Time-Varying Beta: A Boundedly Rational Equilibrium Approach

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Plan of Talk

- Literature and Motivation
- A Dynamic CAPM Framework with Heterogeneous Beliefs
  - Heterogeneous Beliefs and Consensus Belief
  - Equilibrium Return Relation, Betas, and Equilibrium Prices
- A Model with Classical Heterogeneous Agent-Types: Fundamentalists, Trend Followers and Noise traders
- A Numerical Analysis of Time-Varying Beta
  - Deterministic Dynamics and Stationary Benchmark CAPM
  - Trend Extrapolation and Time-Varying Beta: Ex-ante verse Ex-post betas
- Conclusion

1 Literature and Motivation

- **CAPM, Conditional CAPM and Time-Varying Beta Models**
  - CAPM and homogeneous beliefs;
  - Factor models: Fama-French.
  - Conditional expectations: Bollerslev, Engle & Wooldridge (1988);
  - Dependence on micro- and macro-economic factors;
  - Beta stability has been rejected—it varies from 2.5 in 1940s and fell to -0.5 in 2001 for the book-to-market portfolios, e.g. Kothari, Shanken & Sloan (1995); Campbell & Vuolteenaho (2004);
  - Conditional CAPM provides a convenient way to incorporate time-varying beta and displays superiority in explaining the cross-section of returns and anomalies, e.g. Jagannathan & Wang (1996).

- **Econometric Models of Time-Varying Beta**
  - GARCH and M-GARCH: Engle (1982) and Bollerslev (1986), Bollerslev (1990);
  - EGARCH: asymmetric and nonlinear effects of beta on conditional volatility of positive and negative shocks: Braun, Nelson & Sunier (1990);
  - The random walk model: Fabozzi & Francis (1978) and Collins, Ledolter & Rayburn (1987);
  - The mean-reverting model: Bos and Newbold (1984);
  - The Markov switching models: Hamilton (1989);
  - Ang & Chen (2007) treat betas as endogenous variables that vary slowly and continuously over time and find that a single-factor model performs substantially better at explaining the book-to-market premium.

- Estimation:
  * Discrete changes in betas across constant betas within subsamples: Campbell & Vuolteenaho (2004), Fama & French (2006), and Lewellen & Nagel (2006);
  * Rolling window estimates;
- When betas vary over time, the standard OLS inference is misspecified and cannot be used to assess the fit of a conditional CAPM.

- **What are missing**
  - The econometric models lack of economic explanation;
  - Do not take into account agents' behaviour;
  - In the real world, agents have heterogeneous subjective beliefs and they are boundedly rational rather than perfectly rational.
  - The financial markets represent the aggregation of the interaction of the boundedly rational behaviour among heterogeneous agents, which should be reflected in the time-varying betas.

- **Heterogenous Agent Models (HAMs)**
  - Heterogeneous beliefs under learning: Williams (1977), Detemple & Murthy (1994), Zapatero (1998);
  - Applications of the theory of nonlinear dynamical systems, bounded rationality and herding: Day & Huang (1990), Kirman (1992), Lux (1995), Brock & Hommes (1997), Brock & Hommes (1998);
  - The key element: the expectation feedback:
  - Explain various types of market behaviour, such as the long-term swing of market prices from the fundamental price, asset bubbles, market crashes, the stylized facts and various kinds of power law behaviour: Farmer, Gillemot, Lillo, Mike & Sen (2004), Lux (2004), Alfarano, Lux & Wagner (2005), Chiarella, He & Hommes (2006), Gaunersdorfer & Hommes (2007), and He & Li (2007).
  - Surveys: Hommes (2006), LeBaron (2006) and Chiarella, Dieci & He (2009*b*).

- Most of the HAMs are not in the context of the CAPM, except Westerhoff (2004), Böhm & Chiarella (2005) and Chiarella, Dieci & He (2007).

- **Aims of this paper**
  - to model explicitly the stochastic behaviour of beta by incorporating heterogeneity, boundedly rationality and the expectation feedback;
  - to provide some economic explanation and intuition of the mechanism underlying the time variation of beta;
  - to examine the consistency and relationship between ex-ante and ex-post betas.

2 A Dynamic CAPM Framework with Heterogeneous Beliefs

2.1 Heterogeneous Beliefs and Consensus Belief

- **Basic Idea & Framework**: Lintner (1969), Chiarella, Dieci & He (2009*a*);
- **Set up**: repeated one-period mean-variance dynamic framework;
- **Market**:
  - one frisk-free asset (r_f) and N risky assets: $\widetilde{r}_{j,t}, j = 1, 2, \cdots, N$;
  - I investors grouped into H agent-types with fractions: $n_h = I_h/I$.
- **Heterogeneous Beliefs**
  - Assume $\widetilde{r}_{h,t} \sim MVN$;
  - Heterogeneous beliefs $\mathcal{B}_{h,t}(\widetilde{\mathbf{r}}) = (\mathbb{E}_{h,t}(\widetilde{\mathbf{r}}), \Omega_{h,t})$.

- **Portfolio Optimisation**:
  - Portfolio wealth:

$$\widetilde{W}_{h,t+1} = W_{h,t}(1 + r_f + w_{h,t}^T(\tilde{r}_{t+1} - r_f \mathbf{1}));$$

  - Investor i: $\max E_{i,t}(u_i(\widetilde{W}_{i,t+1}))$ with concave utility function $u_i(\cdot)$.
  - The global absolute risk aversion, e.g., CARA utility function

$$\theta_h := -E_{h,t}\left[u_h''(\widetilde{W}_{h,t+1})\right] / E_{h,t}\left[u_h'(\widetilde{W}_{i,t+1})\right]$$

  - The optimal portfolio of investor i:

$$\mathrm{w}_{h,t} = \frac{\theta_h^{-1}}{W_{h,t}} \Omega_{h,t}^{-1} E_{h,t}\left[\widetilde{\mathrm{r}} - r_f \mathbf{1}\right].$$

- **Market Aggregate Demand** in wealth for risky assets

$$\zeta_t := \sum_{h \in H} \zeta_{h,t} = \sum_{h \in H} n_h \theta_h^{-1} \Omega_{h,t}^{-1} [E_{h,t}(\widetilde{\mathrm{r}}_{t+1}) - r_f \mathbf{1}].$$

- **The market clearing condition**: $\zeta_t = \mathbf{S}_t \mathbf{p}_t$ results in a deterministic equation that gives price vector $\mathbf{p}_t$ as

$$\mathbf{p}_t = \mathbf{S}_t^{-1} \sum_{h \in H} n_h \theta_h^{-1} \Omega_{h,t}^{-1} [E_{h,t}(\widetilde{\mathbf{r}}_{t+1}) - r_f \mathbf{1}],$$

where $\mathbf{S}_t :=$*diag*$[s_{1,t}, s_{2,t}, ..., s_{N,t}]$ and $s_{j,t}$ is the supply of asset j

$$E_{h,t}(\widetilde{\mathbf{r}}_{t+1}) = \mathbf{f}_h(\mathbf{r}_{t-1}, \mathbf{r}_{t-2}, ..., \mathbf{p}_{t-1}, \mathbf{p}_{t-2}, ...),$$
$$\Omega_{h,t} = \Omega_h(\mathbf{r}_{t-1}, \mathbf{r}_{t-2}, ..., \mathbf{p}_{t-1}, \mathbf{p}_{t-2}, ...).$$

- **The return**

$$\widetilde{\mathbf{r}}_t = \mathbf{P}_{t-1}^{-1}(\mathbf{p}_t + \widetilde{\mathbf{d}}_t) - \mathbf{1} = \mathbf{F}(\mathbf{r}_{t-1}, \mathbf{r}_{t-2}, ..., \mathbf{p}_{t-1}, \mathbf{p}_{t-2}, ...; \widetilde{\mathbf{d}}_t),$$

where $\mathbf{P}_t :=$*diag*$(p_{1,t}, p_{2,t}, ..., p_{N,t})$ and the dividends are assumed to follow an i.i.d. process with $\mathbb{E}\left[\widetilde{\mathbf{d}}_t\right] = \overline{\mathbf{d}}$.

- **Consensus Belief**: $\mathcal{B}_a = \{\mathbb{E}_a(\widetilde{\mathbf{r}}), \boldsymbol{\Omega}_a\}$
  - Aggregate risk aversion: $\theta_a := \left(\sum_h \theta_h^{-1}\right)^{-1}$.
  - An "aggregate" variance/covariance matrix $\boldsymbol{\Omega}_a$ can be defined as

$$\boldsymbol{\Omega}_{a,t}^{-1} = \theta_a \sum_h n_h \theta_h^{-1} \boldsymbol{\Omega}_{h,t}^{-1}.$$

  - The "aggregate" expected returns on the risky assets $E_a\left(\widetilde{\mathbf{r}}_{t+1}\right)$:

$$E_{a,t}\left(\widetilde{\mathbf{r}}\right) = \theta_a \boldsymbol{\Omega}_{a,t} \sum_h n_h \theta_h^{-1} \boldsymbol{\Omega}_{h,t}^{-1} E_{h,t}\left(\widetilde{\mathbf{r}}_{t+1}\right)$$

  - The consensus belief is a **weighted** average of the heterogeneous beliefs, characterising the relation to the heterogeneous beliefs.
- **The market clearing prices**:

$$\mathbf{p}_t = \mathbf{S}_t^{-1} \theta_a^{-1} \boldsymbol{\Omega}_{a,t}^{-1} [E_{a,t}(\widetilde{\mathbf{r}}_{t+1}) - r_f \mathbf{1}].$$

- **CAMP under the heterogeneous beliefs**

$$E_{a,t}(\widetilde{\mathbf{r}}_{t+1}) - r_f\mathbf{1} = \beta_{a,t}[E_{a,t}(\widetilde{r}_{m,t+1}) - r_f],$$

where

$$\widetilde{r}_{m,t+1} = \frac{[E_{a,t}(\widetilde{\mathbf{r}}_{t+1}) - r_f\mathbf{1}]^\top \Omega_{a,t}^{-1}\widetilde{\mathbf{r}}_{t+1}}{[E_{a,t}(\widetilde{\mathbf{r}}_{t+1}) - r_f\mathbf{1}]^\top \Omega_{a,t}^{-1}\mathbf{1}}$$

denotes the random return on the market portfolio;

- **The ex-ante beta**:

$$\beta_{a,t} = \frac{[E_{a,t}(\widetilde{\mathbf{r}}_{t+1}) - r_f\mathbf{1}]^\top \Omega_{a,t}^{-1}\mathbf{1}}{[E_{a,t}(\widetilde{\mathbf{r}}_{t+1}) - r_f\mathbf{1}]^\top \Omega_{a,t}^{-1}[E_{a,t}(\widetilde{\mathbf{r}}_{t+1}) - r_f\mathbf{1}]}[E_{a,t}(\widetilde{\mathbf{r}}_{t+1}) - r_f\mathbf{1}].$$

- Time variation of aggregate betas is due to agents' time varying beliefs about the first and the second moments of the return distributions.

2.2 Steady State Equilibrium of the Deterministic Model

- Let $\mathbf{s}_t = \mathbf{s}$ and $\widetilde{\mathbf{d}}_t = \overline{\mathbf{d}}$;
- Then the steady state prices $\overline{\mathbf{p}}$ and returns $\overline{\mathbf{r}}$ must satisfy

$$\overline{\mathbf{p}} = \mathbf{S}^{-1} \sum_{h \in H} n_h \theta_h^{-1} \overline{\Omega}_h^{-1} [\overline{\mathbf{f}}_h - r_f \mathbf{1}], \tag{2.1}$$

where $\overline{\Omega}_h := \Omega_h(\overline{\mathbf{r}}, \overline{\mathbf{r}}, ..., \overline{\mathbf{p}}, \overline{\mathbf{p}}, ...)$, $\overline{\mathbf{f}}_h := \mathbf{f}_h(\overline{\mathbf{r}}, \overline{\mathbf{r}}, ..., \overline{\mathbf{p}}, \overline{\mathbf{p}}, ...)$, and $\overline{p}_j = \frac{\overline{d}_j}{\overline{r}_j}, j = 1, 2, ..., N$, representing equilibrium prices through the usual discounted dividend formula;

- The steady state prices, or returns, emerge endogenously from the market dynamics with evolving heterogeneous beliefs
- **Consistency Condition**:

$$\overline{\mathbf{f}}_h := \mathbf{f}_h(\overline{\mathbf{r}}, \overline{\mathbf{r}}, ..., \overline{\mathbf{p}}, \overline{\mathbf{p}}, ...) = \overline{\mathbf{r}}, \quad h \in H.$$

3 A Model with Classical Heterogeneous Agent-Types

- A typical heterogeneous agent model;
- **Three Types**—Fundamentalists, trend followers and noise traders.
- **Fundamentalists**
  – Mean

$$E_{f,t}(\widetilde{\mathbf{r}}_{t+1}) = \rho + \alpha \mathbf{P}^{*-1}(\mathbf{p}^* - \mathbf{p}_{t-1}) = \rho + \alpha(1 - \mathbf{P}^{*-1}\mathbf{p}_{t-1}),$$

  where $\rho = [\rho_1, \rho_2, ..., \rho_N]^\top$ is the long-run component or the fundamental of asset returns, $\mathbf{p}^* = [p_1^*, p_2^*, ..., p_N^*]^\top, p_j^* = \frac{\bar{d}_j}{\rho_j}$ is the fundamental prices;

  – Constant beliefs about the variance/covariance: $\Omega_{f,t} = \overline{\Omega}_f$.

- **Chartists—Trend Followers**
  - The expected return

$$E_{c,t}(\widetilde{\mathbf{r}}_{t+1}) = \mathbf{u}_{t-1},$$

where $\mathbf{u}_{t-1}$ is a vector of sample mean of past realized returns $\mathbf{r}_{t-1}, \mathbf{r}_{t-2}, \ldots$ with geometric decaying weights $(1-\delta)\{1, \delta, \delta^2, \cdots\}$

$$\mathbf{u}_{t-1} = \delta \mathbf{u}_{t-2} + (1-\delta)\mathbf{r}_{t-1}. \tag{3.1}$$

  - The variance/covariance matrix $\Omega_{c,t}$

$$\Omega_{c,t} = \overline{\Omega}_c + \lambda \mathbf{V}_{t-1},$$

where $\lambda \geq 0$ measures the sensitivity of the second-moment estimate to the sample variance $\mathbf{V}_{t-1}$

$$\mathbf{V}_{t-1} = \delta \mathbf{V}_{t-2} + \delta(1-\delta)(\mathbf{r}_{t-1} - \mathbf{u}_{t-2})(\mathbf{r}_{t-1} - \mathbf{u}_{t-2})^{\top}.$$

- The consensus variances/covariances and expected returns are given, respectively, by

$$\Omega_{a,t} = \theta_a^{-1}\left(\frac{n_f}{\theta_f}\overline{\Omega}_f^{-1} + \frac{n_c}{\theta_c}\Omega_{c,t}^{-1}\right)^{-1} = \left(\frac{n_f}{\theta_f} + \frac{n_c}{\theta_c}\right)\left(\frac{n_f}{\theta_f}\overline{\Omega}_f^{-1} + \frac{n_c}{\theta_c}\Omega_{c,t}^{-1}\right)^{-1},$$

$$\begin{aligned} E_{a,t}(\tilde{\mathbf{r}}_{t+1}) &= \theta_a \Omega_{a,t}\left[\frac{n_f}{\theta_f}\overline{\Omega}_f^{-1} E_{f,t}(\tilde{\mathbf{r}}_{t+1}) + \frac{n_c}{\theta_c}\Omega_{c,t}^{-1} E_{c,t}(\tilde{\mathbf{r}}_{t+1})\right]. \\ &= \left(\frac{n_f}{\theta_f}\overline{\Omega}_f^{-1} + \frac{n_c}{\theta_c}\Omega_{c,t}^{-1}\right)^{-1}\left\{\frac{n_f}{\theta_f}\overline{\Omega}_f^{-1}\left[\rho + \alpha(1 - \mathrm{P}^{*-1}\mathrm{p}_{t-1})\right] + \frac{n_c}{\theta_c}\Omega_{c,t}^{-1}\mathrm{u}_{t-1}\right\}. \end{aligned}$$

- **Noise traders**—The demand for the risky assets $\widetilde{\xi}_t := [\widetilde{\xi}_{1,t}, \widetilde{\xi}_{2,t}, ..., \widetilde{\xi}_{N,t}]^\top$, where $\widetilde{\xi}_{j,t}$ are i.i.d. with $E(\widetilde{\xi}_{j,t}) = 0$, $Var(\widetilde{\xi}_{j,t}) = q^2 s_j^2$, $q \geq 0$ capturing the 'intensity' of noise-trading.

- The market clearing condition in the presence of noise traders thus becomes

$$\theta_a^{-1}\Omega_{a,t}^{-1}[E_{a,t}(\tilde{\mathbf{r}}_{t+1}) - r_f\mathbf{1}] + \widetilde{\Xi}_t\mathbf{p}_t = \mathbf{S}\mathbf{p}_t$$

and the market clearing prices

$$\mathbf{p}_t = (\mathbf{S}-\widetilde{\Xi}_t)^{-1}\theta_a^{-1}\Omega_{a,t}^{-1}[E_{a,t}(\widetilde{\mathbf{r}}_{t+1}) - r_f\mathbf{1}], \tag{3.2}$$

where $\widetilde{\Xi}_t := diag(\widetilde{\xi}_{1,t}, \widetilde{\xi}_{2,t}, ..., \widetilde{\xi}_{N,t})$.

- **The complete dynamic model**

$$\mathbf{p}_t = (\mathbf{S}-\widetilde{\Xi}_t)^{-1}\Big\{\frac{n_f}{\theta_f}\overline{\Omega}_f^{-1}\Big[\rho + \alpha(\mathbf{1} - \mathbf{P}^{*-1}\mathbf{p}_{t-1})\Big] + \frac{n_c}{\theta_c}\Omega_{c,t}^{-1}\mathbf{u}_{t-1} - \Big(\frac{n_f}{\theta_f}\overline{\Omega}_f^{-1} + \frac{n_c}{\theta_c}\Omega_{c,t}^{-1}\Big) r_f\mathbf{1}\Big\}, \tag{3.3}$$

$$\widetilde{\mathbf{r}}_t = \mathbf{P}_{t-1}^{-1}(\mathbf{p}_t + \widetilde{\mathbf{d}}_t) - \mathbf{1}, \tag{3.4}$$

where $\Omega_{c,t} = \overline{\Omega}_c + \lambda \mathbf{V}_{t-1}$, and $\mathbf{u}_{t-1}$ and $\mathbf{V}_{t-1}$ are updated according to

$$\mathbf{u}_t = \delta\mathbf{u}_{t-1} + (1-\delta)\mathbf{r}_t, \tag{3.5}$$

$$\mathbf{V}_t = \delta\mathbf{V}_{t-1} + \delta(1-\delta)(\mathbf{r}_t - \mathbf{u}_{t-1})(\mathbf{r}_t - \mathbf{u}_{t-1})^\top. \tag{3.6}$$

- **The steady state of the deterministic model**
  - Let $\widetilde{\mathbf{d}}_t \equiv \overline{\mathbf{d}}$ and $\widetilde{\boldsymbol{\Xi}}_t \equiv \mathbf{0}$ for all t. Then the steady state $(\overline{\mathbf{p}}, \overline{\mathbf{r}}, \overline{\mathbf{u}}, \overline{\mathbf{V}})$ must satisfy

$$\overline{\mathbf{r}} = \overline{\mathbf{P}}^{-1}\overline{\mathbf{d}} = \overline{\mathbf{u}},$$

$$\overline{\mathbf{V}} = \mathbf{0},$$

  - Assume $\overline{\mathbf{p}} = \mathbf{p}^*$. Then

$$\rho := \mathbf{P}^{*-1}\overline{\mathbf{d}} = \overline{\mathbf{P}}^{-1}\overline{\mathbf{d}} = \overline{\mathbf{r}}.$$

    Therefore, $\mathbf{p}^* = \overline{\mathbf{p}}$ is defined implicitly by the equation

$$\mathbf{p}^* = \mathbf{S}^{-1}\left(\frac{n_f}{\theta_f}\overline{\boldsymbol{\Omega}}_f^{-1} + \frac{n_c}{\theta_c}\overline{\boldsymbol{\Omega}}_c^{-1}\right)(\mathbf{P}^{*-1}\overline{\mathbf{d}} - r_f\mathbf{1}). \qquad (3.7)$$

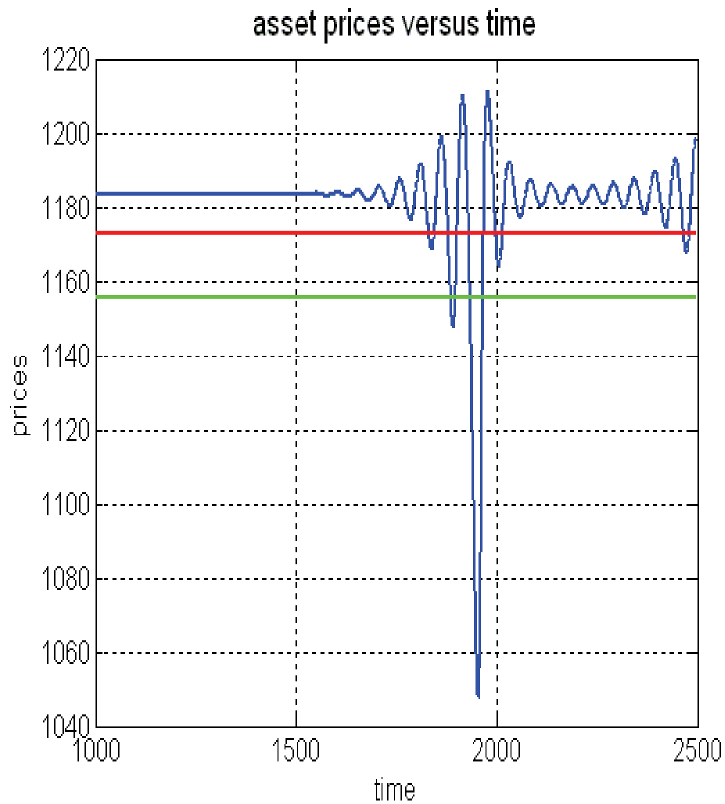
4 A Numerical Analysis of Time-Varying Beta

- **Aim**: to examine the impact of heterogeneous beliefs on the market and time variation of beta;
- **Parameter Selection**
  - A common parameter setting $\theta_f = \theta_c := \theta = 0.005$, $r_f = 0.02$, $\mathbf{s} = (1,1,1)^T$, $\rho = (0.12, 0.15, 0.21)^T$, $\overline{\Omega}_c = \overline{\Omega}_f := \overline{\Omega} = diag(\sigma_1^2, \sigma_2^2, \sigma_3^2)$, $\sigma_1 = 0.13$, $\sigma_2 = 0.15$, $\sigma_3 = 0.18$.
  - In our example $\mathbf{p}^*$ and $\overline{\mathbf{d}}$ turn out to be:

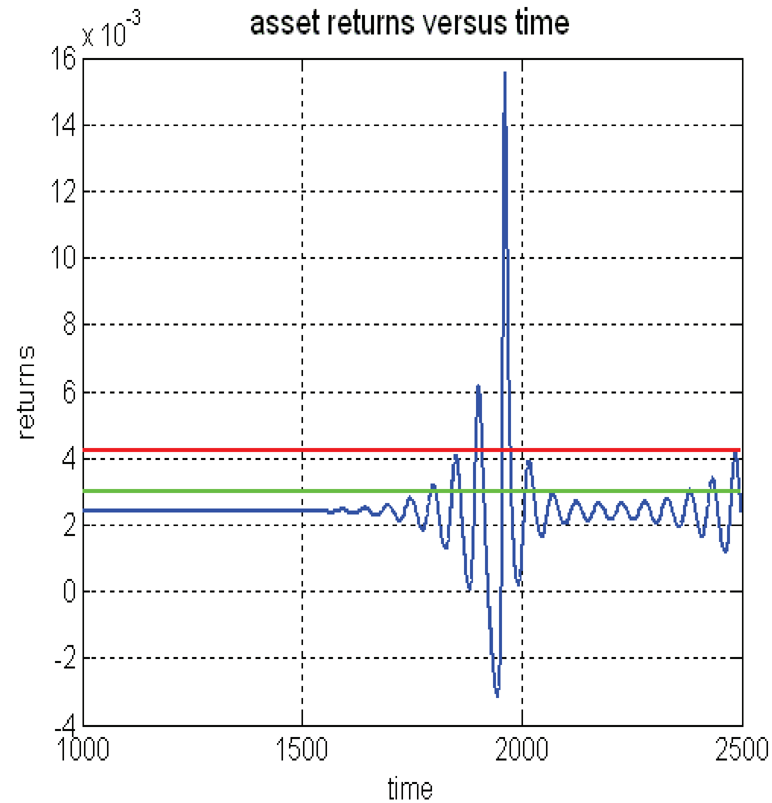
$$\mathbf{p}^* = \begin{bmatrix} p_1^* \\ p_2^* \\ p_3^* \end{bmatrix} = \begin{bmatrix} 1183.43 \\ 1155.56 \\ 1172.84 \end{bmatrix}, \qquad \overline{\mathbf{d}} = \begin{bmatrix} \overline{d}_1 \\ \overline{d}_2 \\ \overline{d}_3 \end{bmatrix} = \begin{bmatrix} 142.012 \\ 173.333 \\ 246.296 \end{bmatrix},$$

  - The parameters α, δ, λ, and n_f vary across examples, as well as q;
  - The parameters ρ, r_f, σ_1, σ_2, σ_3 and α, δ are at annual basis;
  - Trading frequency: $K = 12$ (monthly), 50 (weekly), 250 (daily).

- **Deterministic Dynamics**
  - **Aim**: to better understand the interaction of the nonlinearity and noise;
  - **Intuition**: when the trend followers extrapolate the recent trend in returns strongly (corresponding to a low δ), the market tends to be destabilized.
  - **Verification**: consider the changes in the equilibrium prices of the deterministic model when δ changes.
  - **Parameters**: $\alpha = 0.3$, $n_f = 0.3$, $\lambda = 1.5$ and $\delta = 0.784$ at an annual frequency, $K = 50$.
  - **Market instability**: the steady state equilibrium loses its stability when δ decreases so that $\delta < \widehat{\delta}$, where $\widehat{\delta} \in (0.785, 0.786)$ corresponds to the bifurcation value;



(a) Equilibrium prices



(b) Equilibrium returns

Figure 4.1: The fluctuations of price (a) and return (b). The blue, green and red to represent asset 1, 2, and 3, respectively.

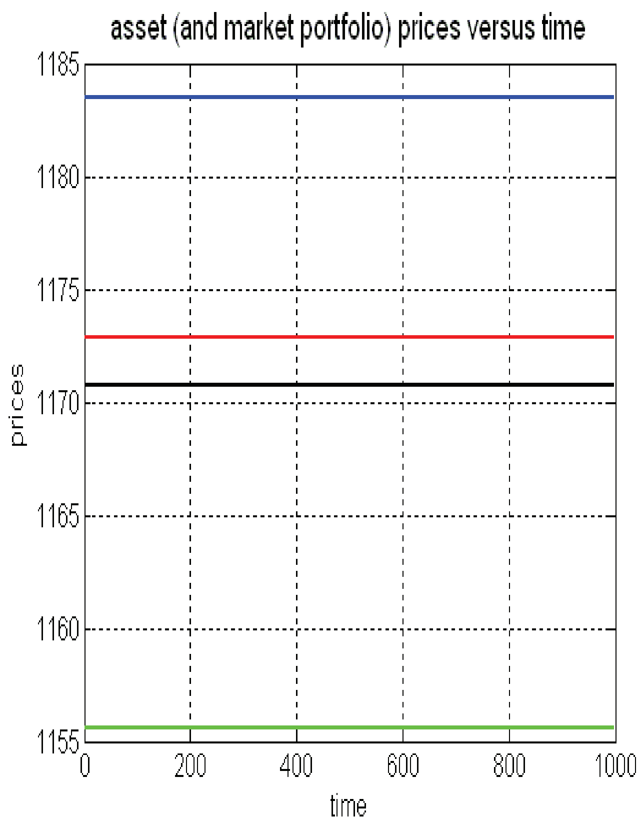
– **Observation**: only asset 1 fluctuates around the steady state equilibrium level, due to the large value of $\boldsymbol{\lambda}$ and the selection of $\boldsymbol{\delta = 0.784}$.

– **Explanation**:

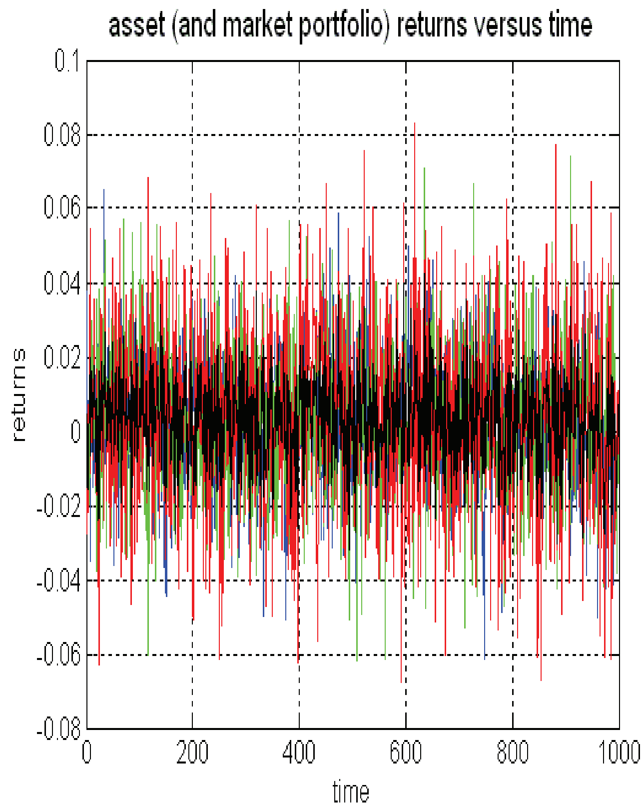
  * The interaction of the strong extrapolation of the trend followers and mean-reverting activity of the fundamentalists leads to the fluctuations of asset 1, and hence the market portfolio.
  * As $\boldsymbol{\delta}$ decreases further, namely as the trend followers extrapolate the recent returns even more strongly, all three assets will be destabilized.

- **A benchmark case of the standard 'stationary' CAPM**
  - The standard CAPM with homogeneous and constant beliefs: $\alpha = 0$ and $\delta = 1$, $\mathbf{u_0} = \rho$ and $\mathbf{V_0} = \mathbf{0}$.
  - Correspondingly, $E_{f,t}(\widetilde{\mathbf{r}}_{t+1}) = E_{c,t}(\widetilde{\mathbf{r}}_{t+1}) = \rho$ and $\overline{\Omega}_{c,t} = \overline{\Omega}_c = \overline{\Omega}$.
  - Assume no noise traders: $q = 0$.
  - Under these assumptions,

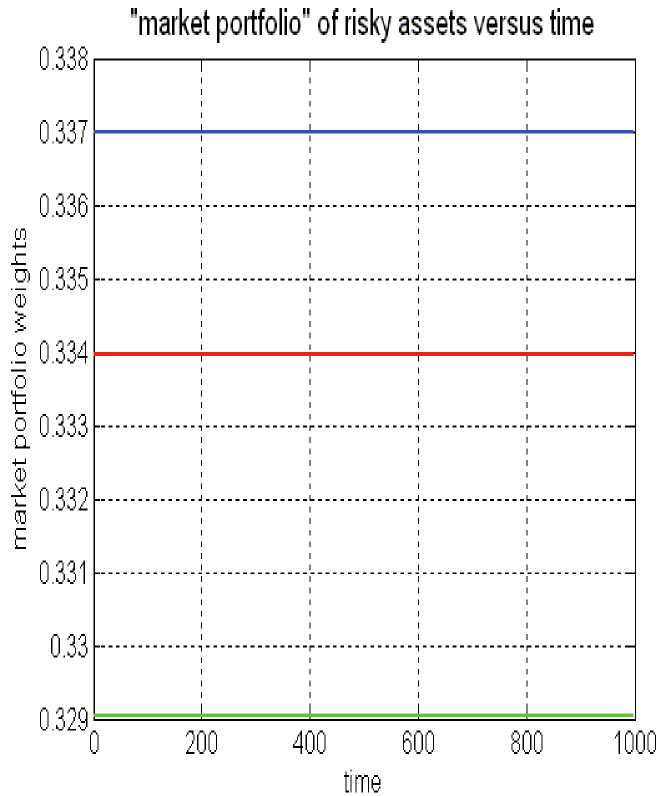
$$\mathbf{p}_t = \frac{1}{\theta}\mathbf{S}^{-1}\overline{\Omega}^{-1}(\rho - r_f\mathbf{1}) = \mathbf{p}^*,$$
$$\widetilde{\mathbf{r}}_t = \mathbf{P}^{*-1}(\mathbf{p}^* + \widetilde{\mathbf{d}}_t) - \mathbf{1} = \mathbf{P}^{*-1}\widetilde{\mathbf{d}}_t,$$
$$\mathbf{u}_t = \rho\,(= \mathbf{P}^{*-1}\overline{\mathbf{d}}\,),$$
$$\mathbf{V}_t = \mathbf{0}.$$



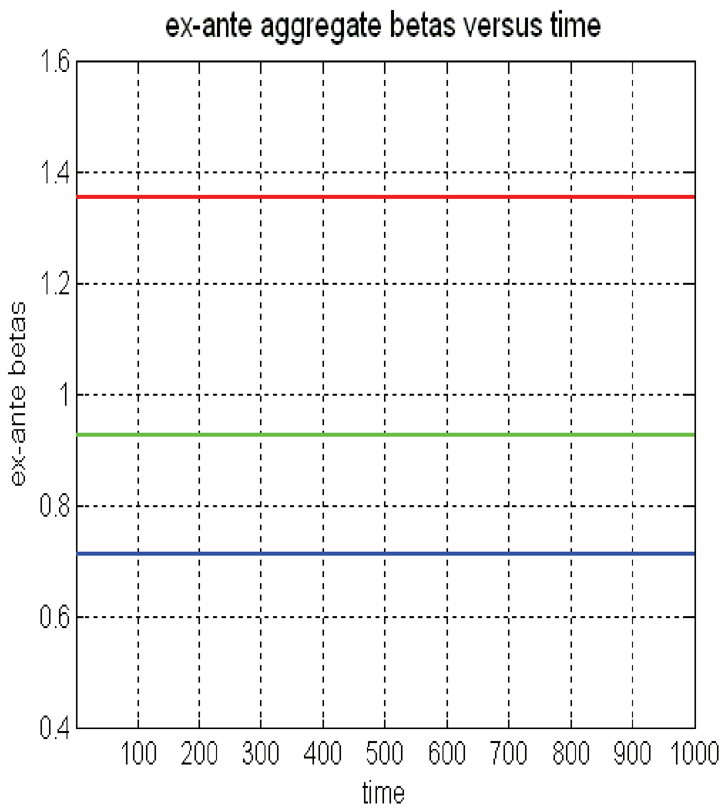
(a) Prices



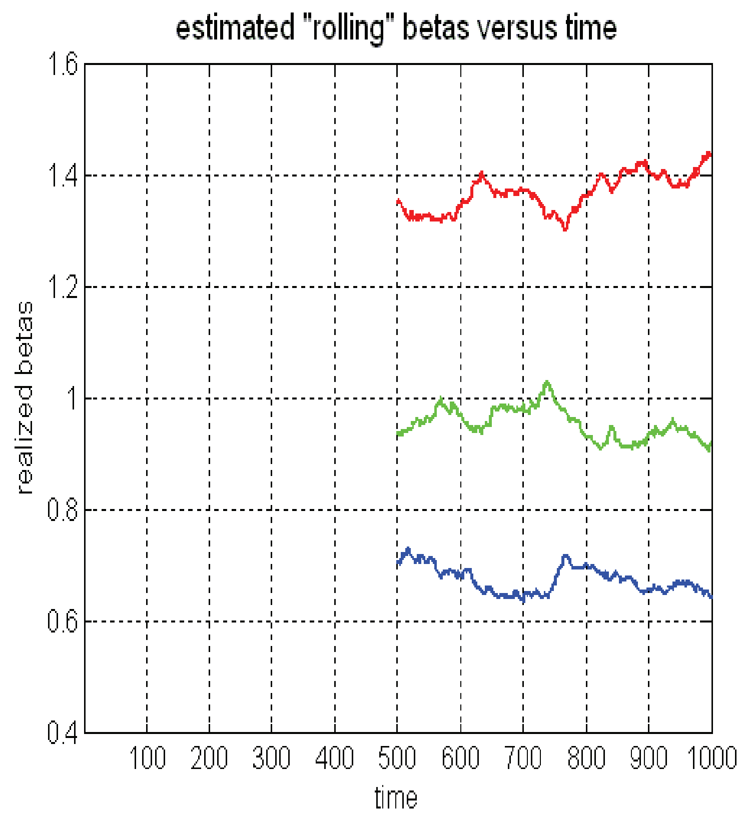
(b) Returns



(c) Market portfolio



(d) Ex-ante betas



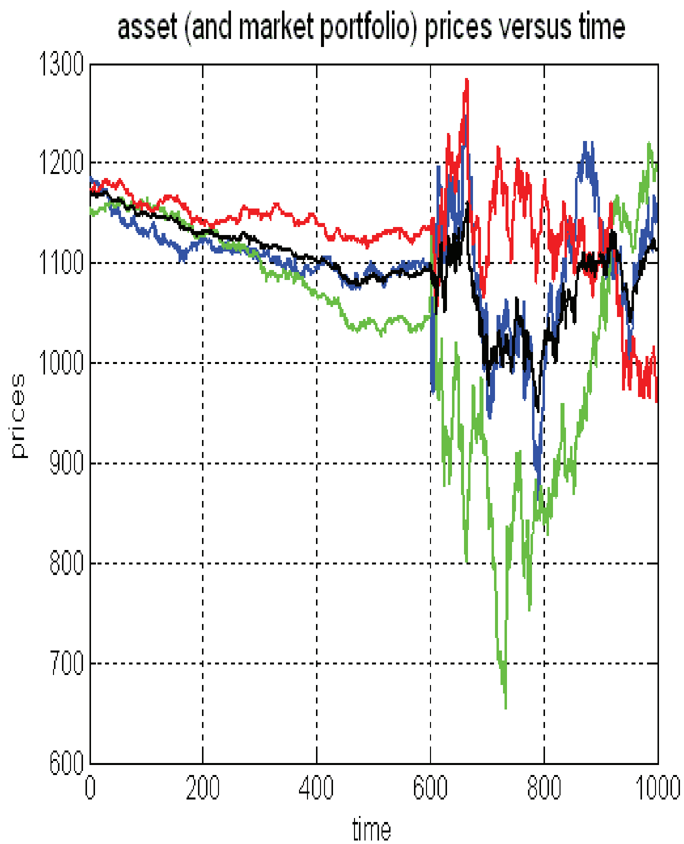
(e) Rolling estimates of betas

Figure 4.2: The dynamics of the benchmark stationary CAPM without noise trader with $\boldsymbol{K = 50}$ and $\boldsymbol{T = 1000}$

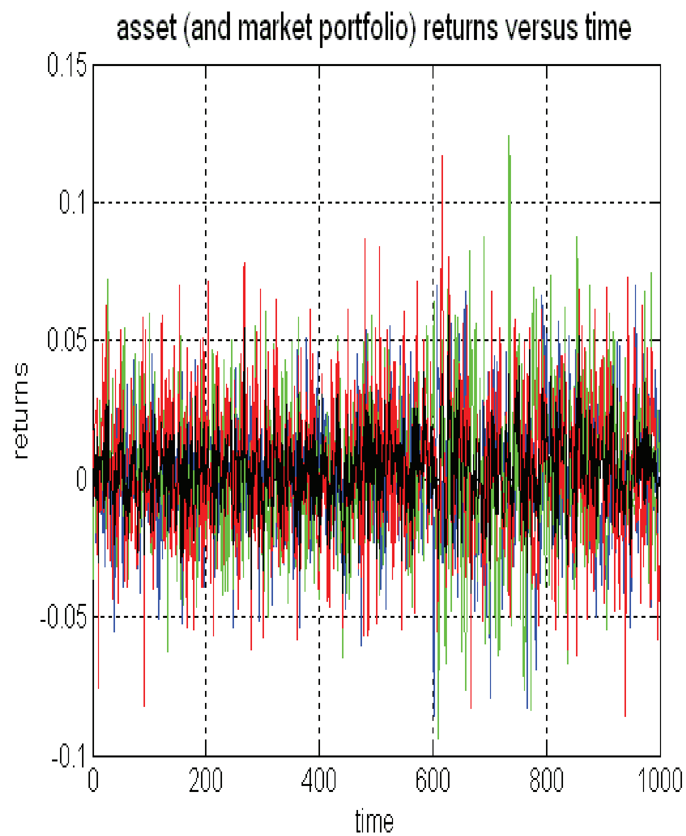
– **Observations**:

* A typical simulation of the benchmark scenario, with a weekly time step $K = 50$ with the length of the simulation $T = 1000$ time periods;
* Constant market equilibrium prices, market portfolio and ex-ante betas;
* Returns are linear function of the random dividend processes;
* The ex-post betas estimated via 'rolling' regression, using a rolling windows of 500 periods, appear to fluctuate randomly around their constant ex-ante beta levels.
* Apart from some small random fluctuations, the rolling window estimates of the betas are consistent with the constant ex-ante betas implied by the market equilibrium.

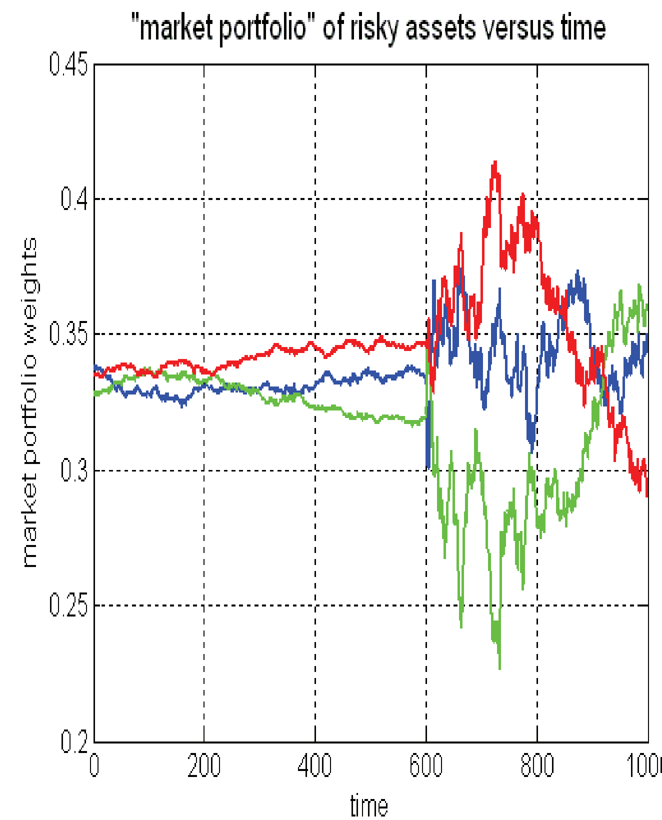
- **Trend following and time-varying betas**
  - **Aim**: To examine the trend extrapolation on the market and betas;
  - **Parameter selection**: $\alpha = 0.3$, $K = 50$, $T = 1000$, $\lambda = 0.5$, $n_f = 0.3$ and allow the decay rate δ to be change at two different levels;
  - The fundamental traders expect a certain degree of mean reversion towards fundamental prices, whereas chartists update their beliefs about the expected returns and volatility/correlations based upon realized returns and observed deviations from sample average returns.
  - Initially, $\delta = \delta_1 = 0.98$, close to 1, the benchmark homogeneous CAPM case;
  - A regime switching in δ occurs just after period $t^* = 600$ corresponding to a decrease of δ from $\delta_1 = 0.98$ to $\delta_2 = 0.85$, so that the chartists putting more weight on recent returns' history when forming their beliefs;



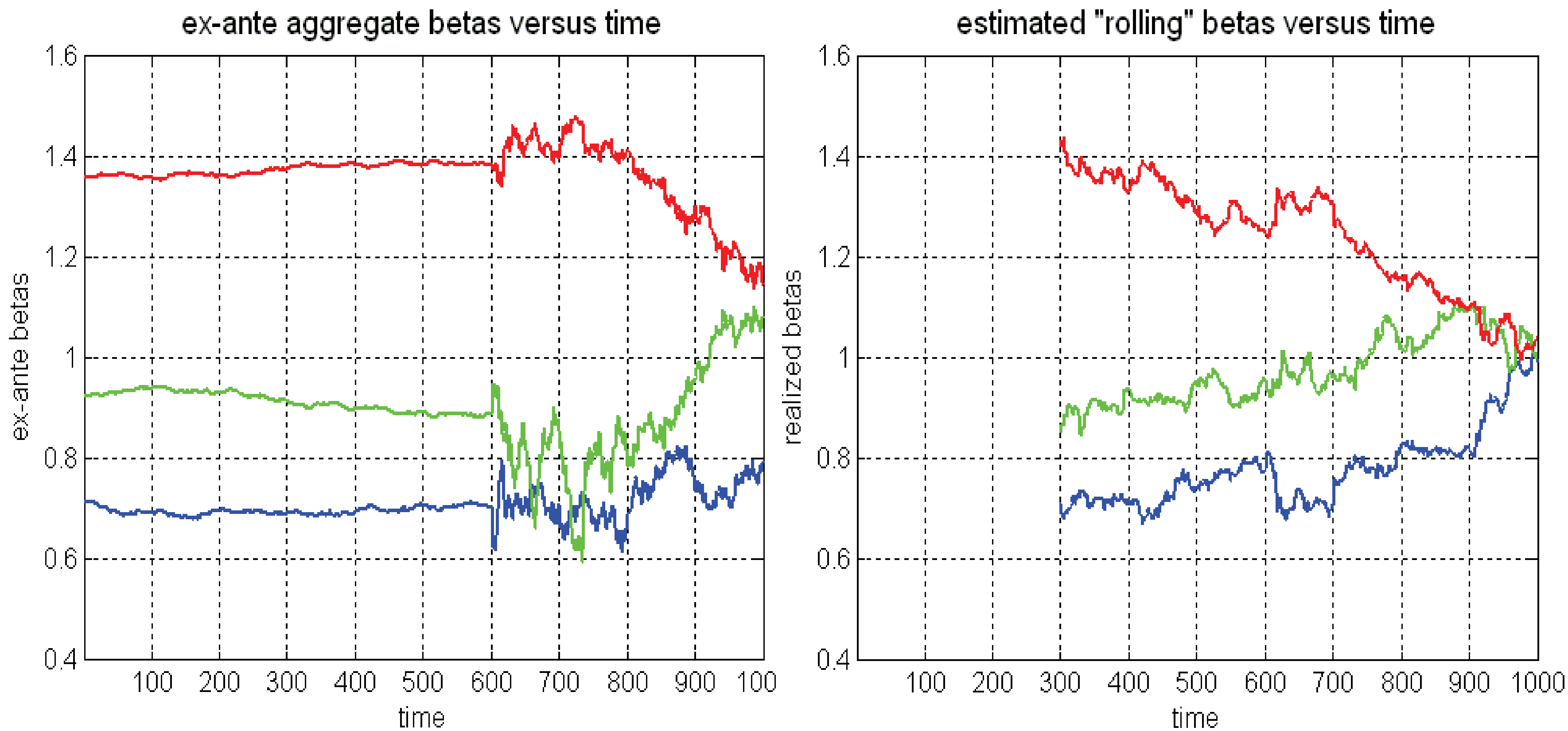
(a) Prices



(b) Returns

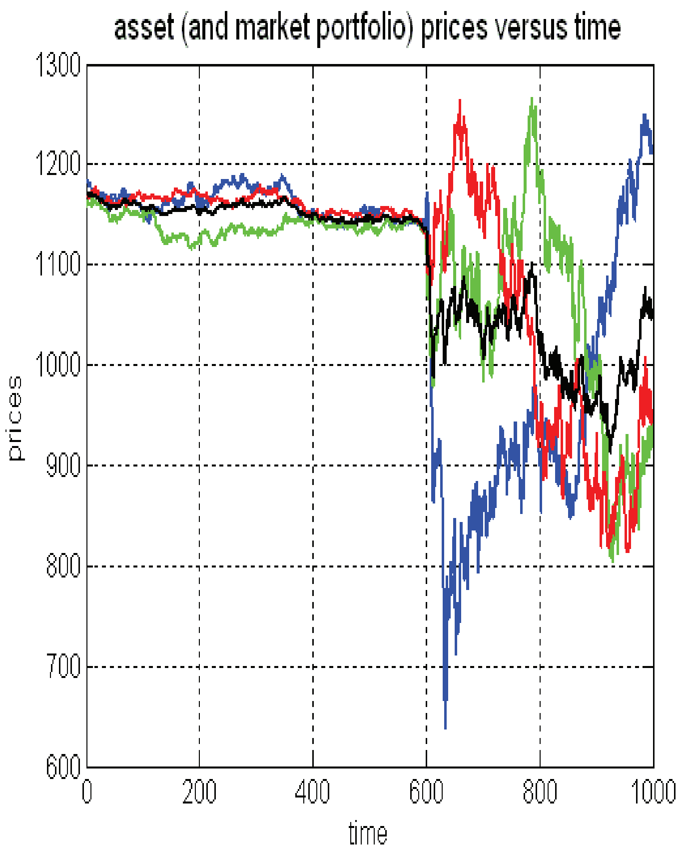


(c) Market portfolio

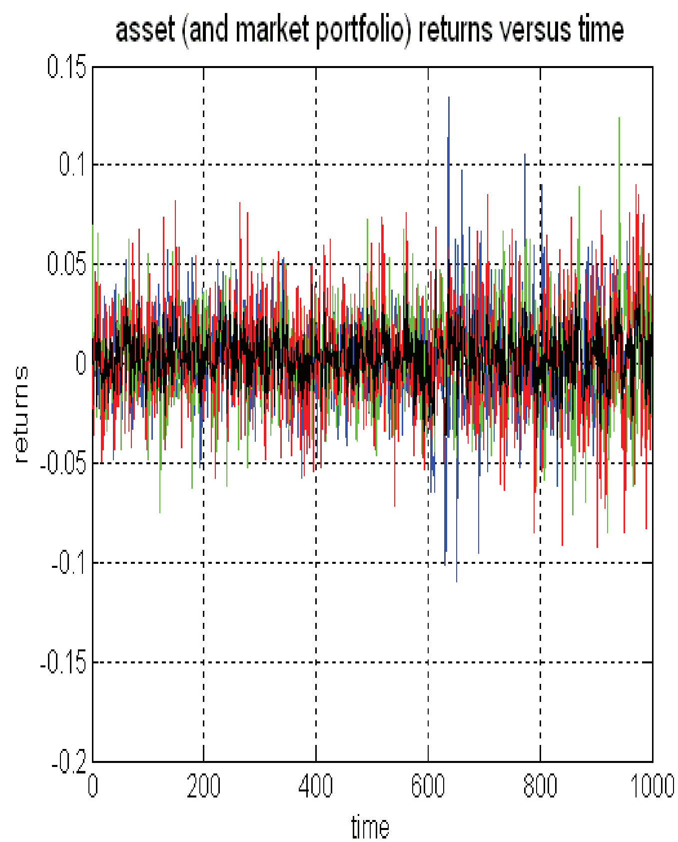


(d) Ex-ante betas (e) 5 years rolling window estimates of betas

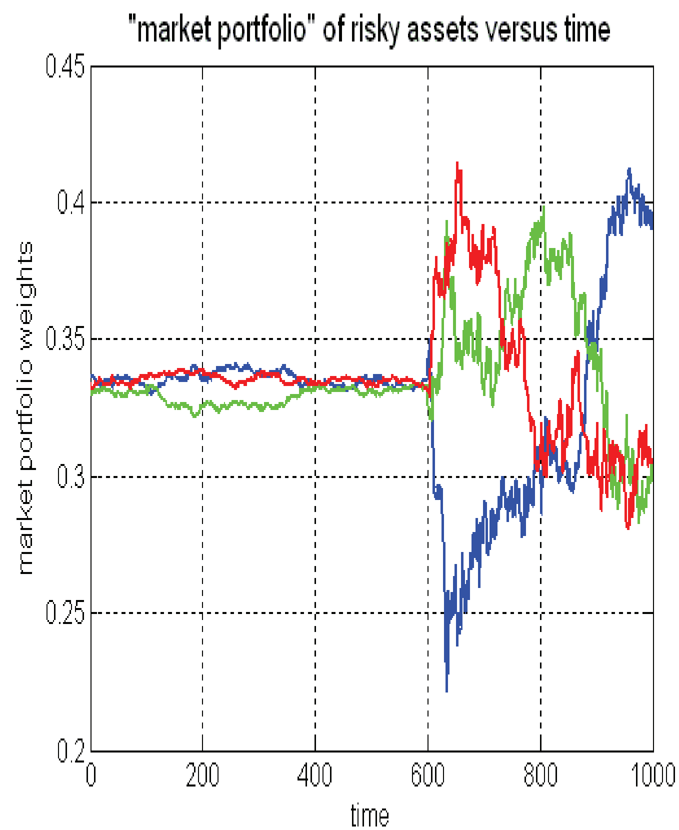
Figure 4.3: Simulation 1: Illustration of the impact of a change in δ at $t = 600$ on the market.



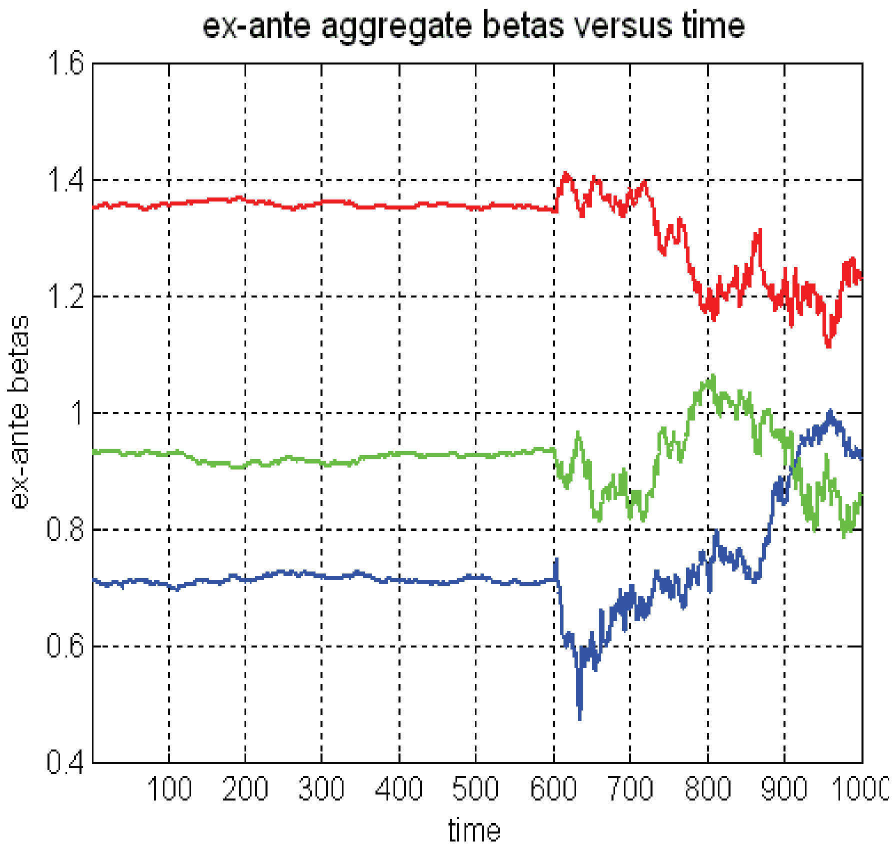
(a) Prices



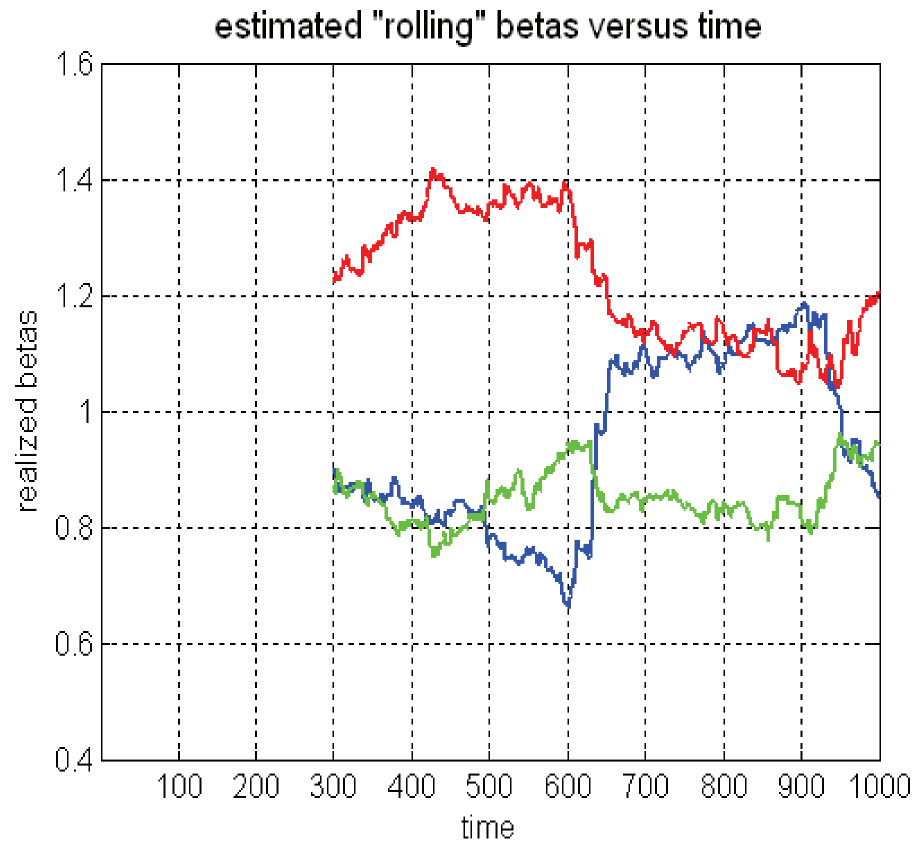
(b) Returns



(c) Market portfolio



(d) Ex-ante betas



(e) 5 years rolling window estimates of betas

Figure 4.4: Simulation 2: Illustration of the impact of a change in δ at $t = 600$ on the market.

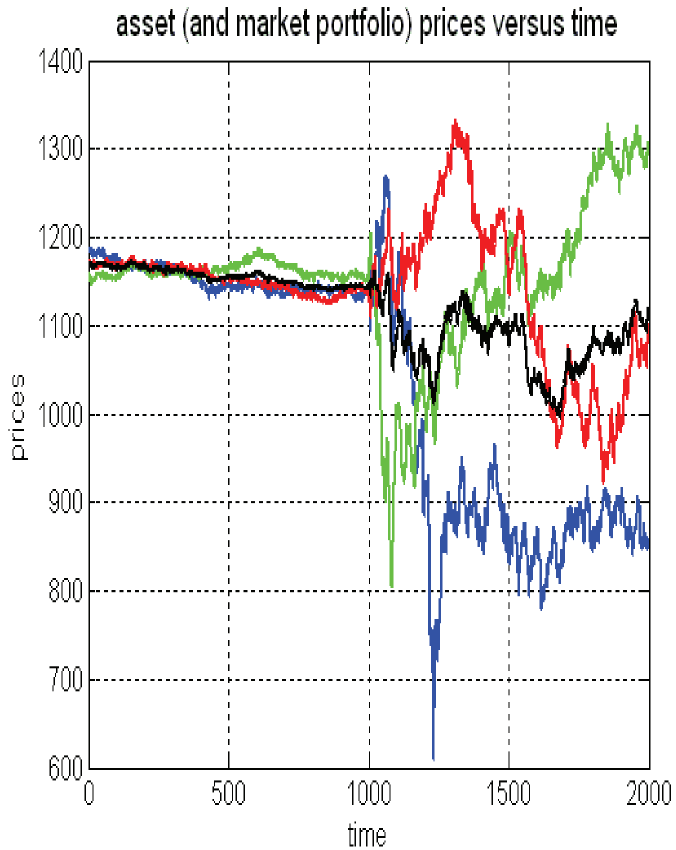
– **Observations**:

- The change in δ has significant impact on the market equilibrium prices.
- Under the change, agents start varying their portfolios over time in order to explore the emerging endogenous correlation patterns between the risky assets, sometimes reinforcing them.
- In the first period with high $\delta = 0.98$, the equilibrium prices, returns, market weights and ex-ante aggregate betas fluctuate around their steady state levels, and that the dynamics in the initial period is not far from the reference case described in the stationary CAPM case.
- The parameter change $\delta = 0.85$ then leads to a new scenario with more pronounced endogenous fluctuations of prices and returns, and consequently on the time-varying ex-ante betas.
- The stochastic nature of the time-varying betas changes significantly when the trend chasing behaviour of the trend followers changes.

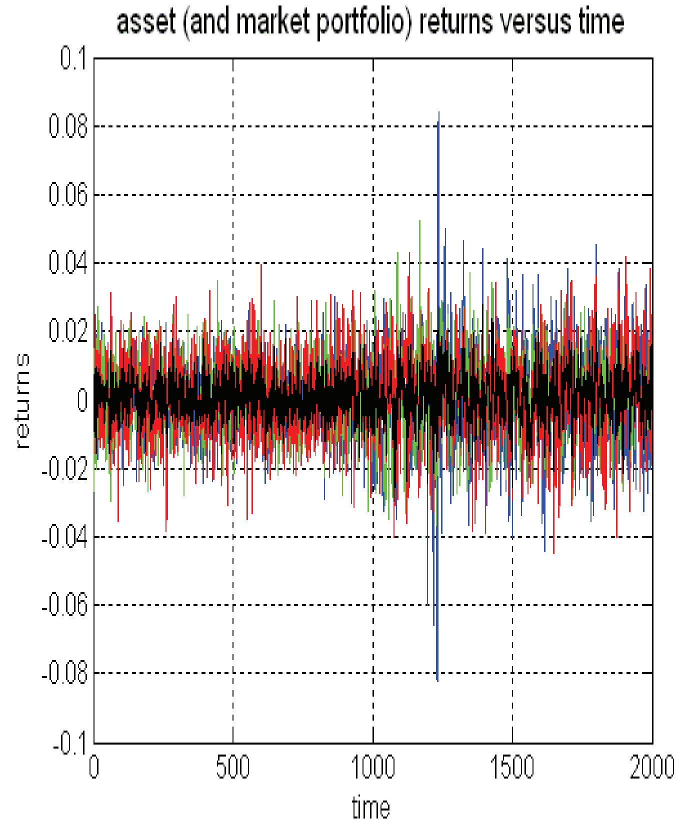
* The expectation feedback mechanism leads to high volatility in the market and the time-varying betas that reflect the change in risk of the risky assets.
* In the period following the change, the extrapolation leads the asset returns to be highly correlated with the market portfolio return;
* Measured by the time-varying ex-ante betas, less risky can become more risky due to the change in extrapolation.
* The ex-post beta 5 year rolling window estimates of betas can display very different patterns from the ex-ante betas, can be misleading in an economy with boundedly rational agents.

- This observation may provide an explanation to why in empirical studies that the time-varying CAPM based on the rolling window estimates of betas may have little or no explanatory power and this may simply due to the way the model is estimated rather than any shortcoming of the underlying equilibrium models.
- Similar experiments could be carried out by assuming that an exogenous shock at time t^* affects other behavioral parameter.

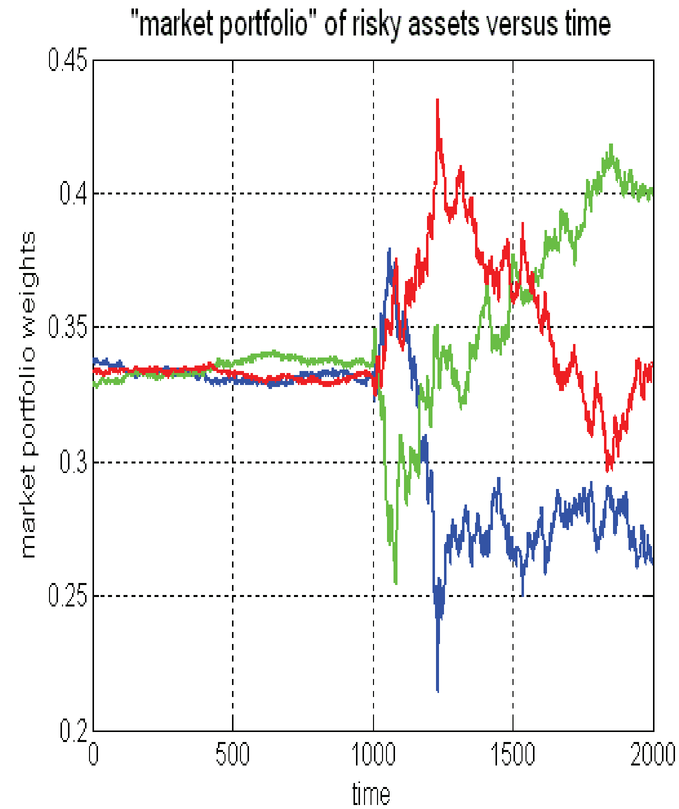
- **Other Results**:
  - Similar results can be obtained for $K = 12$ or 250;
  - With $K = 250$ and $T = 2000$, δ is decreased from $\delta_1 = 0.98$ to $\delta_2 = 0.85$ at time $t^* = 1000$, the 2 year rolling window estimates in betas is inconstant significantly with the ex-ante betas, the estimated betas vary between 0.8 and 1.4 for assets 1 and 3.



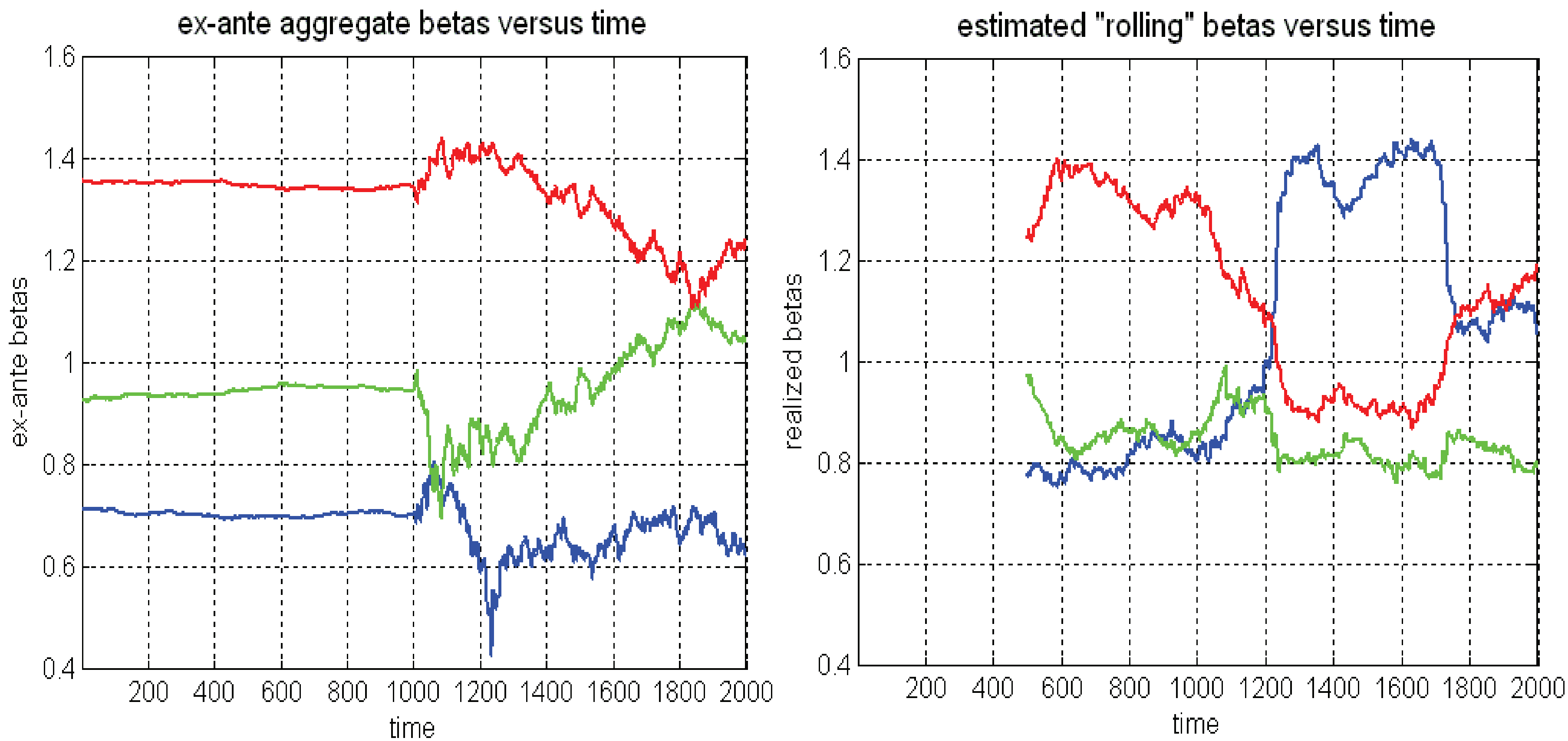
(a) Prices



(b) Returns



(c) Market portfolio

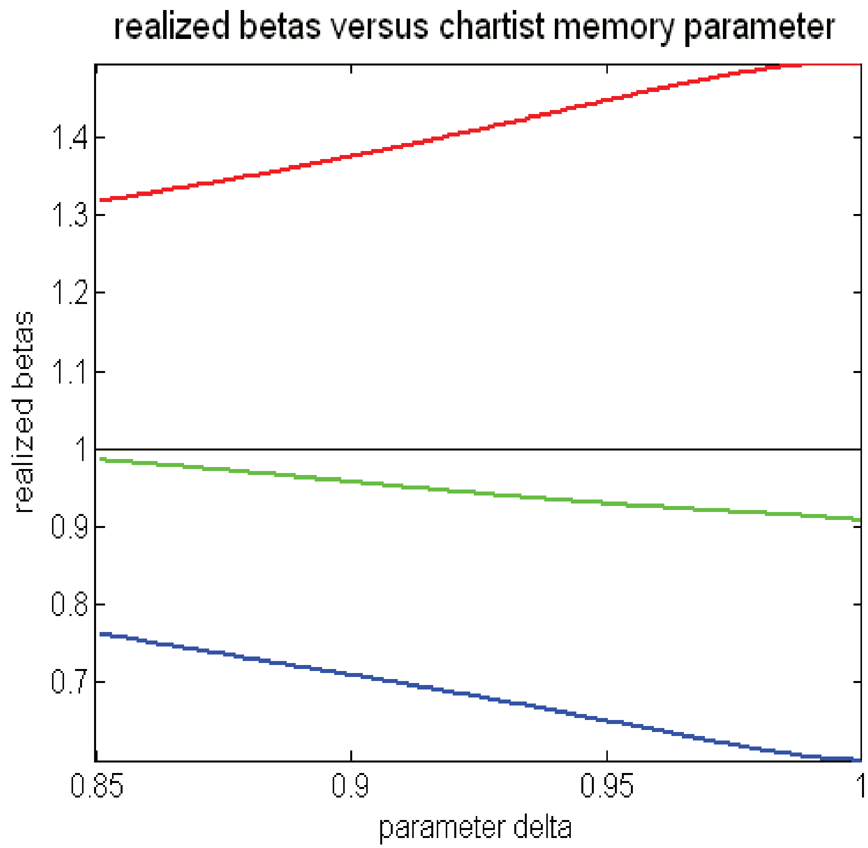


(d) Ex-ante betas (e) Rolling estimates of betas

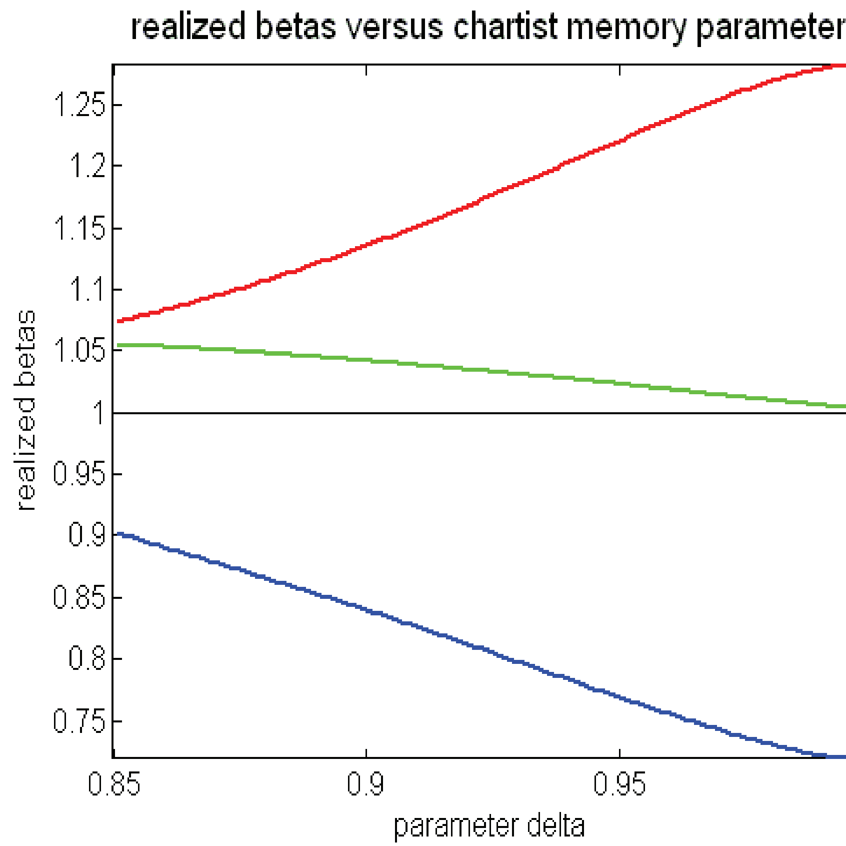
Figure 4.5: Illustration of the impact of a change in δ at $t = 1000$ on the market.

- **Summary**
  - What matters when beliefs are approximately homogeneous and constant over time is the ‘fundamental’ part of agents beliefs about the expected returns and their variance/covariance matrix. As a consequence, when the steady state equilibrium of the underlying deterministic system is stable, the estimated betas are consistent with ex-ante betas.
  - However, when the steady state of the underlying deterministic system is destabilised via a particular bifurcation scenario, or close to the stability boundary, stronger correlation patterns emerge from the noisy model, driven by time varying expectations and by the history-dependent portion of second-moment beliefs.
  - Ex-ante betas are directly related to certain behavioural parameters;
  - The time variation of estimated betas could be related, in principle, to changes in market sentiment, but can be significantly different from the ex-ante betas.

- **Dependence of realized betas on the parameters**
  - To offer a deeper insight into the effect, on the beta coefficients, of the model behavioral parameters,
    - δ—the extrapolation rate or 'memory' of the trend followers;
    - λ—sensitivity of risk beliefs to historical volatility/correlation;
    - α—the fundamentalist mean reversion parameter;
    - q—the strength of the noise traders.
  - **Two time horizons**:
    - Monthly: $K = 12, T = 480$;
    - Weekly: $K = 50, T = 1000$
  - **Parameters**: $\alpha = 0.3$, $n_f = 0.3$, $\delta = 0.9$, $\lambda = 0.5$, $q = 0$
  - **Estimation of ex-post betas**: OLS



Monthly

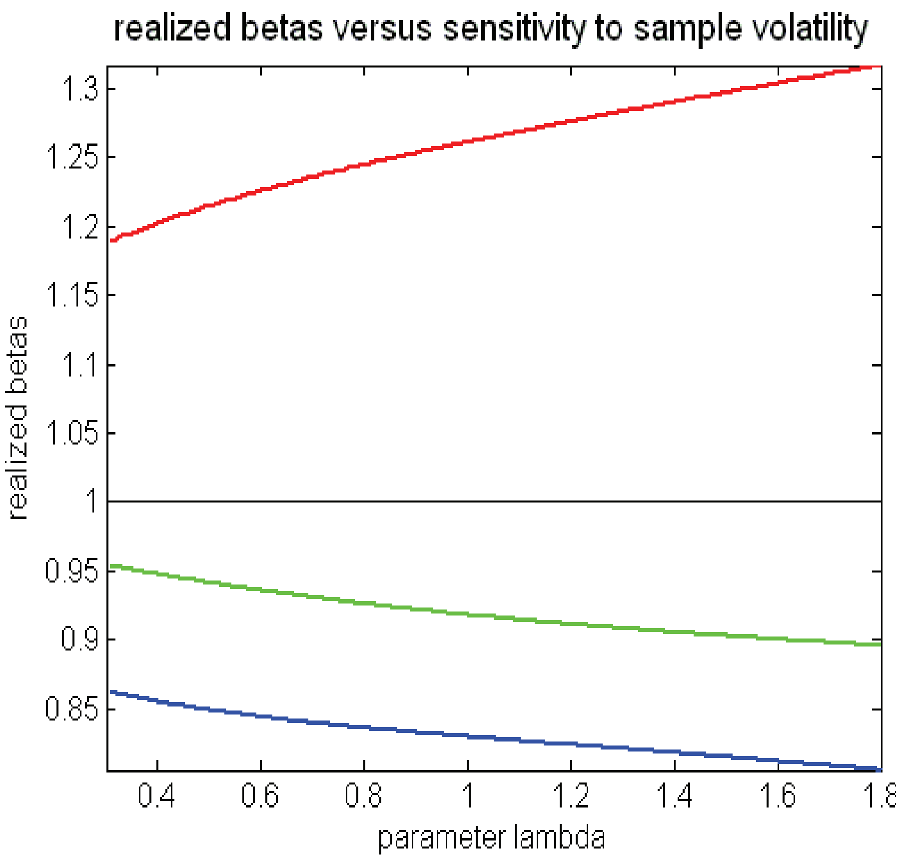


Weekly

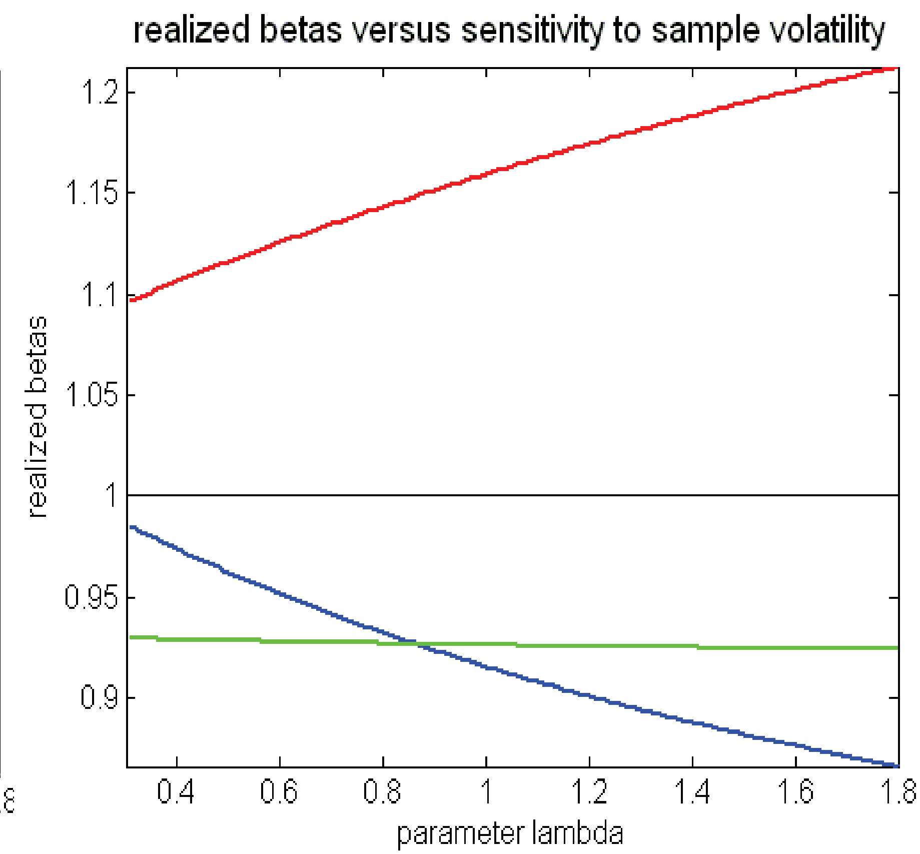
Figure 4.6: Dependence of ex-post β on δ.

Observation:

- The parameter $\boldsymbol{\delta} \in [\mathbf{0.85, 1}]$
- Systematic changes in betas when $\boldsymbol{\delta}$ decreases, in particular, at weekly basis
- A tendency on the beta coefficients to become less dispersed as $\boldsymbol{\delta}$ decreases.
- The dynamic behavior of each asset tends to become increasingly similar to the market.



Monthly



Weekly

Figure 4.7: Dependence of ex-post β on $\boldsymbol{\lambda}$—similar effect to $\boldsymbol{\delta}$.

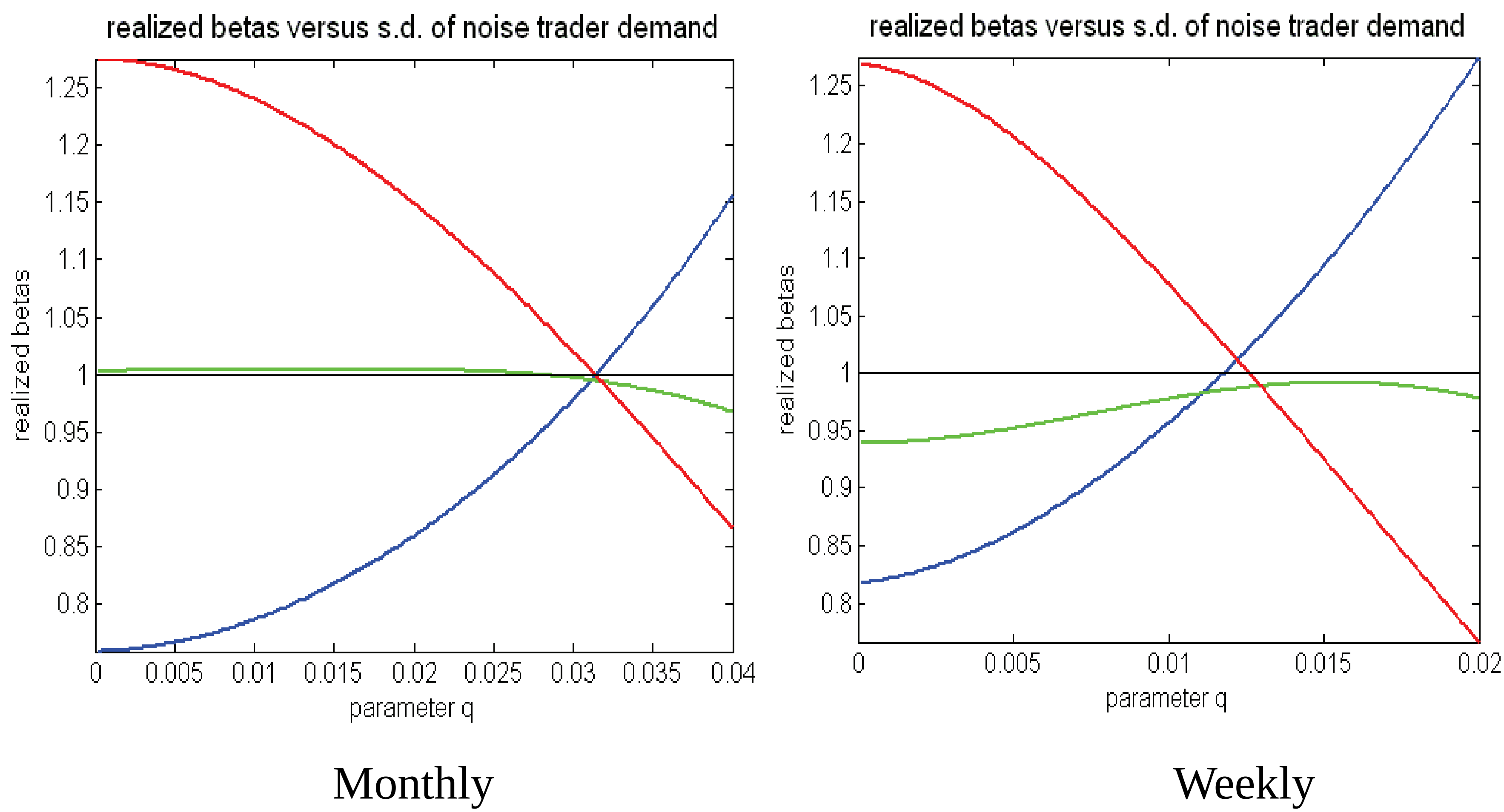
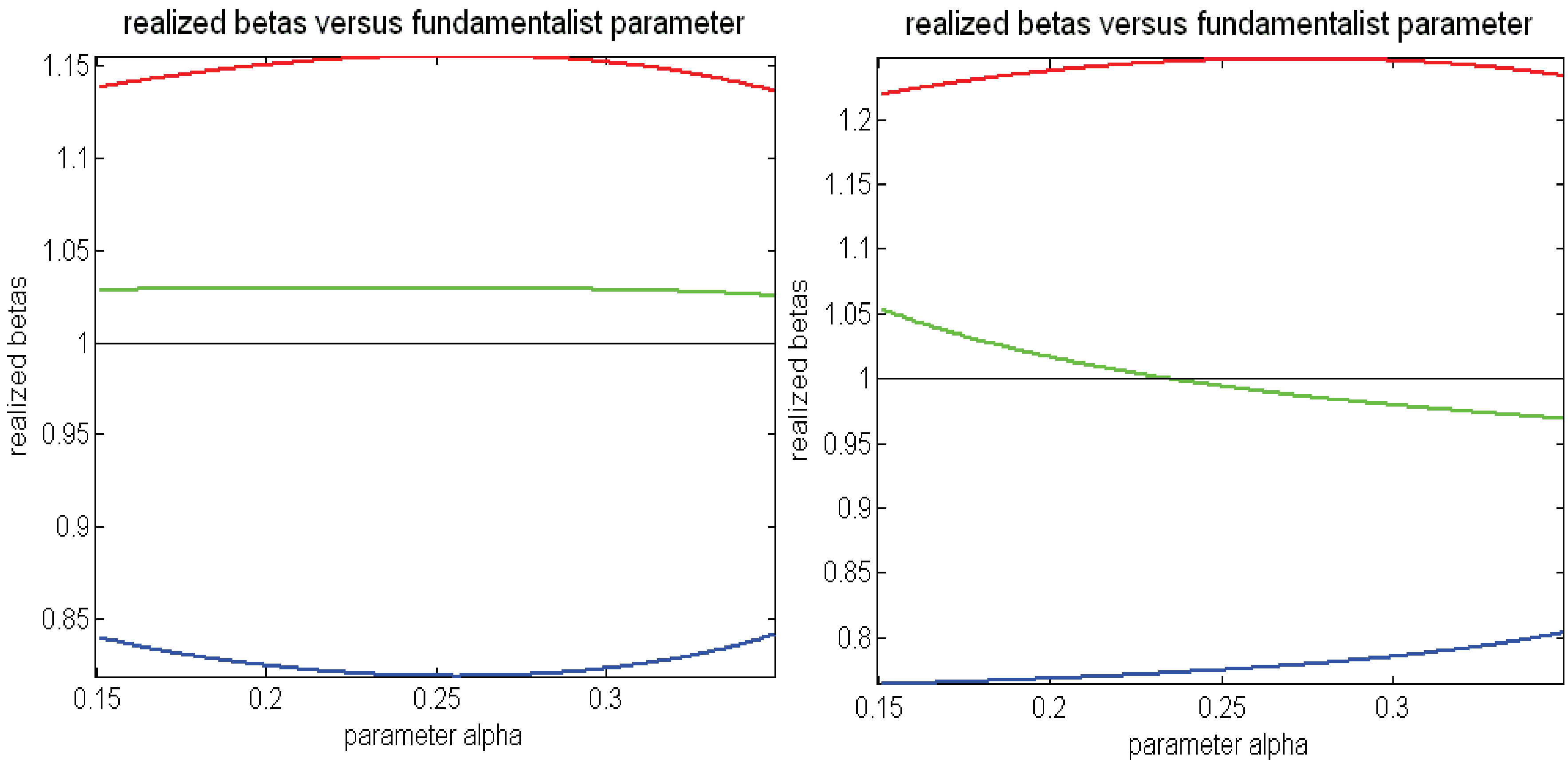


Figure 4.8: Dependence of ex-post β on q.

Observation:

- When the parameter q increases over the range $[0, 0.04]$ for monthly, or in the range $[0, 0.02]$ for weekly data, the beta coefficients become less dispersed initially;
- Large noise may produce large shifts of the betas and reverse their risk levels;



Monthly

Weekly

Figure 4.9: Dependence of ex-post β on α—not very significant.

5 Conclusion

- **Aim**: To model explicitly the stochastic behaviour of betas through agents' behaviour;
- **Approach**: boundedly rational dynamic equilibrium model of a financial market with heterogeneous agents within the mean-variance framework of repeated one-period optimisation;
- **Result**: Dynamic CAPM relation between the expected equilibrium returns and time-varying betas under heterogeneous beliefs;
- **Application**: A model with fundamentalists, trend followers and noise traders;

- **Findings**
  - Independently of the fundamentals, there is a systematic change in the market portfolio, asset prices and returns, and time varying betas when investors change their behaviour;
  - The stochastic nature of time-varying betas;
  - The variation of the estimated betas can be significantly different from that of ex-ante betas.
  - The rolling window estimates of betas may have no explanatory power and this may simply be due to the way we estimated the model rather than some shortcoming of the underlying equilibrium models.
- **Future work**
  - To examine the statistical properties of the asset returns;
  - To study the impact of adaptive behaviour when agents use the combined strategies with updating weights by some fitness measures.

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