The h-principle, the Nash-Kuiper theorem and the Euler equations

The h-principle is a concept introduced by Gromov. Intuitively a system of partial differential equations (or, more generally, of partial differential inequalities) is said to satisfy the h-principle if exactly the opposite of "rigidity" can be proven, i.e. if there is a wide abundance of solutions. This is often the case for underdetermined systems, but one observes occasionally the striking phenomenon that some (geometrically relevant) overdetermined systems of partial differential equations also obey the h-principle. The primary example is the highly counterintuitive Nash-Kuiper C^1 isometric embedding theorem: as a corollary of this result we conclude, for instance, that for every $\varepsilon > 0$ there is a C^1 isometric embedding of the standard 2-dimensional sphere in the 3-dimensional euclidean ball of radius ε.

We will start this course with a first general introduction to the ideas behind the h-principle following the book of Eliashberg and Mischachev (in particular we will cover the famous Smale's sphere eversion theorem). We will then prove a version of the Nash-Kuiper theorem. Finally, we will give ideas on how similar (but much more complicated) techniques can be used in the case of the incompressible Euler equations, giving the first example of continuous solutions which dissipate the kinetic energy. These solutions were conjectured to exist in 1949 by Lars Onsager, in connection with Kolmogorov's theory of turbulence and his "energy cascade". Indeed, the full conjecture of Onsager states the existence of dissipative Hölder solutions for every Hölder exponent strictly smaller than $\frac{1}{3}$ and the absence of dissipation ifor any Hölder solution with exponent strictly larger than $\frac{1}{3}$. The latter part of the conjecture has been proved by Eyink and Constantin-E-Titi in the nineties, whereas the first is still open.