Alternative Benamou-Brenier Transport Problems, with or without Convexity

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It is well known that the classical Monge-Kantorovitch problem for the cost $|x-y|^p$ admits a formulation via the continuity equation, minimizing the cost $\int\int |q|^p \rho^{1-p} dxdt$ among solutions of $d\rho/dt + \div q = 0$. This formulation ha first been proposed by Benamou and Brenier for numerical purposes, since it lead to the minimization of a convex functional under linear constraints.

Recently, some variants have appeared in the same framework : Dolbeault, Nazaret and Savar have proposed to investigate some new distances on measures, directly defined through the minimization of $\int\int |q|^p m(\rho)^{1-p} dxdt$, for a nonlinear "mobility" function m. Under concavity assumption on m this, too, leads to a convex minimization problem, and the existence of a geodesic $(\rho_t)_t$ is straightforward. For absolutely continuous measures, the geodesic is characterized through a system of PDEs, which recalls the continuity and Hamilton-Jacobi equations of a Mean-Field Game (as in the works by Lasry and Lions) and is the subject of a current research with B. Nazaret.

On the other hand, I will also present a recent work in collaboration with L Brasco and G. Buttazzo, where a different model, typically presented in terms of atomic measures instead of absolutely continuous ones, is expressed in the same way. In this case, the existence is less trivial due to the lack of convexity, and no serious PDE may be written for the density of the optimal curve of measures, since those measures turn out to be non-absolutely continuous. Yet, the problem is equivalent to a wide family of models, introduced by Xia, Bernot, Caselles, Morel and nowadays known as "branched transport models", and the solutions represent optimal networks.