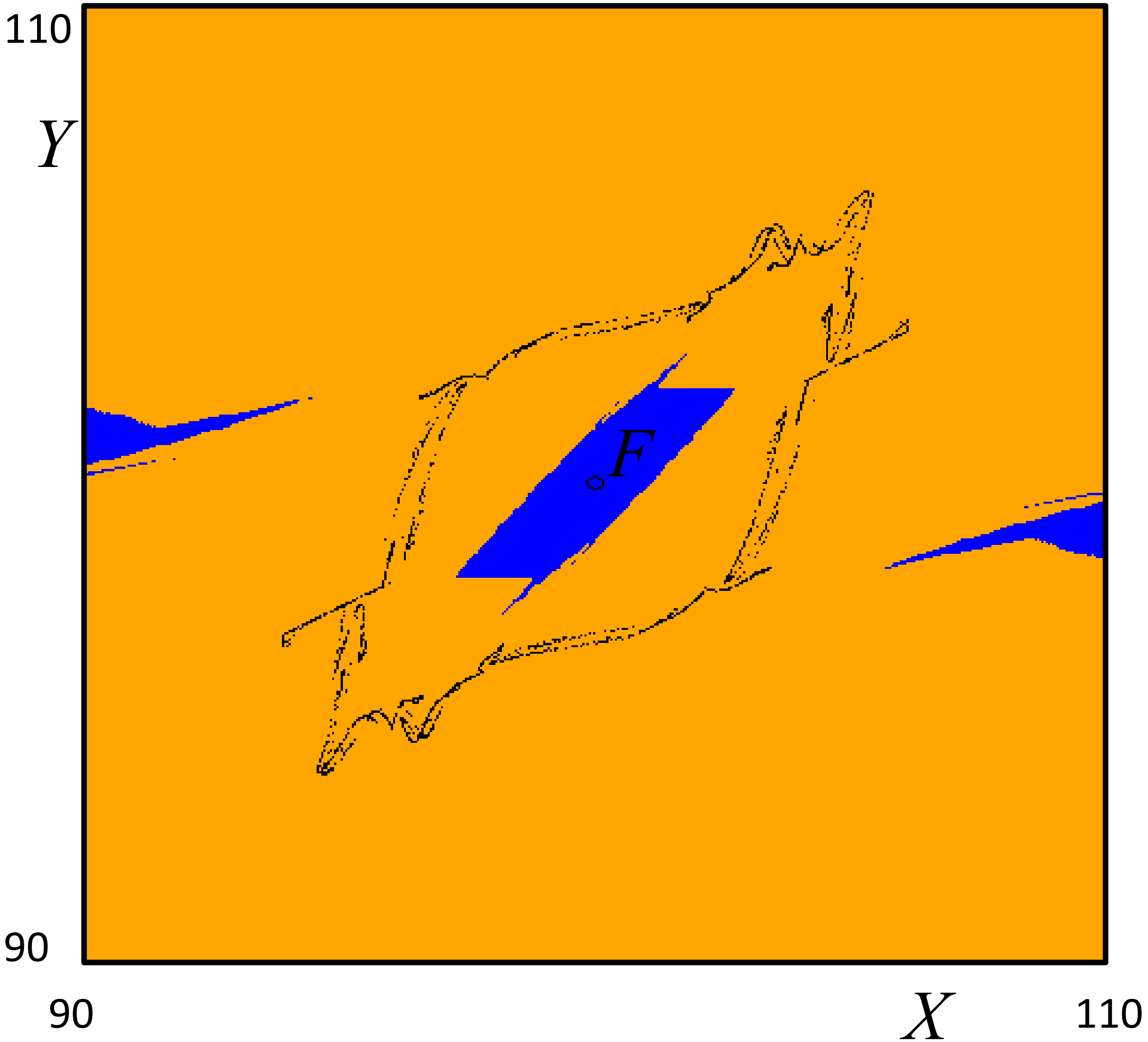
Bifurcation Diagram

$\delta = 1$; m=0.5; α=0.5; c=0.5; F=100



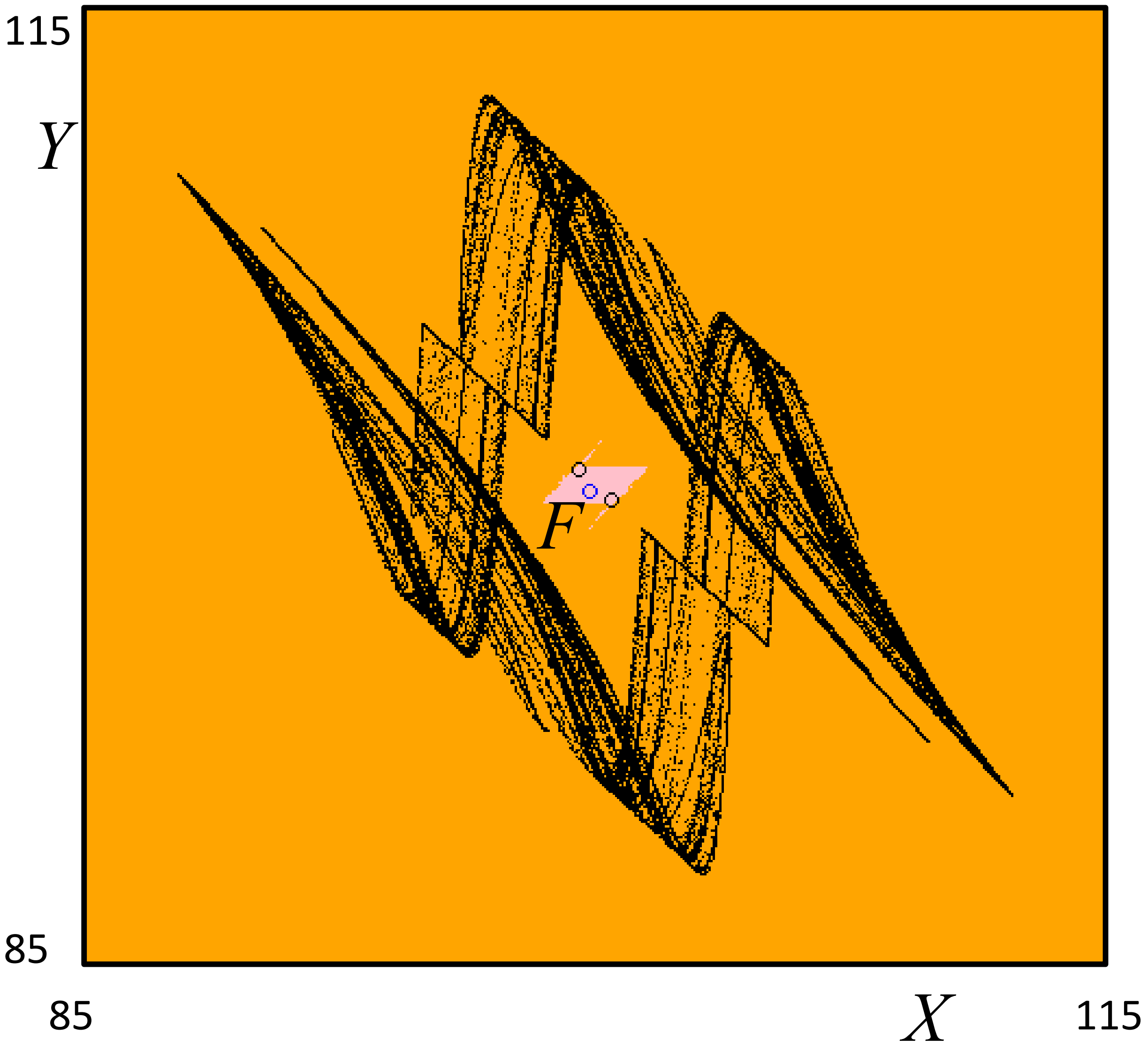
Bifurcation Diagram

$\delta = 0.793$; $\alpha = 0.514$; μ=1.4488; a=7.93; c=0.325; m=0.2872; F=100



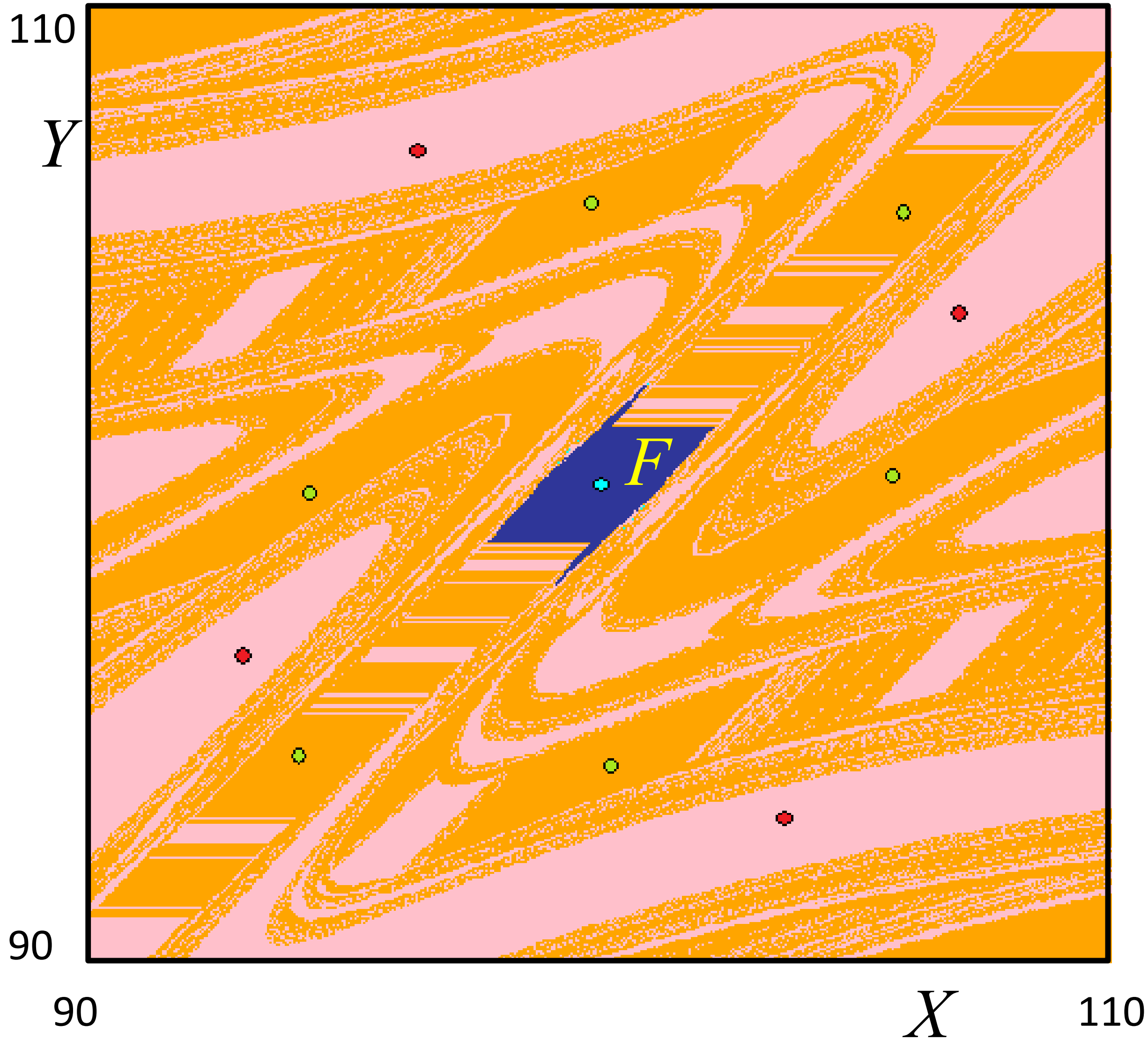
Phase Plane - Bistability

$\delta = 1$; $\alpha = 0.5$; μ=5.1657; a=7.93; c=0.253; m=0.549; F=100



Phase Plane – Bistability and Fundamental Fixed Point Locally Unstable

Triple coexistence



$\delta = 1$; $\alpha = 0.199$; μ=5.4055; a=10.9; c=0.172; m=0.8; F=100

Fundamental price as a random walk

Until now, we have used a fixed value for the fundamental price. Let us consider it variable, in particular we introduce a random component in this way:

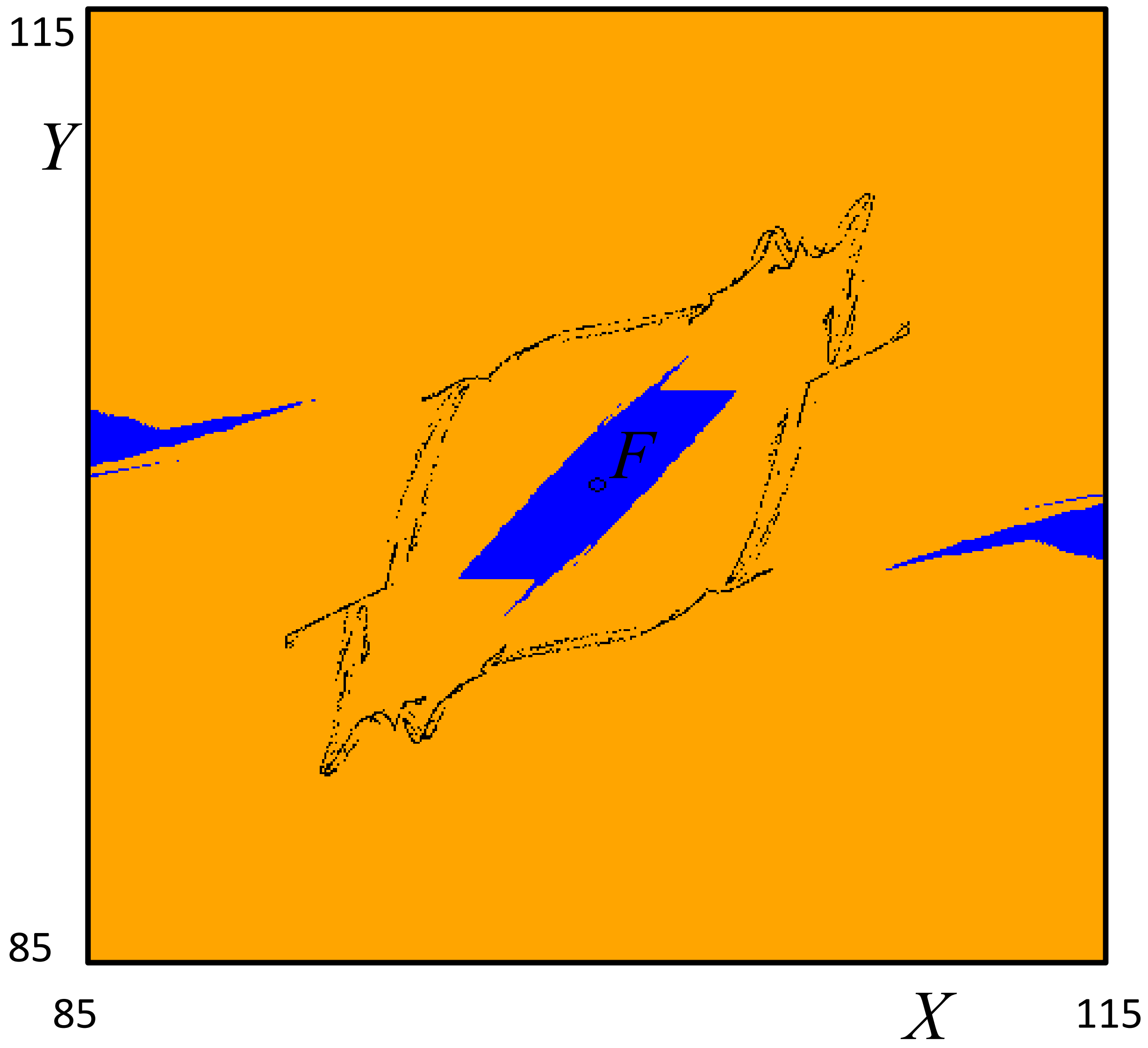
$$F_{t+1} = F_t \left(1 + \frac{\sigma_\delta}{\sqrt{K}} \tilde{\delta}_t\right)$$

where $\tilde{\delta}_t \sim N(0,1)$.

We expect that, in presence of multistability, even starting from the basin of attraction of an attractor, the movements of the fundamental price can lead the orbit to switch, in an almost unpredictable way, between the basins of attractions.

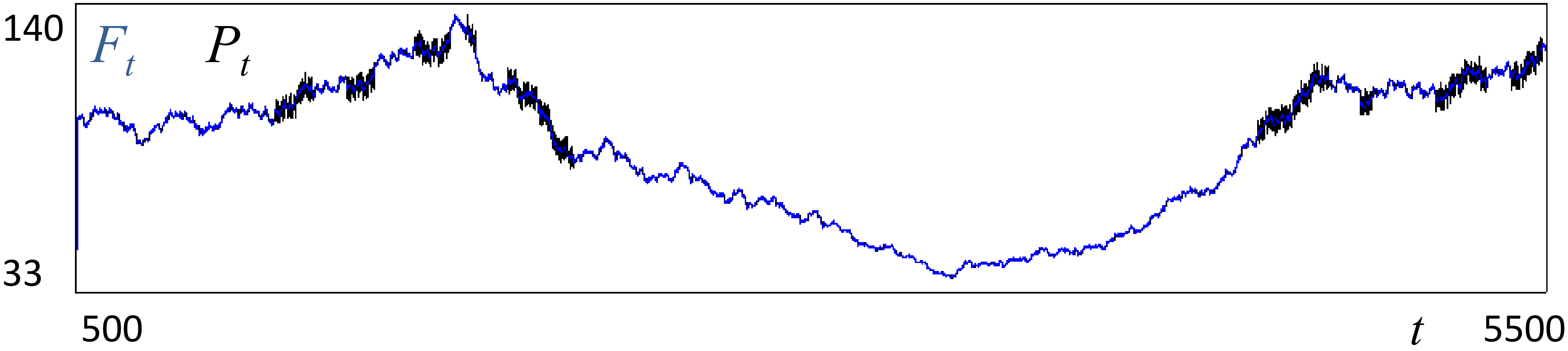
If the coexisting attractor are characterized by quite different levels of stability (for instance the fundamental fixed point and a chaotic attractor) this mechanism permit to reproduce the phenomenon of volatility clusters:

$\delta = 0.793$; $\alpha = 0.514$; μ=1.4488; a=6; c=0.325; m=0.2872; F_0=100

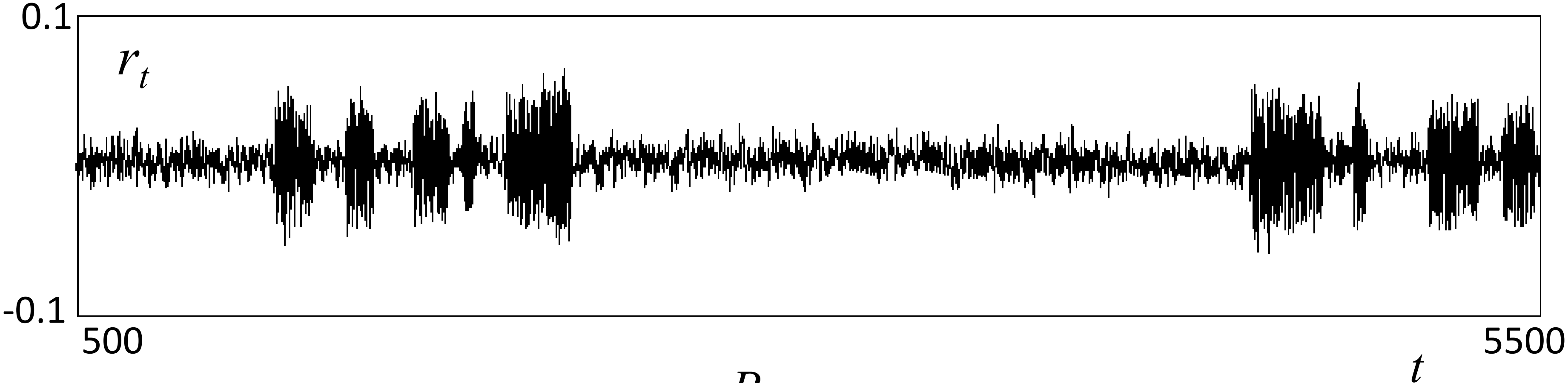


This could be an interesting starting point…

Volatility clusters



$\sigma_\delta = 0.1$ $K = 250$



$$r_t = \frac{P_t}{F_t} - 1$$

Thank you