Excess Covariance and Dynamic Instability in a Multi-Asset Model

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Outline

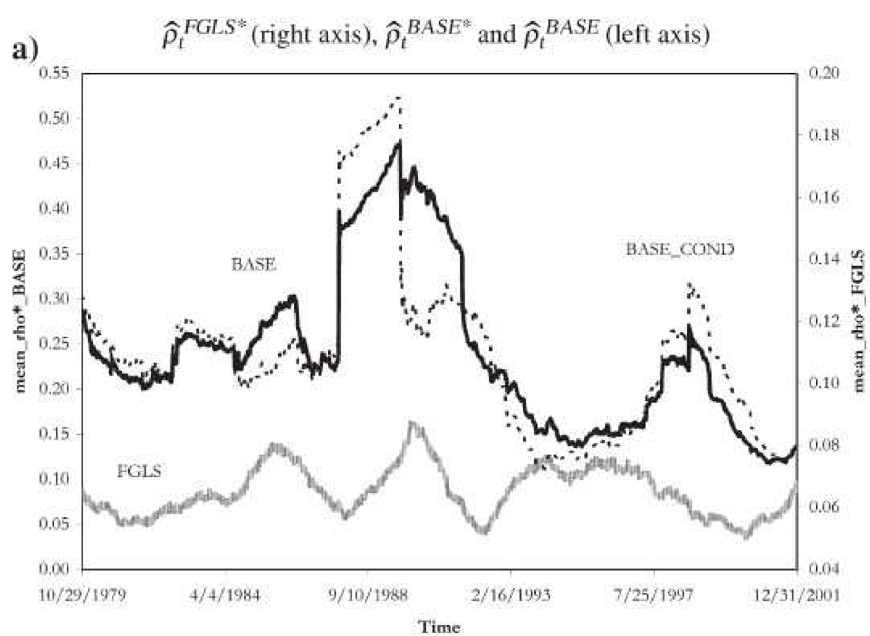
- Motivation
- Literature
  - excess covariance
  - previous modelling
- The model
  - a multi–asset Walrasian market
  - generic heterogeneous agents
  - stationary equilibria
  - the mean–variance CRRA agents case
  - evolutionary stability of strategies
  - evolutionary pressure towards instability

Excess Covariance (1/2)

It is difficult to measure it ('excess' compared to what?)

- Froot and Dabora (1999) and Brealey et al. (2008) analyze the same stock on different markets
- Kallberg and Pasquariello (2008) measure excess idiosyncratic covariance of 82 industries traded at NYSE from 1976 to 2001

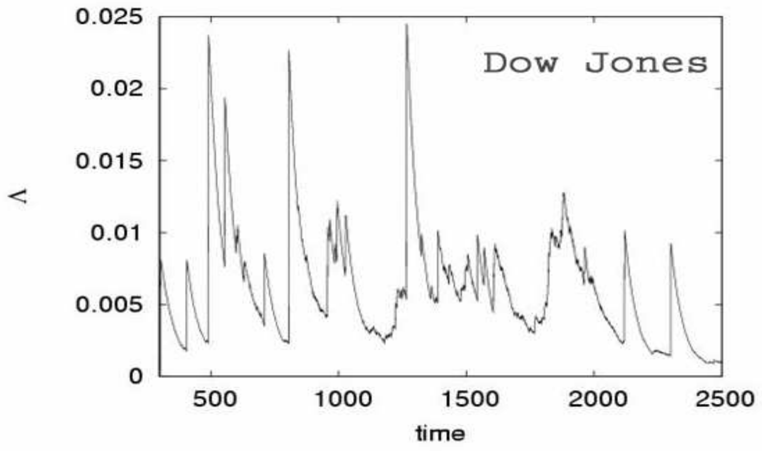
J. Kallberg, P. Pasquariello / Journal of Empirical Finance 15 (2008) 481–502



Excess Covariance (2/2)

“Econophysics” approach: statistical analysis of covariance matrices in real stock markets

- one dominant eigenvalue (Bouchaud & Potters, 2003)
- high fluctuation of this eigenvalue → price instability



Other models

- Why excess covariance?
  - Allen and Gale (2000): financial contagion
  - Kyle and Xiong (2001): wealth effects
  - Kodres and Pritsker (2002): rebalancing of risk–averse agents
  - Yuan (2005): financial constraints
  - Veldkamp (2006): cost of information and herding
  - Raffaelli and Marsili (2006): a stochastic behavioral model
- Our model is related to
  - Heterogeneous Agent Models (a survey is Hommes, 2006)
  - Capital Asset Pricing Model – CAPM (Sharpe, 1964)

Assets

- One riskless asset supplied at price 1 (numeraire) with fixed return r_f
- I assets with price p_t^i, $i \in I$, paying risky dividend d_t^i at time t
- Vectors notation: $\vec{p}_t = \left(p_t^1, \ldots, p_t^I\right)'$, $\vec{d}_t = \left(d_t^1, \ldots, d_t^I\right)'$
- $\vec{d}_t$ is the real "economy", $\vec{p}_t$ is the financial one
- there is no (explicit) consumption

Traders

N heterogeneous (types of) strategies

For strategy $n \in N$:

- $w_{t,n}$ wealth at time t
- $x^i_{t,n}$ wealth fraction invested in risky assets $n \in N$
- $1 - \vec{x}'_{t,n}\vec{1} = 1 - \sum_i x^i_{t,n}$ wealth fraction invested in riskless security
- CRRA: $x^i_{t,n}$ independent on wealth $w_{t,n}$

Strategies

- full information & CRRA
- $\rightarrow$ aggregate wealth for agents with same strategies

$$\vec{x}_{t,n} \equiv f_n\left(\{\vec{p}_\tau, \vec{d}_\tau\} : \tau < t\right) \quad ,$$

- $\hat{X}_t$ is the $I \times N$ matrix of portfolios
- Aggregate portfolio of the representative investor

$$\vec{x}_t \equiv \frac{\hat{X}_t \vec{w}_t}{\sum_n w_{t,n}}$$

Market mechanism

Time is discrete.

Walrasian price setting mechanism
(quantity normalized to 1)

$$p_t^i = \sum_{n \in N} w_{t,n}\, x_{t,n}^i$$

Full information, but “no trade argument” does not apply.

Inter–temporal budget constraint $\forall n \in N$

$$w_{t+1,n} = w_{t,n} \left((1 - \sum_i x_{t,n}^i)(1 + r_f) + \sum_i \frac{x_{t,n}^i}{p_t^i} \left(d_{t+1}^i + p_{t+1}^i\right) \right)$$

Notation

$\hat{X}_t$	$I \times N$	matrix of the investment shares at time t, $x^i_{t,n}$
$\vec{x}_{t,n}$	$I \times 1$	vector of the investment shares of strategy n at time t, $x^i_{t,n}$
$\vec{w}_t$	$N \times 1$	vector of the wealths at time t, $w_{t,n}$
$\vec{\varphi}_t$	$N \times 1$	vector of wealth shares $\varphi_{t,n} = \frac{w_{t,n}}{\sum_m w_{t,m}}$
$\vec{x}_t$	$I \times 1$	vector of the "representative" portfolio at time t, $\vec{x}_t = \hat{X}_t \varphi_t$
$\vec{p}_t$	$I \times 1$	vector of endogenous prices p^i_t
$\vec{r}_t$	$I \times 1$	vector of ex–dividend returns $r^i_t = \frac{p^i_{t+1} - p^i_t}{p^i_t} - r_f$
$\vec{d}_t$	$I \times 1$	vector of random dividends d^i_t
$\vec{\delta}_t$	$I \times 1$	vector of random dividend yields $\delta^i_t \equiv \frac{d^i_t}{p^i_{t-1}}$

Every variable at time t can be expressed from variables at time $t-1$ and $\vec{d}_t$.

The system is well defined under general assumption

"Ex–capital" gains: $z_{t-1,n} = \left((1+r_f)(1-\vec{x}_{t-1,n}\cdot\vec{1}) + \vec{x}_{t-1,n}\cdot\vec{\delta}_t\right)\, w_{t-1,n}$

Does not depend on the investment decisions, prices and wealths at time t.

Positions of different strategies: $\hat{Y}_t$: $\quad y^i_{t,n} \equiv \frac{x^i_{t,n}\, w_{t,n}}{p^i_t} = \frac{x^i_{t,n}\, w_{t,n}}{\sum_m x^i_{t,m}\, w_{t,m}}$

Then we have evolution of total wealth and of wealth shares $\varphi_{t,n}$:

$$\vec{w}_t = \vec{z}_{t-1} + \hat{Y}_{t-1}\vec{p}_t \ ,$$

$$\vec{\varphi}_t = \frac{\left(\mathbb{I}_n - \hat{Y}_{t-1}\hat{X}_t\right)^{-1} \vec{z}_{t-1}}{\vec{1}'\left(\mathbb{I}_n - \hat{Y}_{t-1}\hat{X}_t\right)^{-1} \vec{z}_{t-1}} = \frac{\left(\mathbb{I}_n - \hat{Y}_{t-1}\hat{X}_t\right)^{-1} \vec{\varphi}_{t-1}}{\vec{1}'\left(\mathbb{I}_n - \hat{Y}_{t-1}\hat{X}_t\right)^{-1} \vec{\varphi}_{t-1}}$$

Dividend yields

- Dividend yields: $\delta_t^i \equiv \frac{d_t^i}{p_{t-1}^i}$
- $\vec{\delta}_t \simeq$ multi–normal distribution with mean $\vec{\delta}$ and variance covariance matrix $\hat{D}$
  - this ties real and financial economy
  - established assumption: Chiarella and He (2001) and Anufriev et alii (2006)
  - possible story:
    - a time step takes months
    - firms obtain credit from banks according to financial capitalization
    - the expected return of scale is constant
  - the assumption is strong
    - helps computations
    - seems unnecessary:
      (finance $\longrightarrow$ real economy) $\not\Rightarrow$ finance is more correlated than real economy

Stationary Equilibria

Complicated non-linear dynamics are possible. We focus on special solution.

Stationary equilibria:

1. Agents don't modify investment decision $x^i_{t,n}$ (because expectations are constant)
2. Agents wealth shares $\phi_{t,n} = w_{t,n}/w_t$ are (almost) constant
3. Trade happens because of (small) stochastic wealth fluctuations

Formally:

$$\vec{x}_{t,n} \rightarrow \vec{x}_n \ , \qquad \phi_{t,a} \rightarrow \phi_a \ , \qquad \vec{x}_t \rightarrow \vec{x} \ .$$

It is a self–consistent equilibrium of the model under:

- full rationality
- every agent has a negligible impact
- wealth does not influence choices (CRRA)
- coexistence of strategies is possible

Excess covariance

We are interested in first and second moments, mean

$$\vec{c}_t = E_t\left[\vec{r}_{t+1} + \vec{\delta}_{t+1}\right]$$

and variance–covariance matrix

$$\hat{C}_t = Cov\left(\vec{r}_{t+1} + \vec{\delta}_{t+1}, \vec{r}_{t+1} + \vec{\delta}_{t+1}\right) \quad .$$

In equilibrium (all variables are constant)

$$\vec{c} = \frac{\vec{x}'\vec{\delta}}{1 - \vec{x}'\vec{1}}\vec{1} + \vec{\delta}$$

and

$$\hat{C} = \underbrace{\frac{\vec{x}'\hat{D}\vec{x}}{(1 - \vec{x}'\vec{1})^2}\vec{1}\vec{1}' + \frac{\vec{1}\vec{x}'\hat{D} + \hat{D}\vec{x}\vec{1}'}{1 - \vec{x}'\vec{1}}}_{\text{Excess covariance}} + \hat{D}$$

Relation to Capital Asset Pricing Model (CAPM)

The ex-dividend return r_t^x of the portfolio $\vec{x}$ is

$$r_t^x = \vec{x}' \vec{r_t} + r_f = r_f + \vec{x}' \vec{1} \frac{\vec{\delta_t}' \vec{x}}{1 - \vec{x}' \vec{1}}$$

the excess return (equity premium) of each asset i is

$$r_t^i = \frac{\vec{\delta_t}' \vec{x}}{1 - \vec{x}' \vec{1}} = \frac{1}{\vec{x}' \vec{1}} (r_t^x - r_f)$$

it is similar to CAPM: $E[r_t^i - r_f] = \beta^i E[r_t^x - r_f]$

it holds for all returns, not only for expected values (stronger than CAPM)

In CAPM:

$$\beta^i = \frac{\mathrm{Cov}(r_t^i, r_t^x)}{\mathrm{Var}(r_t^x)}$$

For us $\beta_i = \frac{1}{\vec{x}' \vec{1}}$ (for every asset)

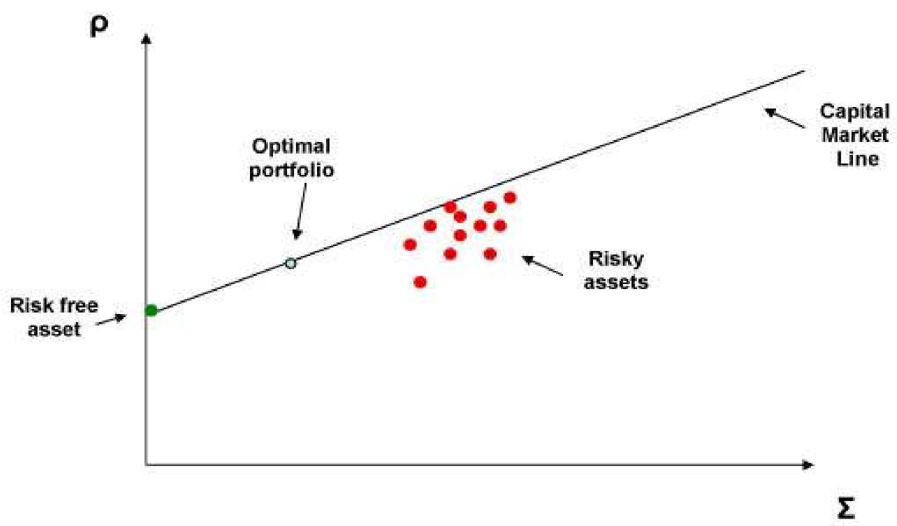
Main example (1/3): Mean–variance CRRA strategies

The agents are Constant Relative Risk Averse (CRRA) and choose a portfolio taking into account expected first and second moments

They compute mean vector $\vec{c}_t = E_t\left[\vec{r}_{t+1} + \vec{\delta}_{t+1}\right]$

and variance–covariance matrix $\hat{C}_t = \mathrm{Cov}_t\left(\vec{r}_{t+1} + \vec{\delta}_{t+1}, \vec{r}_{t+1} + \vec{\delta}_{t+1}\right)$

They choose the portfolio
$\vec{x} = \frac{1}{\gamma}\hat{C}_t^{-1}\vec{c}_t$, where γ is a positive parameter of risk aversion.



Equilibrium
(no more t–dependence):

$$\vec{x} = \frac{1}{\gamma + \vec{1}'\hat{D}^{-1}\vec{\delta}}\hat{D}^{-1}\vec{\delta}$$

No short selling:
$x_i > 0$, $\quad 1 - \vec{x}'\vec{1} > 0$

Main example (2/3): Analytical results in equilibrium

Action of investors $\Rightarrow$ three main results:

1. The dynamics of log-prices of different assets do not behave as random walks, rather they are bounded one to the other, the difference $\log \frac{p_i}{p_j}$ stays finite and it does not diverge as $\sqrt{t}$ as for correlated random walks.

2. The market develops a volatility of returns which is much higher than the volatility of dividend yields (by a factor N)

3. The market develops a market mode, i.e. an eigenvalue of the correlation matrix which is much larger than the others (Of the order of N)

Main example (3/3): Dynamic under a learning process

Investors update first and second moments with a parameter μ

$$
\begin{array}{lrcl}
i) & \vec{x}_t & = & \frac{1}{\gamma}\hat{C}_{t-1}^{-1}\vec{c}_{t-1} \\
ii) & r_t^i & = & \frac{\vec{x}'_{t-1}\vec{\delta}_t x_t^i + (1+r_f)(1-\vec{x}'_{t-1}\vec{1})x_t^i}{(1-\vec{x}'_t\vec{1})x_{t-1}^i} - 1 - r_f \\
iii) & \vec{c}_t & = & (1-\mu)\vec{c}_{t-1} + \mu\left[\vec{r}_{t-1} + \vec{\delta}_{t-1}\right] \\
iv) & \hat{C}_t & = & (1-\mu)\hat{C}_{t-1} + \mu \mathrm{Cov}(\vec{r}_{t-1} + \vec{\delta}_{t-1}, \vec{r}_{t-1} + \vec{\delta}_{t-1})
\end{array}
$$

Deterministic skeleton of the system is stable

but fluctuations of $\vec{\delta}_t$ have tremendous effects as $\vec{x}'\vec{1} \to 1$ (from (ii))

Back to general Stationary Equilibria

Consider the $\vec{x}$ vector of (constant) aggregate investments

$\vec{c}$ and $\vec{\delta} = E(\vec{\delta})$ are in the same plane generated by $\vec{1}$ and $\vec{\delta}$ (call $\vec{2}$ the orthogonal part of $\vec{\delta}$ with respect to $\vec{1}$)

It means that any other dimension brings unrewarded risk

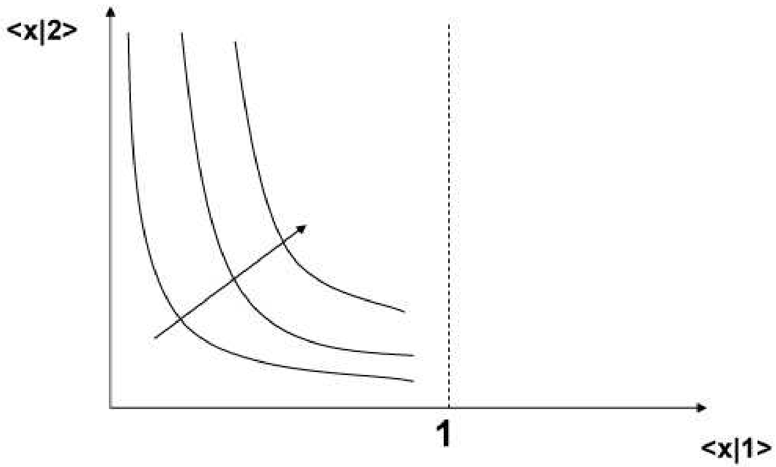
Any rational risk–averse agent n, if she has learnt $\vec{\delta}$ would not use a strategy $\vec{x}_n$ outside this plane

Invading strategies

A population is in equilibrium with $\vec{x}$ in the $\vec{1}$–$\vec{2}$ plane

Suppose that a new population of agents, with negligible wealth, enters with a new strategy $\vec{z}$ (also in the $\vec{1}$–$\vec{2}$ plane)

If $E\left[\log\left(\frac{w_{t+1,z}}{w_{t,z}} \Big/ \frac{w_{t+1}}{w_t}\right)\right] > 0$, then $\vec{z}$ will overthrown $\vec{x}$



We have iso-curves on the $\vec{1}$–$\vec{2}$ projections:

no circular situations.

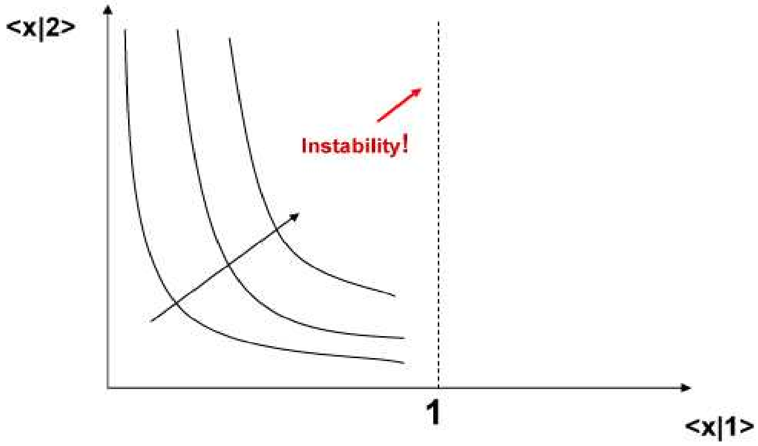
Evolutionary stability

A part of the population changes slightly from $\vec{x}$ to $\vec{x}+\vec{\epsilon}$ (still in the $\vec{1}$–$\vec{2}$ plane)

We can compute the gradient of a variation that brings higher expected wealth growth

$$\frac{\partial E\left[\log\left(\frac{w_{t+1,\epsilon}}{w_{t,\epsilon}}\Big/\frac{w_{t+1}}{w_t}\right)\right]}{\partial\vec{\epsilon}} = E\left[\frac{\vec{\delta}_{t+1}+r_{t+1}\vec{1}}{1+r_f+r_{t+1}+\vec{\epsilon}'\vec{\delta}_{t+1}+r_{t+1}\vec{\epsilon}'\vec{1}}\right]$$

$$\simeq E\left[\frac{\vec{\delta}_{t+1}+r_{t+1}\vec{1}}{1+r_f+r_{t+1}}\right]$$



If expected yields $\vec{\delta}$ are (as expected) all positive:

Evolutionary pressure brings towards instability

Main results

We have a multi–asset model

- it allows for any set of heterogeneous strategies (aggregation)
- the only source of noise is from dividend yields
- under Stationary Equilibria it explicitly shows excess covariance
- it has relation to CAPM
- it is unstable as the risk–free investment $\rightarrow 0$
- evolutionary pressure pushes towards instability

Under mean–variance strategies

- it explicitly shows a dominant eigenvalue
- it has a unique stable equilibrium
- there is no short–selling in equilibrium