Evolution and market behavior with endogenous strategies

Giulio Bottazzi Pietro Dindo

LEM, Scuola Superiore Sant'Anna, Pisa

Scuola Superiore Sant'Anna, 2 October 2009

Research questions

Consider an asset market and an ecology of investment strategies competing to gain superior returns. The open questions are:

- ⇒ which strategies do survive in the long run?
- ⇒ is a strategy dominating all the others?
- ⇒ is it possible to establish an order relationship among them?

Answer to these questions help to clarify specific issues (think of financial markets) as well as general issues (“as if” point).

Where do we stand?

On this issue

- Behavioral Finance (De Long at al 90ab etc)

  Pros Ecology of strategies behaviorally grounded
  Cons No wealth-driven strategy selection
  Focus Market biases

- HAM Finance (a survey is Hommes 2006)

  Pros Focus on price feedbacks
  Cons No wealth-driven strategy selection (mostly CARA), deterministic
  Focus Stylized facts

- Evolutionary Finance (Blume and Easley, Evstigneev et al.)

  Pros Wealth-driven strategy selection
  Cons Absence of price feedbacks (no endogenous strategies)
  Focus Market selection

⇒ Our approach: evolutionary finance with endogenous strategies.

Framework

- Trading is repeated and occurs in discrete time
- Many assets in constant supply with uncertain dividends
- Market is complete (Arrow's securities)
- A strategy is a portfolio of wealth fractions (CRRA)
- Walrasian market clearing
- Intertemporal budget constraint
- Market dynamics is formalized as a random dynamical system

Market model
Market clearing and intertemporal budget constraint

Cash numeraire $P_{0,t} = 1$. K assets. I agents. Walrasian market clearing equations set prices

$$P_{k,t} = \sum_{i=1}^{I} \alpha^i_{k,t} W^i_t \quad k \in \{1, \ldots, K\} \, ,$$

$P_{k,t}$ and W^i_t price of asset k and wealth of agent i at time t. α^i_t strategy of agent i at time t, $\sum_{k=1}^{K} \alpha^i_{k,t} = 1 - \alpha^i_{0,t}$.

$s = 1, \ldots, S = K$ states with probabilities π_s. If at time $t+1$ state $s(t+1)$ is realized

$$W^i_{t+1} = \sum_{k=1}^{K} \frac{\alpha^i_{k,t} W^i_t}{P_{k,t}} \left(\delta_L P_{k,t+1} + \delta_{k,s(t+1)} \right) \quad i \in \{1, \ldots, I\} \, .$$

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Known results

	EXO STRATEGIES $\alpha_t = \alpha(d_{t-1}, \ldots, d_0)$	ENDO STRATEGIES $\alpha_t = \alpha(p_{t-1}, \ldots, p_0)$
Short Lived Assets	Blume and Easley (1992) $\alpha_s^* \div \pi_s$ $I_\pi(\alpha) = \sum_{s=1}^{S} \pi_s \log\left(\frac{\pi_s}{\alpha_s}\right)$	dominance? survival? ordering?
Long Lived Assets	Evstigneev et al (2006, 2008) $\alpha_s^* \div \pi_s$	dominance? survival? ordering?

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Short Lived Assets	Blume and Easley (1992) $\alpha^*_s \div \pi_s$ $I_\pi(\alpha) = \sum_{s=1}^S \pi_s \log\left(\frac{\pi_s}{\alpha_s}\right)$	our business today
Long Lived Assets	Evstigneev et al (2006, 2008) $\alpha^*_s \div \pi_s$	dominance? survival? ordering?

A toy market

- Two states of the world, $S = 2$, which occur with probability π and $1 - \pi$. Bernoulli process $\omega = (\dots, \omega_t, \dots, \omega_0) \in \Omega$.
- Two (short-lived) Arrow's securities, $K = 2$, paying $D_{k,s} = \delta_{k,s}$.
- Consumption rate is constant and uniform, $\alpha_0 = c$.
  $\alpha_2 = 1 - \alpha_1 - c$.
- Define normalized prices $p_{s,t} = \frac{P_{s,t}}{W_t}$ so that $p_{1,t} + p_{2,t} = 1 - \alpha_0$, $\forall t$.
- Two agents, $I = 2$, with wealth fractions $\phi_t = W_t^1 / (W_t^1 + W_t^2)$ and $1 - \phi_t$.
- Endogenous strategis with one memory lag, $L = 1$,
  - $\alpha^1_{1,t} = \alpha^1_1(p_{1,t-1})$ describes the portfolio choice of the first agent,
  - $\alpha^2_{1,t} = \alpha^2_1(p_{1,t-1})$ describes the portfolio choice of the second agent.

Two agents: the random dynamical system

Given $x_t = (\phi_t, p_t, q_t = p_{t-1})$, the state of our market at time t, the random dynamical system is the composition of the following maps

$$
\begin{cases}
\phi_{t+1} & = & \begin{cases} \frac{\alpha_1^1(q_t)\phi_t}{p_t} & \text{with probability} \quad \pi \\ \\ \frac{(1-\alpha_0-\alpha_1^1(q_t))\phi_t}{1-\alpha_0-p_t} & \text{with probability} \quad 1-\pi \end{cases} , \\
\\
p_{t+1} & = & \alpha_1^1(p_t)\phi_{t+1} + (1-\phi_{t+1})\alpha_1^2(p_t) \, , \\
\\
q_{t+1} & = & p_t \, .
\end{cases}
$$

That is, $x_{t+1} = f_\pi(x_t)$ with probability π and $x_{t+1} = f_{1-\pi}(x_t)$ with probability $1-\pi$, depending on the realization of ω_t.

Definition

The state $x^* = (\phi^*, p^*, q^* = p^*)$ is a deterministic fixed point of the random dynamical system generated by the maps f_π and $f_{1-\pi}$, $\varphi(t, \omega, x) = \dots f_\pi \circ \dots \circ f_{1-\pi} \dots$, if it holds

$$\varphi(t, \omega, x^*) = x^* \quad \forall \omega \in \Omega \tag{1}$$

or, in terms of the maps, if it holds both

$$f_\pi(x^*) = x^* \quad \textit{and} \quad f_{1-\pi}(x^*) = x^* \,. \tag{2}$$

Fixed points

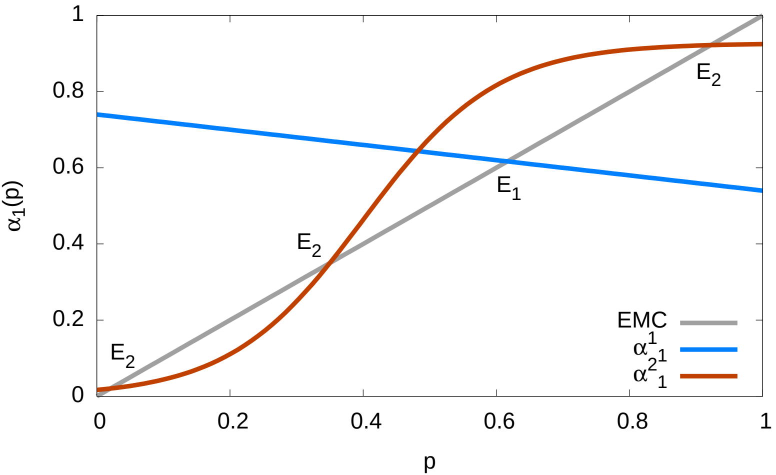
In our toy market

Theorem

Fixed points of the random dynamical system that represents the toy market dynamics are given by

$$
\begin{aligned}
x_1^* &= (\phi^* = 1, p^* = \alpha_1^1(p^*), q^* = p^*) \\
x_2^* &= (\phi^* = 0, p^* = \alpha_1^2(p^*), q^* = p^*) \\
x_{1/2}^* &= (\phi^*, p^* = \alpha_1^1(p^*) = \alpha_1^2(p^*), q^* = p^*)
\end{aligned}
$$

Fixed points on a plot: the Equilibrium Market Curve



Definition

A fixed point x^* of the random dynamical system $\varphi(t, \omega, x)$ is called locally stable if $\lim_{t \to \infty} ||\varphi(t, \omega, x) - x^*|| \to 0$ for all x in a neighborhood $U(\omega)$ of x and for all $\omega \in \Omega$.

Theorem

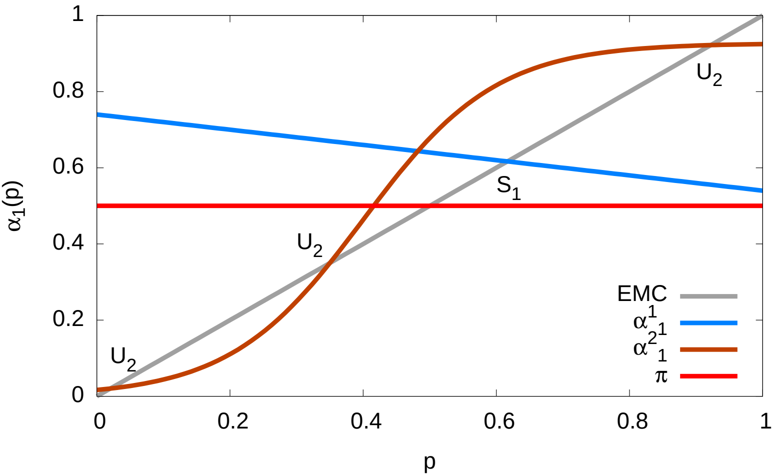
Provided that the eigenvalues of the iterated map are inside the unit circle the deterministic fixed point is locally stable (use Multiplicative Ergodic Theorem and Local Hartman-Grobman Theorem). For fixed points of the type $(1, \alpha_1^2(p^*), p^*)$ *eigenvalues are*

$$\mu = \left(\frac{\alpha_1^2(p*)}{\alpha_1^1(p^*)}\right)^{\pi} \left(\frac{\alpha_2^2(p*)}{\alpha_2^1(p^*)}\right)^{1-\pi} \quad and \quad \lambda = \left.\frac{\partial \alpha_1^1(p)}{\partial p}\right|_{p^*} \tag{3}$$

and for fixed points of the type $(\phi^*, \alpha_1^1(p^*) = \alpha_1^2(p^*), p^*)$

$$\mu = 1 \quad and \quad \lambda = \phi^* \left.\frac{\partial \alpha_1^1(p)}{\partial p}\right|_{p^*} + (1-\phi^*) \left.\frac{\partial \alpha_1^2(p)}{\partial p}\right|_{p^*} \tag{4}$$

Local stability on the EMC plot



Dominance and equivalence

Strategy i dominates strategy j, $i > j$, if

$$\forall \epsilon > 0\ , \quad \exists T \quad \text{s.t.} \quad \text{Prob}\left\{ \frac{\phi_t^j}{\phi_t^i} < \epsilon, \quad \forall t > T \right\} = 1\ .$$

Strategy i is equivalent to strategy j, $i \sim j$, if $\exists$ a $c \in (0, \infty)$

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or if $i > j$ for every j.

With no price feedback the entropy $I_\pi(\alpha)$ rules

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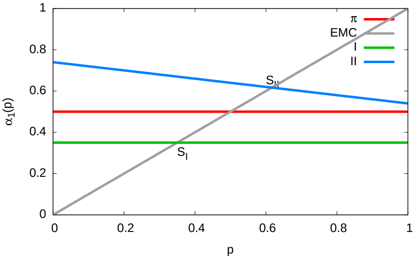
With no price feedback the entropy $I_\pi(\alpha)$ rules

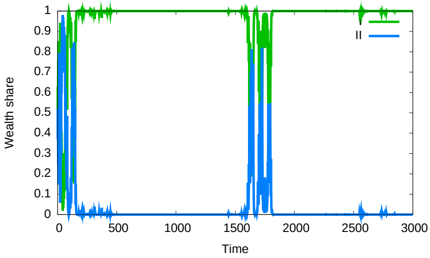
Ecology of investment strategies

- Unbiased: $\alpha_{1,t}^{UB} = (1 - \alpha_0)\pi$.
- Biased: $\alpha_{1,t}^{B} \neq (1 - \alpha_0)\pi'$ with $\pi' \neq \pi$.
- Price follower: $\alpha_{1,t}^{PF}$ such that $\frac{\partial \alpha_1^{PF}(p)}{\partial p} > 0$.
- Contrarian: $\alpha_{1,t}^{C}$ such that $\frac{\partial \alpha_1^{C}(p)}{\partial p} < 0$.
- Logit investor: $\alpha_{1,t}^{LG}$ such that $\alpha_1^{LG}(p) \sim \exp(\beta U(p))$.

Ordering is not complete

Coexistance of stable equilibria, *I?II*





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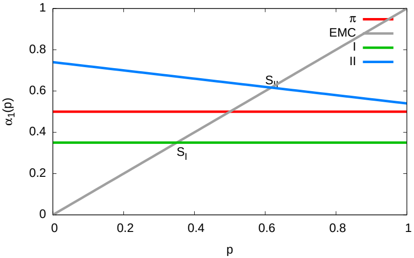
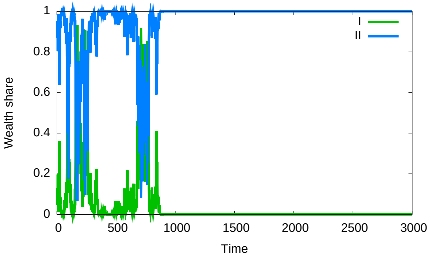
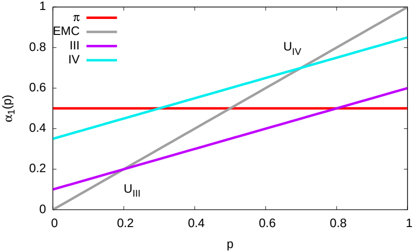
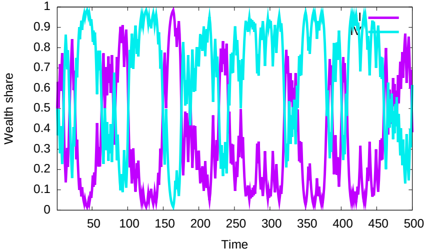




Ordering is not complete

Multiple unstable equilibria, *III*?*IV*





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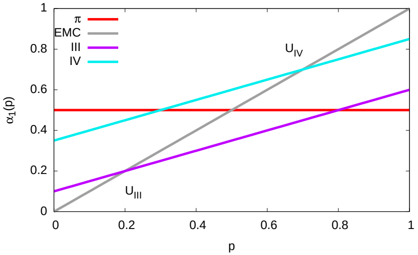
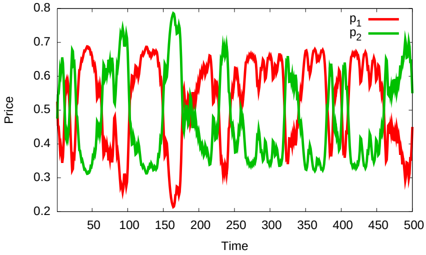
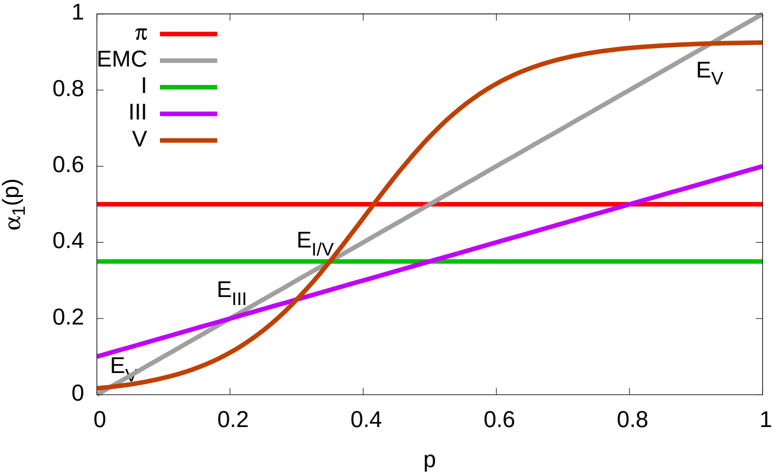




Ordering is not transitive

$I > III > V \sim I$



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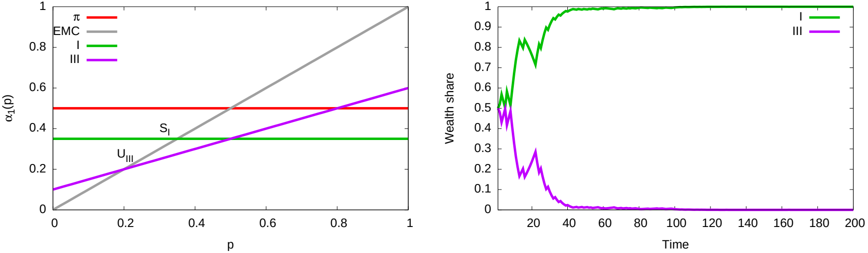
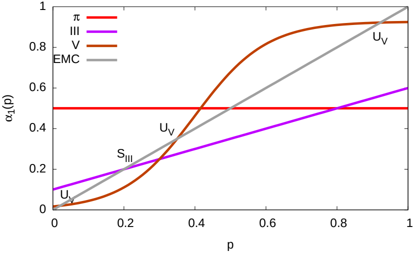
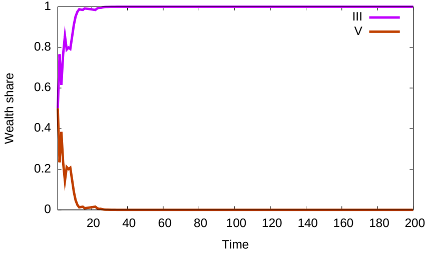


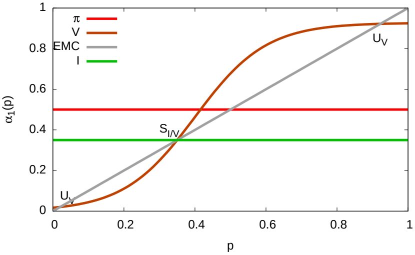
Ordering is not transitive

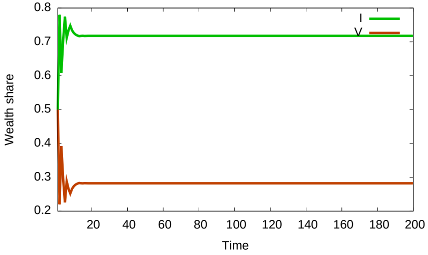
III > V



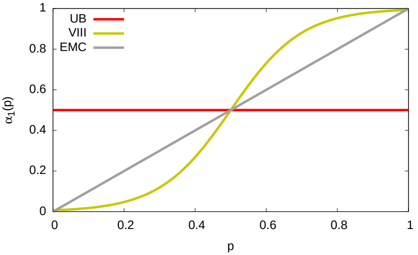


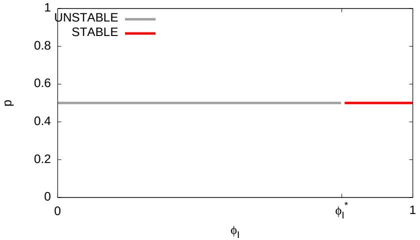
Ordering is not transitive
$V \sim I$





Does it exist a dominant strategy?
Yes, but not strictly





Beyond toy market

Same type of results holds with I agents, L memory lag, $S = K$ assets.
For x^* with $\phi^I = 1$ and $p^* = \alpha^I(p^*)$, eigenvalues are
$\Lambda = (\mu_1, ..., \mu_{I-1}, \lambda_{1,1}, ..., \lambda_{k,l}, \ldots, \lambda_{K-1,L})$, with

$$\mu_i = \prod_{k=1}^{K} \left(\frac{\alpha_k^i(p^*)}{\alpha_k^I(p^*)} \right)^{\pi_k} , \tag{5}$$

and, for a any given k, $\lambda_{k,l}$ one of the L solutions of the following equation

$$\lambda^L + \sum_{l=0}^{L-1} \lambda^l (\alpha_k^I)^{(L-1-l,k)} = 0 \, , \tag{6}$$

where

$$(\alpha_k^I)^{(0,k)} = \left. \frac{\partial \alpha_k^I}{\partial p_k} \right|_{p^*} , \quad (\alpha_k^I)^{(l,k)} = \left. \frac{\partial \alpha_k^I}{\partial p_k^l} \right|_{p^*} \qquad l = 1, \ldots, L \, , \quad k = 1, \ldots, K-$$

Conclusion

- Many fixed points, located on the Equilibrium Market Curve, whose local stability depends both on
  - Entropy w.r.t. dividend payment process
  - Price feedbacks being not too strong

$\Rightarrow$ No ordering relation based on market dominance can be established