THE INFINITY LAPLACIAN AND TUG-OF-WAR GAMES (GAMES THAT PDE PEOPLE LIKE TO PLAY).

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ABSTRACT. In these notes we review some recent results concerning the inifnity Laplacian, Tug-of-War games and their relation to some well known PDEs. In particular, we show that solutions to certain PDEs can be obtained as limits of values of Tug-of-War games when the parameter that controls the length of the possible movements goes to zero. Since the equations under study are nonlinear and not in divergence form we will make extensive use of the concept of viscosity solutions.

1. INTRODUCTION

The main goal of these notes is to introduce and study the infinity Laplacian that is the second order elliptic operator given by

$$\Delta_\infty u(x) := \big(D^2 u \,\nabla u\big) \cdot \nabla u(x) = \sum_{i,j=1}^N \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_i x_j}(x).$$

There are two excellent surveys concerning the infinity Laplacian operator, see [5] and [17].

Here we will deal with functions defined in a bounded domain in $\mathbb{R}^N$ equipped with the Euclidean norm. Some of the properties and results presented here can be extended to a general norm but we prefer to avoid this sort of generality and instead refer to [5].

Existence of viscosity solutions to the Dirichlet problem can be done with a simple approximation by $p-$harmonic functions and we will include the details here (see Section 2). A different proof using Perron's method can be also obtained easily (we refer to [5] for details).

Uniqueness is more difficult, in the sense that it took a long time for Jensen [30] to give the first, quite tricky, proof and then another proof, still tricky, but more in line with standard viscosity solution theory, was given by Barles and Busca, see [7]. A third proof was given in [17].

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Some history. It all began in 1967 with Gunnar Aronsson's paper [2]. Aronsson looked for optimal Lipschitz extensions of a given datum. Recall that a function $u : \Omega \mapsto \mathbb{R}$ is Lipschitz if

$$\mathrm{Lip}(u, \Omega) = \inf\{L \,:\, |u(x) - u(y)| \le L|x - y|, \, \forall x, y \in \Omega\}$$

is finite. Aronsson observed that the Lipschitz constant of a function in a domain coincides with the L^∞-norm of the gradient if the domain is convex, while this is not generally the case if the domain is not convex. The problem of minimizing the Lipschitz constant subject to a Dirichlet condition was known to have a largest and a smallest solution, given by explicit formulas, from the works of McShane and Whitney [40], [55]. In fact,

$$u^*(x) = \min_{y \in \partial\Omega} F(y) + \mathrm{Lip}(F, \partial\Omega)|x - y|$$

and

$$u_*(x) = \max_{y \in \partial\Omega} F(y) - \mathrm{Lip}(F, \partial\Omega)|x - y|.$$

Aronsson derived, among other things, interesting information about the set on which these two functions coincide and the derivatives of any solution on this "contact set". In particular, he established that minimizers for the Lipschitz constant are unique iff there is a function $u \in C^1(\Omega) \cap C(\overline{\Omega})$ which satisfies

$$|Du| \equiv \mathrm{Lip}(F, \partial\Omega) \text{ in } \Omega, \qquad u = F \text{ on } \partial\Omega,$$

which is then the one and only solution. This is a very special circumstance.

The following question naturally arose: is it possible to find a canonical Lipschitz constant extension of F into Ω that would enjoy comparison and stability properties? Furthermore, could this special extension be unique once the boundary data is fixed? The point of view was that the problem was an extension problem. Aronsson's clever proposal in this regard was to introduce the class of absolutely minimizing functions for the Lipschitz constant. During these research Aronsson was led to the now famous pde $\Delta_\infty u = 0$. He showed existence of a C^2 solution under special circumstances. but, the question of uniqueness of the function whose existence Aronsson proved would be unsettled for 26 years (until the work by Jensen [30]).

The best known explicit irregular absolutely minimizing function - outside of the relatively regular solutions of eikonal equations - was exhibited again by Aronsson, who showed in 1984, see [3], that

$$u(x, y) = x^{4/3} - y^{4/3}$$

is absolutely minimizing in $\mathbb{R}^2$ for the Lipschitz constant and for the L^∞-norm of the gradient.

A major advance was the introduction by Jensen of viscosity solutions to the equation. The theory of viscosity solutions of much more general equations was born in first order case the 1980's. The developers of these results in the second order case were Jensen, Ishii, Caffarelli, Crandall, Evans, Lions, Souganidis, etc. The main features of the theory are summarized in [16], which contains a detailed history.

Jensen proves uniqueness of viscosity solutions and the validity of a comparison principle using approximations to the equation (and variants of it) by $p-$Laplacian type problems as $p \to \infty$. Jensen's work generated considerable interest in the theory. Among other contributions was a new uniqueness proof by Barles and Busca in [7]. Moreover, Ishii introduced the Perron method in the theory of viscosity solutions, which is a powerful tool to prove existence of solutions.

After existence and uniqueness, one wants to know about regularity. One of the key ideas to prove regularity results was the fact that solutions have the property of comparison with cones. One of the challenging open problems in the subject is concerned with regularity: are ∞-harmonic functions C^1? (note that the explicit solution $u(x, y) = x^{4/3} - y^{4/3}$ prevents for general C^2 regularity results. Savin, see [53] proved that they are C^1 in the case $N = 2$, see also [22]. Differentiability in any dimension was recently proved by Evans and Smart in [21].

The second main goal of these notes is to introduce the reader (expert or not) to some important techniques and results in the theory of second order elliptic PDEs and their connections with game theory.

The fundamental works of Doob, Hunt, Kakutani, Kolmogorov and many others have shown the profound and powerful connection between the classical linear potential theory and the corresponding probability theory. The idea behind the classical interplay is that harmonic functions and martingales share a common origin in mean value properties. This approach turns out to be useful in the nonlinear theory as well.

First, our aim is to explain through elementary examples a way in which elliptic PDEs arise in Probability. For instance, first we show how simple is the relation between probabilistic issues on random walks and the Laplace operator and also other elliptic operators, as well as the heat equation.

Next, we will enter in what is the core of these notes, the approximation by means of values of games of solutions to nonlinear problems like $p-$harmonic functions, that is, solutions to the PDE, $\mathrm{div}(|\nabla u|^{p-2}\nabla u) = 0$ (including the nowadays popular case $p = \infty$).

We will assume that the reader is familiar with basic tools from probability theory (like conditional expectations) and with the (not so basic) concept of viscosity solutions for second order elliptic and parabolic PDEs (we refer to the book [11] for this last issue).

The Bibliography of these notes does not escape the usual rule of being incomplete. In general, we have listed those papers which are closer to the topics discussed here (some of them are not explicitly cited in the text). But, even for those papers, the list is far from being exhaustive and we apologize for omissions.

2. The infinity Laplacian

As we mentioned in the introduction, the infinity Laplacian is given by

$$\Delta_\infty u(x) := \big(D^2 u \,\nabla u\big)\cdot \nabla u(x) = \sum_{i,j=1}^{N} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i x_j}(x). \tag{1}$$

In these notes we present an outline of the theory of this operator that can be seen as the archetypal L^∞ variational problem in the calculus of variations.

2.1. **Passing to the limit as $p \to \infty$ in the equation $\Delta_p u = 0$. Viscosity solutions.** In this section we present a way of obtaining existence of viscosity solutions to

$$\begin{cases} \Delta_\infty u(x) = 0, & x \in \Omega, \\ u(x) = F(x), & x \in \partial\Omega, \end{cases} \tag{2}$$

taking the limit as $p \to \infty$ along subsequences of solutions u_p to

$$\begin{cases} \Delta_p u_p(x) = 0, & x \in \Omega, \\ u(x) = F(x), & x \in \partial\Omega. \end{cases} \tag{3}$$

For this problem, let us state the definitions of a weak and a viscosity solution.

Definition 2.1. *A function $u\colon \Omega \to \mathbb{R}$ is a* weak subsolution *of* (3) *in Ω if $u \in W^{1,p}(\Omega)$, verifies*

$$-\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\varphi = 0, \tag{4}$$

for every $\varphi \in C_0^\infty(\Omega)$ and

$$u = F \qquad \text{on } \partial\Omega$$

in the sense of traces.

Now, concerning viscosity solutions we have the following definition. In our case, we have to consider the following expression

$$F(x,u,\xi,S) = |\xi|^{p-2}\text{trace}(S) + (p-2)|\xi|^{p-4}\langle S\xi,\xi\rangle.$$

Definition 2.2. *An upper semicontinuous function $u\colon \Omega \to \mathbb{R}$ is a* viscosity subsolution *of* (3) *in Ω if, whenever $\hat{x} \in \Omega$ and $\varphi \in C^2(\Omega)$ are such that $u - \varphi$ has a strict local maximum at $\hat{x}$, then*

$$F(\hat{x},\varphi(\hat{x}),\nabla\varphi(\hat{x}),D^2\varphi(\hat{x})) = \Delta_p\varphi(\hat{x}) \geq 0. \tag{5}$$

A lower semicontinuous function $v\colon \Omega \to \mathbb{R}$ is a viscosity supersolution *of* (3) *in Ω if $-v$ is a viscosity subsolution, that is, whenever $\hat{x} \in \Omega$ and $\varphi \in C^2(\Omega)$ are such that $v - \varphi$ has a strict local minimum at $\hat{x}$, then*

$$F(\hat{x},\varphi(\hat{x}),\nabla\varphi(\hat{x}),D^2\varphi(\hat{x})) = \Delta_p\varphi(\hat{x}) \leq 0. \tag{6}$$

Finally, a continuous function $h\colon \Omega \to \mathbb{R}$ *is a* viscosity solution *of* (3) *in* Ω *if it is both a viscosity subsolution and a viscosity supersolution.*

Lemma 2.3. *There exists a unique weak solution to* (3) *and it is characterized as being a minimizer for the functional*

$$F_p(u) = \int_\Omega \frac{|\nabla u|^p}{p}$$

in the set $\{u \in W^{1,p}(\Omega) \ : \ u = F \ \text{on } \partial\Omega\}$.

Proof. The functional F_p is coercive and weakly semicontinuous, hence the mimimum is attained. It is easy to check that this mimimum is a weak solution to (3) in the sense of Definition 2.1. Uniqueness comes from the strict convexity of the functional. □

Proposition 2.4. *A continuous weak solution of* (3) *is a viscosity solution.*

Proof. Let $x_0 \in \Omega$ and a test function ϕ such that $u(\hat{x}) = \phi(\hat{x})$ and $u - \phi$ has a strict minimum at $\hat{x}$. We want to show that

$$F(\hat{x}, \phi(\hat{x}), \nabla\phi(\hat{x}), D^2\phi(\hat{x})) \leq 0,$$

that is,

$$(p-2)|D\phi|^{p-4}\Delta_\infty\phi(\hat{x}) + |D\phi|^{p-2}\Delta\phi(\hat{x}) \leq 0.$$

Assume that this is not the case, then there exists a radius $r > 0$ such that

$$(p-2)|D\phi|^{p-4}\Delta_\infty\phi(x) + |D\phi|^{p-2}\Delta\phi(x) > 0,$$

for every $x \in B(\hat{x}, r)$. Set $m = \inf_{|x-\hat{x}|=r}(u-\phi)(x)$ and let $\psi(x) = \phi(x) + m/2$. This function ψ verifies $\psi(\hat{x}) > u(\hat{x})$ and

$$\operatorname{div}(|D\psi|^{p-2}D\psi) > 0.$$

Multiplying by $(\psi - u)^+$ extended by zero outside $B(\hat{x}, r)$ we get

$$-\int_{\{\psi > u\}} |D\psi|^{p-2}D\psi D(\psi - u) > 0.$$

Taking $(\psi - u)^+$ as test function in the weak form of (3) we get

$$-\int_{\{\psi > u\}} |Du|^{p-2}Du D(\psi - u) = 0.$$

Hence,

$$C(N,p)\int_{\{\psi>u\}} |D\psi - Du|^p \quad \leq \int_{\{\psi>u\}} \langle |D\psi|^{p-2}D\psi - |Du|^{p-2}Du, D(\psi - u)\rangle < 0,$$

a contradiction.

This proves that u is a viscosity supersolution. The proof of the fact that u is a viscosity subsolution runs as above, we omit the details. □

Now we prove that there is a subsequence of u_p that converges uniformly.

Lemma 2.5. *There exists a subsequence of u_p and a function $u_\infty \in W^{1,\infty}(\Omega)$ such that*

$$\lim_{p_j \to \infty} u_{p_j}(x) = u_\infty(x)$$

uniformly in $\overline{\Omega}$.

Proof. Using that u_p is a minimizer of the associated energy functional we obtain, for any Lipschitz extension v of F,

$$\int_\Omega |Du_p|^p \leq \int_\Omega |Dv|^p \leq (\mathrm{Lip}(v,\Omega))^p |\Omega|. \tag{7}$$

Hence, we obtain that

$$\left(\int_\Omega |Du_p|^p \right)^{1/p} \leq \mathrm{Lip}(v,\Omega)|\Omega|^{1/p}.$$

Next, fix m, and take $p > m$. We have,

$$\left(\int_\Omega |Du_p|^m \right)^{1/m} \leq |\Omega|^{\frac{1}{m}-\frac{1}{p}} \left(\int_\Omega |Du_p|^p \right)^{1/p} \leq |\Omega|^{\frac{1}{m}-\frac{1}{p}} \mathrm{Lip}(v,\Omega)|\Omega|^{1/p},$$

where $|\Omega|^{\frac{1}{m}-\frac{1}{p}} \to |\Omega|^{\frac{1}{m}}$ as $p \to \infty$. Hence, there exists a weak limit (and hence uniform since we can assume that $m > N$) in $W^{1,m}(\Omega)$ that we will denote by u_∞. This weak limit has to verify

$$\left(\int_\Omega |Du_\infty|^m \right)^{1/m} \leq |\Omega|^{\frac{1}{m}} \mathrm{Lip}(v,\Omega).$$

As the above inequality holds for every m, we get that $u_\infty \in W^{1,\infty}(\Omega)$ and moreover, $\|Du_\infty\|_{L^\infty(\Omega)} \leq \mathrm{Lip}(v,\Omega)$. □

Theorem 2.6. *A uniform limit u_∞ of u_p as $p \to \infty$ is a viscosity solution to* (2).

Proof. From the uniform convergence it is clear that u_∞ is continuous and verifies $u_\infty = F$ on $\partial\Omega$.

Next, to look for the equation that u_∞ satisfies in the viscosity sense, assume that $u_\infty - \phi$ has a strict minimum at $x_0 \in \Omega$. We have to check that

$$\Delta_\infty \phi(\hat{x}) \leq 0. \tag{8}$$

By the uniform convergence of u_{p_i} to u_∞ there are points x_{p_i} such that $u_{p_i} - \phi$ has a minimum at x_{p_i} with $x_{p_i} \to \hat{x}$ as $p_i \to \infty$. At those points we have

$$(p_i - 2)|D\phi|^{p_i-4}\Delta_\infty \phi(x_{p_i}) + |D\phi|^{p_i-2}\Delta\phi(x_{p_i}) \leq 0.$$

If $D\varphi(\hat{x}) = 0$ then (8) is verified, hence we may assume that $D\varphi(\hat{x}) \neq 0$, and hence $D\varphi(x_{p_i}) \neq 0$ for every p_i large enough.

Therefore, we get

$$\Delta_\infty \phi(x_{p_i}) \leq \frac{1}{p_i - 2}|D\phi|^2 \Delta\phi(x_{p_i}). \tag{9}$$

Then passing to the limit in (9) we obtain

$$\Delta_\infty \phi(\hat{x}) \leq 0.$$

That is, u_∞ is a viscosity supersolution of (2).

The fact that it is a viscosity subsolution of (2) is analogous, using a test function ψ such that $u_\infty - \psi$ has a strict maximum at x_0. □

2.2. **L^∞ minimization problems in the calculus of variations.** Let us consider the functionals

$$G_\infty(u) = \|Du\|_{L^\infty(\Omega)}$$

and

$$L(u) = \mathrm{Lip}(u, \Omega)$$

where $\mathrm{Lip}(u, \Omega)$ stands for the Lipschitz constant of u in Ω, that is,

$$\mathrm{Lip}(u, \Omega) = \sup_{x,y\in\Omega} \frac{|u(x) - u(y)|}{|x - y|}.$$

Note that we can also write

$$\mathrm{Lip}(u, \Omega) = \inf\{L \, : \, |u(x) - u(y)| \leq L|x - y|, \, \forall x, y \in \Omega\}.$$

Also note that one has

$$G_\infty(u) = \mathrm{Lip}(u, \Omega)$$

if Ω is convex, but equality does not hold in general.

Our goal will be to minimize these functionals.

First, to give an idea that this task is not easy in general, let us present an example of nonuniqueness of the minimum.

Let us consider the optimal Lpschitz extension problem, that is, given F defined on $\partial\Omega$ find a solution to

$$\text{find } u \text{ that minimizes } \mathrm{Lip}(u, \overline{\Omega}) \text{ among functions such that } u = F \text{ on } \partial\Omega.$$

Assume that F is Lipschitz (otherwise this problem does not have a minimizer). Then we have

$$\mathrm{Lip}(u, \overline{\Omega}) \geq \mathrm{Lip}(F, \partial\Omega)$$

for every u that extends F. Therefore, any Lipschitz extension u of F with $\mathrm{Lip}(u, \overline{\Omega}) = \mathrm{Lip}(F, \partial\Omega)$ is a solution to our minimization problem. Now, it is easy to construct such extensions, in fact, let

$$u^*(x) = \min_{y\in\partial\Omega} F(y) + \mathrm{Lip}(F, \partial\Omega)|x - y|$$

and

$$u_*(x) = \max_{y\in\partial\Omega} F(y) - \mathrm{Lip}(F, \partial\Omega)|x - y|.$$

Note that we have

$$u_*(x) \leq u^*(x), \qquad \forall x \in \Omega.$$

EX 1. *Prove that u^* and u_* are solutions to our minimization problem. Moreover, show that there are the maximal and the minimal solution in the sense that any other solution u verifies*

$$u_*(x) \le u(x) \le u^*(x), \qquad \forall x \in \Omega.$$

From this property we have a clear criteria for uniqueness, uniqueness for minimizers of our problem holds if and only if

$$u^*(x) = u_*(x).$$

There is no reason for these extremal solutions to coincide, and it is rare that they do. The example below shows this, no matter how nice Ω might be.

Let $\Omega = B(0,1) \subset \mathbb{R}^2 = \{x \in \mathbb{R}^2 : |x| < 1\}$ and let $F : \partial\Omega \mapsto \mathbb{R}$ be such that $-1 \le F \le 1$ and the Lipschitz constant $L := \mathrm{Lip}(F, \partial\Omega)$ is large. Then, according to the definitions of u^* and u_*, we have

$$u^*(0) > u_*(0)$$

if there exists a $\delta > 0$ such that

$$F(z) - L|z| + \delta = F(z) - L + \delta < F(z) + L + \delta = F(z) + L|z| + \delta, \qquad \forall z \in \partial\Omega.$$

Since $-1 \le F \le 1$ this holds for $L > 1$ (taking $0 < \delta < L - 1$), in fact, we have

$$F(z) - L + \delta < 1 - L + (L-1) = 0 < \delta \le F(z) + L + \delta.$$

EX 2. *Let $\Omega = (-1,0) \cup (0,1)$ and let $F(-1) = F(0) = 0$, $F(1) = 1$. Find u^* and u_*.*

Modify this example to show that even if Ω is bounded, then it is not necessarily true that

$$\max_{\overline{\Omega}} u^* \le \max_{\partial\Omega} F.$$

Moreover, show that $F_1 \le F_2$ does not necessarily imply that $u_1^* \le u_2^*$ (here u_i is the maximal solution to the extension problem associated with F_i).

While this sort of nonuniqueness only takes place if the functional involved is not strictly convex, it is more significant here that the previously mentioned functionals are "not local". In fact, look at the local functional

$$G_2(u) = \int_\Omega |Du|^2\, dx.$$

For this functional it holds that if u minimizes G_2 among functions that verify $u = F$ on $\partial\Omega$ then u restricted to a subset of Ω, D, minimizes the functional in D among functions that coincide with u on ∂D. This is what we mean by "local". This property does not hold for minimizers of G_∞ of for minimizers of $\mathrm{Lip}(u, \Omega)$.

This lack of locality can be corrected by a notion which is directly build from locality. Given a general nonnegative functional $G(u, D)$ which makes sense for each open subset D of the domain Ω, it is said that $u : \Omega \mapsto \mathbb{R}$ is *absolutely minimizing* for G in Ω provided that

$$G(u, D) \le G(v, D), \qquad \text{for every } v \text{ such that } u|_{\partial D} = v|_{\partial D}.$$

That is, u is also a minimizer for G in every subdomian D of Ω taking boundary data $u|_{\partial D}$.

When we take $G(u,\Omega)$ to be $\mathrm{Lip}(u,\Omega)$ we said that we are dealing with an *absolutely minimizing Lipschitz extension* (AMLE for short) of $F = u|_{\partial\Omega}$ in Ω.

The theory of absolutely minimizing functions and for the functional Lip is quicker than that for G_∞, and we present this first, ignoring G_∞ for a while. However, it is shown in these notes that a function which is absolutely minimizing for Lip is also absolutely minimizing for G_∞ and conversely. It turns out that the absolutely minimizing functions for Lip and G_∞ are precisely the viscosity solutions of the famous partial differential equation whose study its out main goal, the infinity Laplacian, given by (1).

2.3. u **is AMLE if and only if** u **has comparison with cones.** Let us start by introducing what is a cone.

Definition 2.7. *The function*

$$C(x) = a|x - z|$$

is called a cone with slope a and vertex z.

We also need the definition of u enjoying comparison with cones.

Definition 2.8. *A continuous function u enjoys comparison with cones from above in Ω iff for every $a \in \mathbb{R}$, $V \subset\subset \Omega$ and $z \notin V$, it holds*

$$u(x) - a|x - z| \le \max_{y \in \partial V} u(y) - a|y - z|, \qquad x \in V. \tag{10}$$

A continuous function u has comparison with cones from below iff $-u$ has comparison with cones from above.

When both conditions hold we say that the continuous function u has comparison with cones.

Note that the condition to have comparison with cones from above can be written as

$$u(x) - C(x) \le \max_{y \in \partial V} u(y) - C(y), \qquad x \in V,$$

for every cone C with vertex $z \notin V$. That is, the maximum of $u - C$ is attained on ∂V.

Assume now that u is a continuous function that has comparison with cones.

First, remark that comparison with cones from above can be rewritten as follows: for every $a, c \in \mathbb{R}$ and $z \notin V$ it holds

$$u(x) \le c + a|x - z|, \text{ for } x \in \Omega, \text{ if it holds for } x \in \partial\Omega.$$

Similarly comparison with cones from below can be written as, for every $a, c \in \mathbb{R}$ and $z \notin V$ it holds

$$u(x) \ge c + a|x - z|, \text{ for } x \in \Omega, \text{ if it holds for } x \in \partial\Omega.$$

Now, our aim is to show that, if u has comparison with cones then, for any $x \in V$,

$$\mathrm{Lip}(u, \partial(V \setminus \{x\})) = \mathrm{Lip}(u, \partial V \cup \{x\}) = \mathrm{Lip}(u, \partial V). \tag{11}$$

To prove this we have to show that when $y \in \partial V$,

$$u(y) - \mathrm{Lip}(u, \partial V)|x-y| \leq u(x) \leq u(y) + \mathrm{Lip}(u, \partial V)|x-y|.$$

These inequalities hold since they hold for every $x \in \partial V$ and, from he fact that u has comparison with cones, they hold for every $x \in V$.

Now, let $x, y \in V$, using (11) twice we obtain that

$$\mathrm{Lip}(u, \partial V) = \mathrm{Lip}(u, \partial(V \setminus \{x\})) = \mathrm{Lip}(u, \partial(V \setminus \{x, y\})).$$

Since $x, y \in \partial(V \setminus \{x, y\}$ we get that

$$|u(x) - u(y)| \leq \mathrm{Lip}(u, \partial V)|x-y|,$$

and we conclude that u is AMLE in V.

Now, let us prove that u has comparison with cones if u is AMLE.

To this end let us observe that the Lipschitz constant of a cone $C(x) = a|x-z|$ is given by

$$\mathrm{Lip}(C, V) = |a|$$

and moreover, if $z \notin V$ we have

$$\mathrm{Lip}(C, \partial V) = |a|.$$

Now, assume that $z \notin V$ and let

$$W = \left\{ x \in V \ : \ u(x) - a|x-z| > \max_{w \in \partial V} (u(w) - a|w-z|) \right\}.$$

Our goal is to show that W is empty. If it is not empty, then it is an open set and

$$u(x) = a|x-z| + \max_{w \in \partial V}(u(w) - a|w-z|) := C(x)$$

for $x \in \partial W$. Therefore $u = C$ on ∂W and since u is AMLE we have $\mathrm{Lip}(u, W) = \mathrm{Lip}(C, \partial W) = |a|$. Now, if $x_0 \in W$ the ray of C that contain x_0, i.e., $t \mapsto z + t(x_0 - z)$, contains a segment in W that contains x_0 and its endpoints are on ∂W. Since $t \mapsto C(z + t(x_0 - z)) = at|x_0 - z|$ is linear on the segment with slope $a|x_0 - z|$ (hence its Lipschitz constant is $|a||x_0 - z|$) while $t \mapsto u(z + t(x_0 - z))$ also has $|a||x_0 - z|$ as Lipschitz constant on the segment and has the same boundary values at the endpoints; hence both functions are the same. Therefore,

$$C(z + t(x_0 - z)) = u(z + t(x_0 - z))$$

on the segment. In particular, $C(x_0) = u(x_0)$, a contradiction with the fact that $x_0 \in W$.

This proves that u has comparison with cones.

2.4. **If u has comparison with cones then $\Delta_\infty u = 0$.** From the previous section we know that when u has comparison with cones from above it holds

$$u(x) \leq u(y) + \max_{w\in\partial B_r(y)} \left(\frac{u(w)-u(y)}{r}\right)|x-y| = \max_{w\in\partial B_r(y)} \frac{u(w)}{r}|x-y| + u(y)\left(1-\frac{|x-y|}{r}\right),$$

for any $x \in B_r(y) \subset\subset \Omega$. This inequality follows since it holds trivially for $x \in \partial B_r(y)$.

Now rewrite it as

$$u(x) - u(y) \leq \max_{w\in\partial B_r(y)} (u(w)-u(x)) \frac{|x-y|}{r-|x-y|}, \tag{12}$$

for any $x \in B_r(y) \subset\subset \Omega$.

Assume that u is twice differentiable at x, that is, there are a vector p and a matrix X such that

$$u(z) = u(x) + \langle p; z-x\rangle + \frac{1}{2}\langle X(z-x); z-x\rangle + o(|z-x|^2). \tag{13}$$

In fact,

$$p = Du(x), \qquad X = D^2u(x).$$

We will prove that

$$\Delta_\infty u(x) = \langle D^2u(x)Du(x); Du(x)\rangle = \langle Xp; p\rangle \geq 0. \tag{14}$$

That is, comparison with cones from above implies $\Delta_\infty u \geq 0$ at points where u is twice differentiable.

We can assume that $p \neq 0$ (otherwise the inequality that we want to prove holds trivially).

We use (13) in (12) with two choices of z. First, let us take

$$z = y = x - \lambda p$$

and expand (12) according to (13), we have,

$$-\langle p; y-x\rangle - \langle X(y-x); y-x\rangle + o(|y-x|^2) \leq \max_{w\in\partial B_r(y)} (u(w)-u(x)) \frac{|x-y|}{r-|x-y|}.$$

Now, consider the point $w_{r,\lambda}$ at which the maximum in the right hand side is attained and use it as z in (13) to obtain, after dividing by $\lambda > 0$,

$$|p|^2 + \lambda\frac{1}{2}\langle Xp; p\rangle + o(\lambda) \leq \left(\langle p; w_{r,\lambda}-x\rangle + \frac{1}{2}\langle X(w_{r,\lambda}-x); (w_{r,\lambda}-x)\rangle + o((r+\lambda)^2)\right) \times \frac{|p|}{r-\lambda|p|}.$$

Taking $\lambda \to 0$ we obtain

$$|p|^2 \leq \left(\langle p; \frac{w_r-x}{r}\rangle + \frac{1}{2}\langle X(\frac{w_r-x}{r}); (w_r-x)\rangle + o(r)\right)|p| \tag{15}$$

where w_r is a limit point of $w_{r,\lambda}$ and hence we have $w_r \in \partial B_r(x)$, that is, $\frac{w_r - x}{r}$ is a unit vector. From the previous inequality, it follows that

$$\frac{w_r - x}{r} \to \frac{p}{|p|} \tag{16}$$

as $r \to 0$. Again from the inequality (15), using that,

$$\langle p; \frac{w_r - x}{r} \rangle |p| \leq |p|^2$$

we obtain

$$0 \leq \lim_{r \to 0} \frac{1}{2} \langle X(\frac{w_r - x}{r}); (\frac{w_r - x}{r}) \rangle$$

that is,

$$0 \leq \langle X \frac{p}{|p|}; \frac{p}{|p|} \rangle,$$

which implies (14).

EX 3. *Show that this argument prove that when $\nabla u(x) = p = 0$ we obtain*

$$D^2 u(x) = X \text{ has a nonnegative eigenvalue.}$$

Now, assume that φ is a smooth test function, that is, $u - \varphi$ has a local maximum at x, then

$$\varphi(x) - \varphi(y) \leq u(x) - u(y)$$

and

$$u(w) - u(x) \leq \varphi(w) - \varphi(x).$$

Hence we have that (12) holds with u replaced by φ and from our previous argument, using that φ is smooth we get

$$\Delta_\infty \varphi(x) \geq 0, \qquad \text{if } D\varphi(x) \neq 0,$$

and

$$D^2 \varphi(x) \text{ has a nonnegative eigenvalue} \qquad \text{if } D\varphi(x) = 0.$$

In any case, we have

$$u - \varphi \text{ has a local maximum at } x \Rightarrow \Delta_\infty \varphi(x) \geq 0,$$

that is, if u has comparison with cones from above, then u is a viscosity subsolution to $\Delta_\infty u = 0$ in Ω.

EX 4. *Show that if u has comparison with cones from below then it is a viscosity supersolution to $\Delta_\infty u = 0$ in Ω.*

2.5. $\Delta_\infty u = 0$ **implies comparison with cones.** First, let us compute the $\infty-$Laplacian of a radial function, we get

$$\Delta_\infty G(|x|) = G''(|x|)(G'(|x|))^2$$

when $x \neq 0$. Hence, for any small $\gamma > 0$

$$\Delta_\infty(a|x-z| - \gamma|x-z|^2) = -2\gamma(a - 2\gamma|x-z|)^2 < 0,$$

for $x \neq z$.

Now, if u verifies $\Delta_\infty u \geq 0$ in the viscosity sense (that is, u is a viscosity subsolution to $\Delta_\infty u = 0$), then we have that $u(x) - (a|x-z| - \gamma|x-z|^2)$ cannot have a maximum in V different from z (if it has then we get a contradiction with the fact that $\Delta_\infty u \geq 0$ in the viscosity sense). Hence, if $z \notin V$ and $x \in V$ we must have

$$u(x) - (a|x-z| - \gamma|x-z|^2) \leq \max_{y\in\partial V} u(y) - (a|y-z| - \gamma|y-z|^2).$$

Now, just take $\gamma \to 0$ to obtain

$$u(x) - a|x-z| \leq \max_{y\in\partial V} u(y) - a|y-z|,$$

that is, we have that u has comparison with cones from above.

Analogously, one can show that if u is a viscosity supersolution to $\Delta_\infty u = 0$ then it has comparison with cones from below.

2.6. **Minimizers of** $G_\infty(u) = \|Du\|_{L^\infty(\Omega)}$**.** Let us observe right away that if u is absolutely minimizing for G_∞, then it is absolutely minimizing for Lip.

Proposition 2.9. *Let $u \in C(\Omega)$ be absolutely minimizing for G_∞, that is, whenever $V \subset\subset \Omega$, $v \in C(\overline{V})$ and $u = v$ on ∂V, then $G_\infty(u,V) \leq G_\infty(v,V)$. Then u is absolutely minimizing for Lip (and hence, from our previous results, ∞-harmonic).*

Proof. Let $v \in C(\overline{V})$ and $u = v$ on ∂V. Assume $\mathrm{Lip}(v,\partial V) < \infty$ and replace v by the minimal Lipchitz extension of its boundary values, that is, v_*, so that we may assume that $\mathrm{Lip}(v,V) = \mathrm{Lip}(v,\partial V)$. Then, by assumption, $G_\infty(u,V) \leq G_\infty(v,V)$, which is at most $\mathrm{Lip}(v,V)$. Now we observe that if $u \in C(\overline{V})$ and $F_\infty(u,V) \leq \mathrm{Lip}(u,\partial V)$, then $\mathrm{Lip}(u,V) = \mathrm{Lip}(u,\partial V)$. □

Let us define the Lipschitz constant of a function of u at a point x as follows:

Definition 2.10. *Let $v : \Omega \to \mathbb{R}$ and $x \in \Omega$. Then*

$$L(v,x) := \lim_{r\to 0} Lip(v, B_r(x)) = \inf_{0<r<dist(x,\partial\Omega)} Lip(v, B_r(x)). \tag{17}$$

Of course, $L(u,x) = \infty$ is possible.

Lemma 2.11. *Let $u : \Omega \to \mathbb{R}$ be upper-semicontinuous. Assume that*

$$u(x) \leq u(y) + \max_{w\in\overline{B}_r(y)} \left(\frac{u(w) - u(y)}{r}\right)|x-y|, \tag{18}$$

for $y \in \Omega$, $r > 0$, $x \in \overline{B}_r(y)$. Then,

$$\max_{w \in \partial B_r(y)} u(w) = \max_{w \in \overline{B}_r(y)} u(w).$$

Moreover, u is locally Lipschitz continuous and it is ∞-subharmonic and enjoys comparison with cones from above. In addition the quantity

$$S^+(y,r) := \max_{w \in \overline{B}_r(y)} \left(\frac{u(w) - u(y)}{r} \right) = \max_{w \in \partial B_r(y)} \left(\frac{u(w) - u(y)}{r} \right) \tag{19}$$

is nonnegative and nondecreasing in r, $0 < r < dist(y, \partial\Omega)$. Moreover,

$$\begin{gathered} \text{if } |w - y| = r, \quad \text{and } S^+(y,r) := \left(\frac{u(w) - u(y)}{r} \right) \text{ then} \\ S^+(y,r) \leq S^+(w,s) \text{ for } 0 < s < dist(y, \partial\Omega) - r. \end{gathered} \tag{20}$$

Proof. Assume that $y \in \Omega$, (18) holds, $\overline{B}_r(y) \subset \Omega$, $|x-y| < r$ and $u(x) = \max_{\overline{B}_r(y)} u$. Then we may replace $u(w)$ by $u(x)$ in (18) to conclude that

$$u(x)(1 - |x-y|/r) \leq u(y)(1 - |x-y|/r),$$

which implies that $u(x) \leq u(y)$. Since also $u(x) \geq u(y)$, we conclude that $u(x) = u(y)$. Since this is true for all y such that $x \in B_r(y) \subset\subset \Omega$ and $u(x) = \max_{\overline{B}_r(y)} u$, it is true if $\overline{B}_R(x) \subset\subset \Omega$, $u(x) = \max_{\overline{B}_R(x)} u$ and $|y - x| < R/2$. Thus, if u has a local maximum point, it is constant in a ball around that point. Let us state this, if $\overline{B}_R(x) \subset\subset \Omega$,

$$u \text{ satisfies (18) and } u(x) = \max_{\overline{B}_R(x)} u, \text{ then } u \text{ is constant on } B_{R/2}(x). \tag{21}$$

This guarantees that if u assumes its maximum value at any point of a connected open set, then it is constant in that set, and hence that the maximum of u over any closed ball is attained on the boundary. The first assertion is proved.

Now we deal with the Lipschitz continuity of u. Assume, to begin with, that $u \leq 0$. Then, as $u(w) \leq 0$ in (18), the $u(w)$ on the right can be dropped. Thus we have, written in three equivalent ways,

$$\begin{aligned} u(x) &\leq \left(1 - \frac{|x-y|}{r} \right) u(y), \\ -u(y) &\leq -u(x) \left(\frac{r}{r - |x-y|} \right), \\ u(x) - u(y) &\leq -\frac{|x-y|}{r} u(y). \end{aligned} \tag{22}$$

Any of the first two inequalities is a Harnack inequality. If $u(x) \neq 0$, either estimates the ratio $u(y)/u(x)$ by quantities not depending on u. Taking the inferior limit as $y \to x$ on the right of the first inequality, we find that u is lower-semicontinuous as well as upper-semicontinuous, so it is continuous. If also $\overline{B}_r(x) \subset\subset \Omega$, we may interchange x and y in the third inequality in (22) and conclude, from the two relations, that

$$|u(x) - u(y)| \leq -\min(u(x), u(y)) \frac{|x-y|}{r - |x-y|}.$$

As u is locally bounded, being continuous, we conclude that it is also locally Lipschitz continuous. If $u \leq 0$ does not hold and $x, y \in B_r(z)$, where $\overline{B}_{2r}(z) \subset \Omega$, replace u by $u - \max_{\overline{B}_{2r}(z)} u$. We then have that u is Lipschitz continuous in $B_r(z)$ if $\overline{B}_{2r}(z) \subset \Omega$.

We turn now our attention to the third statement. The assumptions imply that if $|x-y| = s \leq r$, then

$$\frac{u(x) - u(y)}{s} \leq \max_{w \in \partial B_r(y)} \left(\frac{u(w) - u(y)}{r} \right).$$

The monotonicity of $S^+(r, y)$ in r follows upon maximizing the left-hand side with respect to x, $|x - y| = s$. The quantity $S^+(y, r)$ is nonnegative by what was already shown $-u$ attains its maximum over a ball on the boundary.

Now, let the assumptions of (20) hold: $r < \operatorname{dist}(y, \partial\Omega)$ and

$$|w - y| = r, \qquad u(w) = \max_{\overline{B}_r(y)} u. \tag{23}$$

Let $0 < s < \operatorname{dist}(y, \partial\Omega)$ and for $0 \leq t \leq 1$ put $y_t := y + t(w - y)$. By our assumptions,

$$u(y_t) - u(y) \leq \left(\frac{u(w) - u(y)}{r} \right) |y_t - y| = t(u(w) - u(y));$$

equivalently,

$$\frac{u(w) - u(y)}{|w - y|} \leq \frac{u(w) - u(y_t)}{|w - y_t|}$$

which implies, using the choice of w and monotonicity of S^+, that

$$S^+(y, r) = \frac{u(w) - u(y)}{|w - y|} \leq \frac{u(w) - u(y_t)}{|w - y_t|} \leq S^+(y_t, s)$$

for $s \leq |w - y_t| = (1 - t)|w - y|$. Letting $t \nearrow 1$ and using the continuity of $S^+(x, s)$ in x, this gives

$$S^+(y, r) \leq S^+(w, s).$$

This ends the proof. □

Now our aim is to prove the converse to the proposition above. To this aim we need the following result:

Proposition 2.12. *Let u be ∞-subharmonic in Ω and $x \in \Omega$. Then there is a $T > 0$ and Lipschitz continuous curve $\gamma : [0, T) \mapsto \Omega$ with the following properties:*

1. $\gamma(0) = x$,
2. $|\gamma'(t)| \leq 1$, *a.e.* $t \in [0, T)$,
3. $L(u, \gamma(t)) \geq L(u, x)$, $t \in [0, T)$,
4. $u(\gamma(t)) \geq u(x) + tL(u, x)$, $t \in [0, T)$,
5. $t \mapsto u(\gamma(t))$ *is convex on* $[0, T)$,
6. *Either* $T = +\infty$ *or* $\lim_{t \nearrow T} \gamma(t) \in \partial\Omega$.

Proof. We may assume that $L(u,x)>0$, for otherwise we may take $\gamma(t)=x$. Fix $0<\delta<\operatorname{dist}(x,\partial\Omega)$ and take a sequence $\{x_\delta^j\}_{j=0}^J$ (J can be finite or infinite) such that $x_\delta^0=x$, and

$$|x_\delta^{j+1}-x_\delta^j|=\delta, \qquad \text{and} \qquad u(x_\delta^{j+1})=\max_{\overline{B}_\delta(x_\delta^j)} u.$$

According to the increasing slope estimate (20), we then have

$$S^+(x_\delta^{j+1})\geq \frac{u(x_\delta^{j+1})-u(x_\delta^j)}{\delta}=S^+(x_\delta^j,\delta)\geq S^+(x_\delta^j). \tag{24}$$

Hence

$$S^+(x_\delta^j)\geq S^+(x), \qquad j=1,...,J,$$

and

$$u(x_\delta^{j+1})-u(x_\delta^j)\geq \delta S^+(x), \qquad \text{hence} \qquad u(x_\delta^{j+1})-u(x)\geq \delta j S^+(x).$$

Now, we take the piecewise linear curve defined by $\gamma_\delta(0)=x$ and

$$\gamma_\delta(t)=x_\delta^j+(t-j\delta)\left(\frac{x_\delta^{j+1}-x_\delta^j}{\delta}\right) \qquad \text{for } j\delta\leq t\leq (j+1)\delta.$$

By construction we have

$$\begin{aligned}
&\gamma_\delta(0)=x,\\
&|\gamma_\delta'(t)|=1, \ \text{a.e. } t\in[0,\delta J),\\
&L(u,\gamma_\delta(j\delta)\geq L(u,x),\\
&u(\gamma_\delta(j\delta))\geq u(x)+j\delta L(u,x).
\end{aligned} \tag{25}$$

Also by construction we get $J\delta\geq \operatorname{dist}(x,\partial\Omega)-\delta$. By compactness, there is a sequence $\delta_k\searrow 0$ and a function $\gamma:[0,\operatorname{dist}(x,\partial\Omega))\to\Omega$ such that $\gamma_k(t)\to\gamma(t)$ uniformly on compact subsets of $[0,\operatorname{dist}(x,\partial\Omega))$. Clearly $\operatorname{dist}(\gamma(t),\partial\Omega)\geq \operatorname{dist}(x,\partial\Omega)-t$. Moreover, if $0\leq t<\operatorname{dist}(x,\partial\Omega)$, there exist j_k such that $j_k\delta_k\to t$. Passing to the limit in the relations (25) with $\delta=\delta_k$ and $j=j_k$, using the upper-semicontinuity of $L(u,\cdot)$, yields all of the relations of the statement except (5) and (6).

To see that (5) holds, note that the piecewise linear function $g_k(t)$ whose value at $j\delta_k$ is $u(\gamma_{\delta_k}(j\delta_k))$ is convex by (24). By the continuity of u and the uniform convergence of $\gamma_{\delta_k}\to\gamma$, g_k converges to $u(\gamma(t))$, which is therefore convex.

The property (6) can be obtained by the standard continuation argument of ordinary differential equations. There is a curve γ with the properties of the statement defined on a maximal interval of existence of the form $[0,T)$. Assume now that $T<\infty$ and $\lim_{t\nearrow T}\gamma(t)=:\gamma(T)$ and $\gamma(T)\notin\partial\Omega$. The proof concludes by arguing that then was not maximal. We refer to [17] for details. □

Theorem 2.13. *Let u be an AMLE and let $V\subset\subset\Omega$, $v\in C(\overline{V})$ with $u=v$ on ∂V. Then* $\sup_V L(u,x)\leq sup_V L(v,x)$.

Proof. If the conclusion does not hold, then there exists $z \in V$ and $\delta > 0$ such that $L(u,z) > L(v,x) + \delta$ for $x \in V$. Let γ be the curve provided by Proposition 2.12 which starts at z (taking V as Ω in the proposition). Since u is bounded in $\overline{V}$ it follows that $\lim_{t \nearrow T} \gamma(t) = \gamma(T) \in \partial\Omega$. By (3) of the proposition

$$L(u, \gamma(t)) \geq L(u,z) > \sup_V L(v,x)$$

and hence using (2), a. e.,

$$\frac{\partial}{\partial t} v(\gamma(t)) \leq L(v(\gamma(t))) < L(u,z).$$

Integrating over $[0,T]$ and using (2) and (4) we get

$$v(\gamma(T)) - v(z) < TL(u,z) \leq u(\gamma(T)) - u(z).$$

Now, also $-u$ is $\infty-$subharmonic and is related to $-v$ as u is related to v hence there is a second curve $\tilde{\gamma}$ such that

$$-v(\tilde{\gamma}(\tilde{T})) + v(z) < \tilde{T}L(u,z) \leq -u(\tilde{\gamma}(\tilde{T})) + u(z).$$

Adding the two inequalities we get

$$v(\gamma(T)) - v(\tilde{\gamma}(\tilde{T})) < u(\gamma(T)) - u(\tilde{\gamma}(\tilde{T})).$$

This contradicts the fact that $u = v$ on ∂V. □

EX 5. *Show that if u is ∞-harmonic, then it is absolutely minimizing for G_∞.*

3. Linear PDEs and probability

3.1. **The probability of hitting the exit and harmonic functions.** Let us begin by considering a bounded and smooth two-dimensional domain $\Omega \subset \mathbb{R}^2$ and assume that the boundary, $\partial\Omega$ is decomposed in two parts, Γ_1 and Γ_2 (that is, $\Gamma_1 \cup \Gamma_2 = \partial\Omega$ with $\Gamma_1 \cap \Gamma_2 = \emptyset$). We begin with a position $(x,y) \in \Omega$ and ask the following question: assume that you move completely at random beginning at (x,y) (we assume that we are in an homogeneous environment and that we do not privilege any direction. In addition, we assume that every time the particle moves independently of its past history) what is the probability $u(x,y)$ of hitting the first part of the boundary Γ_1 the first time that the particle hits the boundary ?.

We will call Γ_1 the "open part" of the boundary and think that when we hit this part we can "exit" the domain, while we will call Γ_2 the "closed part" of the boundary, when we hit it we are dead.

This problem of describing the random movement in a precise mathematical way is the central subject of Brownian Motion. It originated in 1827, when the botanist Robert Brown observed this type of random movement in pollen particles suspended in water.

A clever and simple way to get some insight to solve the question runs as follows: First, we simplify the problem and approximate the movement by random increments of step h in each of the axes directions, with $h > 0$ small. From (x,y) the particle can move to

$(x+h,y)$, $(x-h,y)$, $(x,y+h)$, or $(x,y-h)$, each movement being chosen at random with probability 1/4.

Starting at (x,y), let $u_h(x,y)$ be the probability of hitting the exit part $\Gamma_1 + B_\delta(0)$ the first time that $\partial\Omega + B_\delta(0)$ is hitt when we move on the lattice of side h. Observe that we need to enlarge a little the boundary to capture points on the lattice of size h (that do not necessarily lie on $\partial\Omega$).

Applying conditional expectations we get

$$u_h(x,y) = \frac{1}{4}u_h(x+h,y) + \frac{1}{4}u_h(x-h,y) + \frac{1}{4}u_h(x,y+h) + \frac{1}{4}u_h(x,y-h). \tag{26}$$

That is,

$$0 = \{u_h(x+h,y) - 2u_h(x,y) + u_h(x-h,y)\} + \{u_h(x,y+h) - 2u_h(x,y) + u_h(x,y-h)\}. \tag{27}$$

Now, assume that u_h converges as $h \to 0$ to a function u uniformly in $\overline{\Omega}$. Note that this convergence can be proved rigorously.

Let ϕ be a smooth function such that $u - \phi$ has a strict minimum at $(x_0, y_0) \in \Omega$. By the uniform convergence of u_h to u there are points (x_h, y_h) such that

$$(u_h - \phi)(x_h, y_h) \le (u_h - \phi)(x,y) + o(h^2) \qquad (x,y) \in \Omega$$

and

$$(x_h, y_h) \to (x_0, y_0) \qquad h \to 0.$$

Note that u_h is not necessarily continuous.

Hence, from (27) at the point (x_h, y_h) and using that

$$u_h(x,y) - u_h(x_h,y_h) \ge \phi(x,y) - \phi(x_h,y_h) + o(h^2) \qquad (x,y) \in \Omega,$$

we get

$$\begin{aligned} 0 \ge\ & \{\phi(x_h+h,y_h) - 2\phi(x_h,y_h) + \phi(x_h-h,y_h)\} \\ & + \{\phi(x_h,y_h+h) - 2\phi(x_h,y_h) + \phi(x_h,y_h-h)\} + o(h^2). \end{aligned} \tag{28}$$

Now, we just observe that

$$\{\phi(x_h+h,y_h) - 2\phi(x_h,y_h) + \phi(x_h-h,y_h)\} = h^2 \frac{\partial^2\phi}{\partial x^2}(x_h,y_h) + o(h^2)$$

$$\{\phi(x_h,y_h+h) - 2\phi(x_h,y_h) + \phi(x_h,y_h-h)\} = h^2 \frac{\partial^2\phi}{\partial y^2}(x_h,y_h) + o(h^2).$$

Hence, substituting in (28), dividing by h^2 and taking limit as $h \to 0$ we get

$$0 \ge \frac{\partial^2\phi}{\partial x^2}(x_0,y_0) + \frac{\partial^2\phi}{\partial y^2}(x_0,y_0).$$

Therefore, a uniform limit of the approximate values u_h, u, has the following property: *each time that a smooth function ϕ touches u from below at a point (x_0, y_0) the derivatives of ϕ must satisfy,*

$$0 \geq \frac{\partial^2 \phi}{\partial x^2}(x_0, y_0) + \frac{\partial^2 \phi}{\partial y^2}(x_0, y_0).$$

An analogous argument considering ψ a smooth function such that $u - \psi$ has a strict maximum at $(x_0, y_0) \in \Omega$ shows a reverse inequality. Therefore, *each time that a smooth function ψ touches u from above at a point (x_0, y_0) the derivatives of ψ must verify*

$$0 \leq \frac{\partial^2 \psi}{\partial x^2}(x_0, y_0) + \frac{\partial^2 \psi}{\partial y^2}(x_0, y_0).$$

But at this point we realize that this is exactly the definition of being u a **viscosity solution to the Laplace equation**

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Hence, we obtain that the uniform limit of the sequence of solutions to the approximated problems u_h, u is the unique viscosity solution (that is also a classical solution in this case) to the following boundary value problem

$$\begin{cases} -\Delta u(x) = 0 & x \in \Omega, \\ u(x) = 1 & x \in \Gamma_1, \\ u(x) = 0 & x \in \Gamma_2. \end{cases} \tag{29}$$

The boundary conditions can be easily obtained from the fact that $u_h \equiv 1$ in a neighborhood (of width h) of Γ_1 and $u_h \equiv 0$ in a neighborhood of Γ_2.

Note that we have only required *uniform* convergence to get the result, and hence no requirement is made on derivatives of the approximating sequence u_h. Moreover, we do not assume that u_h is continuous.

Now, we just notice that in higher dimensions $\Omega \subset \mathbb{R}^N$, the discretization method described above leads in the same simple way to viscosity solutions to the Laplace operator in higher dimensions and then to the fact that exiting probabilities are harmonic functions.

Another way to understand this strong relation between probabilities and the Laplacian is through the *mean value property of harmonic functions.* In the same context of the problem solved above, assume that a closed ball $B_r(x_0, y_0)$ of radius r and centered at a point (x_0, y_0) is contained in Ω. Starting at (x_0, y_0), the probability density of hitting first a given point on the sphere $\partial B_r(x_0, y_0)$ is constant on the sphere, that is, it is uniformly distributed on the sphere. Therefore, the probability $u(x_0, y_0)$ of exiting through Γ_1 starting at (x_0, y_0) is the average of the exit probabilities u on the sphere, here we are using again the formula of conditional probabilities. That is, u satisfies the mean value property on spheres:

$$u(x_0, y_0) = \frac{1}{|\partial B_r(x_0, y_0)|} \int_{\partial B_r(x_0,y_0)} u(x, y)\, dS(x, y)$$

with r small enough. It is well known that this property leads to u being harmonic.

We can also say that, if the movement is completely random and equidistributed in the ball $B_h(x_0, y_0)$, then, by the same conditional expectation argument used before, we have

$$u_h(x_0, y_0) = \frac{1}{|B_h(x_0, y_0)|} \int_{B_h(x_0,y_0)} u_h(x, y)\, dx\, dy.$$

Again one can take the limit as $h \to 0$ and obtain that a uniform the limit of the u_h, u, is harmonic (in the viscosity sense).

3.2. **Counting the number of steps needed to reach the exit.** Another motivating problem is the following: with the same setting as before (Ω a bounded smooth domain in $\mathbb{R}^2$) we would like to compute the expected time, that we call T, that we have to spend starting at (x, y) before hitting the boundary $\partial\Omega$ for the first time.

We can proceed exactly as before computing the time of the random walk at the discrete level, that is, in the lattice of size h. This amounts to adding a constant (the unit of time that we spend in each movement), which depends on the step h, to the right hand side of (26). We have

$$T_h(x, y) = \frac{1}{4}T_h(x + h, y) + \frac{1}{4}T_h(x - h, y) + \frac{1}{4}T_h(x, y + h) + \frac{1}{4}T_h(x, y - h) + t(h).$$

That is,

$$\begin{aligned} 0 = \ & \{T_h(x + h, y) - 2T_h(x, y) + T_h(x - h, y)\} \\ & + \{T_h(x, y + h) - 2T_h(x, y) + T_h(x, y - h)\} + t(h). \end{aligned}$$

Proceeding as we did before, and since we need to divide by h^2, a natural choice is to set that $t(h)$ is of order h^2. Choosing

$$t(h) = Kh^2$$

and letting $h \to 0$, we conclude that a uniform limit of the approximate solutions T_h, T is the unique solution to

$$\left\{ \begin{array}{ll} -\Delta T(x) = 4K & x \in \Omega, \\ T(x) = 0 & x \in \partial\Omega. \end{array} \right. \tag{30}$$

The boundary condition is natural since if we begin on the boundary the expected time needed to reach it is zero.

From the previous probabilistic interpretations for the solutions of problems (29) and (30) as limits of random walks, one can imagine a probabilistic model for which the solution of the limit process is a solution to the general Poisson problem

$$\begin{cases} -\Delta u(x) = g(x) & x \in \Omega, \\ u(x) = F(x) & x \in \partial\Omega. \end{cases} \tag{31}$$

In this general model the functions g and F can be thought as costs that one pays, respectively, along the random movement and at the stopping time on the boundary.

3.3. **Anisotropic media.** Suppose now that the medium in which we perform our random movements is neither isotropic (that is, it is directionally dependent) nor homogeneous (that is, it differs from one point to another).

We can imagine a random discrete movement as follows. We move from a point (x, y) to four possible points at distance h located at two orthogonal axis forming a given angle α with the horizontal, and with different probabilities $q/2$ for the two points on the first axis and $(1-q)/2$ for the two points on the second axis. The angle α (that measures the orientation of the axis) and the probability q depend on the point (x, y). After the same analysis as above, we encounter now the general elliptic equation

$$\sum_{ij} a_{ij}(x, y) u_{x_i x_j}(x, y) = 0.$$

3.4. **The heat equation.** Now assume that we are on the real line $\mathbb{R}$ and consider that we move in a two dimensional lattice as follows: when we are at the point (x_0, t_0), the time increases by $\delta t := h^2$ and the spacial position moves with an increment of size $\delta x = h$ and goes to $x_0 - h$ or to $x_0 + h$ with the same probability. In this way the new points in the lattice that can be reached starting from (x_0, t_0) are $(x_0 - h, t_0 + h^2)$ or to $(x_0 + h, t_0 + h^2)$, each one with probability $1/2$.

As we will see, the choice $\delta t = (\delta x)^2$ is made to ensure that a certain limit as $h \to 0$ exists.

Let us start at $x = 0$, $t = 0$ and let $u_h(x, t)$ be the probability that we are at x at time t (here $x = kh$ and $t = lh^2$ is a point in the two dimensional lattice).

As in the previous subsection, conditional probabilities give the identity

$$u_h(x, t) = \frac{1}{2} u_h(x - h, t - h^2) + \frac{1}{2} u_h(x + h, t - h^2).$$

That is,

$$\frac{u_h(x, t) - u_h(x, t - h^2)}{h^2} = \frac{1}{h^2} \Big\{ \frac{1}{2} u_h(x - h, t - h^2) + \frac{1}{2} u_h(x + h, t - h^2) - u_h(x, t - h^2) \Big\}.$$

Now, as before, we let $h \to 0$ and, assuming uniform convergence (that can be proved !), we arrive to the fact that the limit should be a viscosity solution to

$$u_t(x, t) = \frac{1}{2} u_{xx}(x, t).$$

It is at this point where the relation $\delta t = (\delta x)^2$ is needed.

3.5. **Comments.**

(1) We want to remark that two facts are crucial in the previous analysis. The first one is the formula for the discrete version of the problem obtained using conditional expectations, (26), and the second one is the use of the theory of viscosity solutions to perform the passage to the limit without asking for more than uniform convergence.

These ideas are going to be used again in the next section.

(2) One can show the required uniform convergence from the following two facts: first, the values u_h are uniformly bounded (they all lie between 0 and 1 in the case of the problem of exiting the domain through Γ_1 and by the maximum and the minimum of the datum F in the general case assuming $g \equiv 0$) and second the family u_h is equicontinuous for "not too close points" (see the following sections), this can be proved using coupling methods. In fact, one can mimic a path of the game starting at x but starting at y and when one of the two paths hits the boundary the other is at a position that is close (at a distance smaller than $|x-y|$) of the boundary. It remains to prove a uniform in h estimate that says that "given $\epsilon > 0$ there exists $\delta > 0$ such that if one begins close to the boundary Γ_1 (with $\text{dist}(x_0, \Gamma_1) < \delta$) then the probability of hitting this part of the boundary is bounded below by $1 - \epsilon$" (and of course an analogous statement for positions that start close to Γ_2). For this argument to work in the general case one can impose that F is uniformly continuous.

(3) The initial condition for the heat equation in subsection 3.4 is $u(x,0) = \delta_0$ and the solution can be explicitly obtained as the Gaussian

$$u(x,t) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}.$$

For an initial distribution of particles given by $u(x,0) = u_0(x)$ we get

$$u(x,t) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} * u_0(x).$$

(4) Also note that the identities obtained using conditional expectations are the same that correspond to the discretization of the equations using finite differences. This gives a well-known second order numerical scheme for the Laplacian. Hence, we remark that probability theory can be used to obtain numerical schemes for PDEs.

(5) Finally, note that when a general Poisson problem is considered in (31) the functions g and F that appear can be thought as costs that one pays. The first one is a *running cost* that is payed at each movement, while the second one is a *final cost* that is payed when the game ends reaching the boundary.

We will use this terminology in the next section.

4. Tug-of-War games and the $\infty-$Laplacian

In this section we will look for a probabilistic approach to approximate solutions to the $\infty-$Laplacian. Recall that this is the nonlinear degenerate elliptic operator, usually denoted by Δ_∞, given by,

$$\Delta_\infty u := \left(D^2u \, \nabla u\right) \cdot \nabla u = \sum_{i,j=1}^{N} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_i x_j},$$

and arises from taking limit as $p \to \infty$ in the p-Laplacian operator in the viscosity sense, see Section 2 and references [5] and [10]. In fact, let us present briefly a formal derivation. First, expand (formally) the $p-$laplacian:

$$\begin{aligned} \Delta_p u &= \operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right) = \\ &= |\nabla u|^{p-2} \Delta u + (p-2)|\nabla u|^{p-4} \sum_{i,j} u_{x_i} u_{x_j} u_{x_i,x_j} = \\ &= (p-2)|\nabla u|^{p-4} \left\{ \frac{1}{p-2} |\nabla u|^2 \Delta u + \sum_{i,j} u_{x_i} u_{x_j} u_{x_i,x_j} \right\} \end{aligned}$$

and next, using this formal expansion, pass to the limit in the equation $\Delta_p u = 0$, to obtain

$$\Delta_\infty u = \sum_{i,j} u_{x_i} u_{x_j} u_{x_i,x_j} = Du \cdot D^2 u \cdot (Du)^t = 0.$$

Note that this calculation can be made rigorous in the viscosity sense (this was done in Section 2).

The ∞-laplacian operator appears naturally when one considers absolutely minimizing Lipschitz extensions of a boundary function F; see [30] and also the survey [5]. A fundamental result of Jensen [30] establishes that the Dirichlet problem for Δ_∞ is well posed in the viscosity sense. Solutions to $-\Delta_\infty u = 0$ (that are called infinity harmonic functions) are also used in several applications, for instance, in optimal transportation and image processing (see, e.g., [19], [24] and the references therein). Also the eigenvalue problem related to the ∞-laplacian has been exhaustively studied, see [13], [33], [34], [35].

Let us recall the definition of an absolutely minimizing Lipschitz extension. Let $F : \partial\Omega \to \mathbb{R}$. We denote by $L(F, \partial\Omega)$ the smallest Lipschitz constant of F in $\partial\Omega$, i.e.,

$$L(F, \partial\Omega) := \sup_{x,y \in \partial\Omega} \frac{|F(x) - F(y)|}{|x-y|}.$$

If we are given a Lipschitz function $F : \partial\Omega \to \mathbb{R}$, i.e., $L(F, \partial\Omega) < +\infty$, then it is well-known that there exists a *minimal Lipschitz extension* (MLE for short) of F to Ω, that is, a function $h : \overline{\Omega} \to \mathbb{R}$ such that $h_{|\partial\Omega} = F$ and $L(F, \partial\Omega) = L(h, \overline{\Omega})$. Extremal extensions were explicitly constructed by McShane [40] and Whitney [56],

$$\Psi(F)(x) := \inf_{y \in \partial\Omega} \left(F(y) + L(F, \partial\Omega)|x-y|\right), \quad x \in \overline{\Omega},$$

and

$$\Lambda(F)(x) := \sup_{y\in\partial\Omega} \left(F(y) - L(F,\partial\Omega)|x-y| \right), \quad x \in \overline{\Omega},$$

are MLE of F to $\overline{\Omega}$ and if u is any other MLE of F to $\overline{\Omega}$ then

$$\Lambda(F) \leq u \leq \Psi(F).$$

The notion of a minimal Lipschitz extension is not completely satisfactory since it involves only the global Lipschitz constant of the extension and ignore what may happen locally. To solve this problem, in the particular case of the euclidean space $\mathbb{R}^N$, Arosson [2] introduce the concept of *absolutely minimizing Lipschitz extension* (AMLE for short) and proved the existence of AMLE by means of a variant of the Perron's method. The AMLE is given by the following definition. Here we consider the general case of extensions of Lipschitz functions defined on a subset $A \subset \overline{\Omega}$, but the reader may consider $A = \partial\Omega$.

Definition 4.1. Let A be any nonempty subset of $\overline{\Omega}$ and let $F : A \subset \overline{\Omega} \to \mathbb{R}$ be a Lipschitz function. A function $u : \overline{\Omega} \to \mathbb{R}$ is an *absolutely minimizing Lipschitz extension* of F to $\overline{\Omega}$ if

(i) u is an MLE of F to $\overline{\Omega}$,
(ii) whenever $B \subset \overline{\Omega}$ and $g : \overline{\Omega} \to \mathbb{R}$ is and MLE of F to $\overline{\Omega}$ such that $g = u$ in $\overline{\Omega} \setminus B$, then

$$L(u, B) \leq L(g, B).$$

Remark 4.2. *The definition of AMLE can be extended to any metric space* (X, d), *and existence of such an extension can be proved when* (X, d) *is a separable length space,* [32].

It turns out (see [5]) that *the unique AMLE of F (defined on $\partial\Omega$) to $\overline{\Omega}$ is the unique solution to*

$$\begin{cases} -\Delta_\infty u(x) = 0 & x \in \Omega, \\ u(x) = F(x) & x \in \partial\Omega. \end{cases}$$

Our main aim in this section is to describe a game that approximates this problem in the same way as problems involving the random walk described in the previous section approximate harmonic functions.

4.1. **Description of the game.** We follow [51] and [12], but we restrict ourselves to the case of a game in a bounded smooth domain $\Omega \subset \mathbb{R}^N$ (the results presented in [51] are valid in general length spaces).

A Tug-of-War is a two-person, zero-sum game, that is, two players are in contest and the total earnings of one are the losses of the other. Hence, one of them, say Player I, plays trying to maximize his expected outcome, while the other, say Player II is trying to minimize Player I's outcome (or, since the game is zero-sum, to maximize his own outcome).

Let us describe briefly the game introduced in [51] by Y. Peres, O. Schramm, S. Sheffield and D. Wilson. Consider a bounded domain $\Omega \subset \mathbb{R}^N$, and take $\Gamma_D \subset \partial\Omega$ and $\Gamma_N \equiv \partial\Omega \setminus \Gamma_D$.

Let $F : \Gamma_D \to \mathbb{R}$ be a Lipschitz continuous function. At an initial time, a token is placed at a point $x_0 \in \overline{\Omega} \setminus \Gamma_D$. Then, a (fair) coin is tossed and the winner of the toss is allowed to move the game position to any $x_1 \in \overline{B_\epsilon(x_0)} \cap \overline{\Omega}$. At each turn, the coin is tossed again, and the winner chooses a new game state $x_k \in \overline{B_\epsilon(x_{k-1})} \cap \overline{\Omega}$. Once the token has reached some $x_\tau \in \Gamma_D$, the game ends and Player I earns $F(x_\tau)$ (while Player II earns $-F(x_\tau)$). This is the reason why we will refer to F as the *final payoff function.* In more general models, it is considered also a *running payoff* $g(x)$ defined in Ω, which represents the reward (respectively, the cost) at each intermediate state x, and gives rise to nonhomogeneous problems. We will assume here that $g \equiv 0$. This procedure yields a sequence of game states $x_0, x_1, x_2, \ldots, x_\tau$, where every x_k except x_0 are random variables, depending on the coin tosses and the strategies adopted by the players.

Now we want to give a precise definition of the *value of the game.* To this end we have to introduce some notation and put the game into its normal or strategic form (see [52] and [47]). The initial state $x_0 \in \overline{\Omega} \setminus \Gamma_D$ is known to both players (public knowledge). Each player i chooses an *action* $a_0^i \in \overline{B_\epsilon(0)}$ which is announced to the other player; this defines an action profile $a_0 = \{a_0^1, a_0^2\} \in \overline{B_\epsilon(0)} \times \overline{B_\epsilon(0)}$. Then, the new state $x_1 \in \overline{B_\epsilon(x_0)}$ (namely, the current state plus the action) is selected according to a probability distribution $p(\cdot|x_0, a_0)$ in $\overline{\Omega}$ which, in our case, is given by the fair coin toss. At stage k, knowing the history $h_k = (x_0, a_0, x_1, a_1, \ldots, a_{k-1}, x_k)$, (the sequence of states and actions up to that stage), each player i chooses an action a_k^i. If the game ends at time $j < k$, we set $x_m = x_j$ and $a_m = 0$ for $j \le m \le k$. The current state x_k and the profile $a_k = \{a_k^1, a_k^2\}$ determine the distribution $p(\cdot|x_k, a_k)$ (again given by the fair coin toss) of the new state x_{k+1}.

Denote $H_k = (\overline{\Omega} \setminus \Gamma_D) \times \left(\overline{B_\epsilon(0)} \times \overline{B_\epsilon(0)} \times \overline{\Omega}\right)^k$, the set of *histories up to stage* k, and by $H = \bigcup_{k \ge 1} H_k$ the set of all histories. Notice that H_k, as a product space, has a measurable structure. The *complete history space* H_∞ is the set of plays defined as infinite sequences $(x_0, a_0, \ldots, a_{k-1}, x_k, \ldots)$ endowed with the product topology. Then, the final payoff for Player I, i.e. F, induces a Borel-measurable function on H_∞. A *pure strategy* $S_i = \{S_i^k\}_k$ for Player i, is a sequence of mappings from histories to actions, namely, a mapping from H to $\overline{B_\epsilon(0)}$ such that S_i^k is a Borel-measurable mapping from H_k to $\overline{B_\epsilon(0)}$ that maps histories ending with x_k to elements of $\overline{B_\epsilon(0)}$ (roughly speaking, at every stage the strategy gives the next movement for the player, provided he win the coin toss, as a function of the current state and the past history). The initial state x_0 and a profile of strategies $\{S_I, S_{II}\}$ define (by Kolmogorov's extension theorem) a unique probability $\mathbb{P}^{x_0}_{S_I,S_{II}}$ on the space of plays H_∞. We denote by $\mathbb{E}^{x_0}_{S_I,S_{II}}$ the corresponding expectation.

Then, if S_I and S_{II} denote the strategies adopted by Player I and II respectively, we define the expected payoff for player I as

$$V_{x_0,I}(S_I, S_{II}) = \begin{cases} \mathbb{E}^{x_0}_{S_I,S_{II}}[F(x_\tau)], & \text{if the game terminates a.s.} \\ -\infty, & \text{otherwise.} \end{cases}$$

Analogously, we define the expected payoff for player II as

$$V_{x_0,II}(S_I, S_{II}) = \begin{cases} \mathbb{E}^{x_0}_{S_I,S_{II}}[F(x_\tau)], & \text{if the game terminates a.s.} \\ +\infty, & \text{otherwise.} \end{cases}$$

Finally, we can define the *ϵ-value of the game for Player I* as

$$u_I^\epsilon(x_0) = \sup_{S_I} \inf_{S_{II}} V_{x_0,I}(S_I, S_{II}),$$

while the ϵ-value of the game for Player II is defined as

$$u_{II}^\epsilon(x_0) = \inf_{S_{II}} \sup_{S_I} V_{x_0,II}(S_I, S_{II}).$$

In some sense, $u_I^\epsilon(x_0), u_{II}^\epsilon(x_0)$ are the least possible outcomes that each player expects to get when the ϵ-game starts at x_0. Notice that, as in [51], we penalize severely the games that never end.

If $u_I^\epsilon = u_{II}^\epsilon := u_\epsilon$, we say that *the game has a value*. In [51] it is shown that, under very general hypotheses, that are fulfilled in the present setting, the ϵ-Tug-of-War game has a value.

All these $\epsilon-$values are Lipschitz functions with respect to the discrete distance d^ϵ given by

$$d_\varepsilon(x,y) = \begin{cases} 0 & \text{if } x = y, \\ \varepsilon \left(\left[\!\left[\frac{|x-y|}{\varepsilon} \right]\!\right] + 1 \right) & \text{if } x \neq y. \end{cases} \tag{32}$$

where $|.|$ is the Euclidean norm and $[\![r]\!]$ is defined for $r > 0$ by $[\![r]\!] := n$, if $n < r \leq n+1,\ n = 0, 1, 2, \dots$, that is,

$$d_\varepsilon(x,y) = \begin{cases} 0 & \text{if } \ x = y, \\ \varepsilon & \text{if } \ 0 < |x-y| \leq \varepsilon, \\ 2\varepsilon & \text{if } \ \varepsilon < |x-y| \leq 2\varepsilon \\ \vdots & \end{cases}$$

see [51] (but in general they are not continuous). Let us present a simple example where we can compute the value of the game.

4.2. **The $1-d$ game.** Let us analyze in detail the one-dimensional game and its limit as $\epsilon \to 0$.

We set $\Omega = (0, 1)$ and play the $\epsilon-$game. To simplify we assume that $\epsilon = 1/2^n$ and that the running payoff is zero. Concerning the final payoff, we end the game at $x = 0$ (with zero final payoff) and at $x = 1$ (with final payoff equals to one). Note that, the general result from [51] applies and hence we can assert the existence of a value for this game. Nevertheless, in this simple $1-d$ case we can obtain the existence of such value by direct computations. For the moment, let us assume that there exists a value that we call u_ϵ and proceed, in several steps, with the analysis of this sequence of functions u_ϵ for ϵ small. All the calculations below hold both for u_I^ϵ and for u_{II}^ϵ.

Step 1. $u_\epsilon(0) = 0$ and $u_\epsilon(1) = 1$. Moreover, $0 \leq u_\epsilon(x) \leq 1$ (the value functions are uniformly bounded).

Step 2. u_ϵ is increasing in x and strictly positive in $(0, 1]$.

Indeed, if $x < y$ then for every pair of strategies S_I, S_{II} for Player I and II beginning at x we can construct strategies beginning at y in such a way that

$$x_{i,x} \leq x_{i,y}$$

(here $x_{i,x}$ and $x_{i,y}$ are the positions of the game after i movements beginning at x and y respectively). In fact, just reproduce the movements shifting points by $y-x$ when possible (if not, that is, if the jump is too large and ends outside the interval, just remain at the larger interior position $x = 1$). In this way we see that the probability of reaching $x = 1$ beginning at y is bigger than the probability of reaching $x = 0$ and hence, taking expectations, infimum and supremum, it follows that

$$u_\epsilon(x) \leq u_\epsilon(y).$$

Now, we just observe that there is a positive probability of obtaining a sequence of $1/\epsilon$ consecutive heads (exactly $2^{-1/\epsilon}$), hence the probability of reaching $x = 1$ when the first player uses the strategy that points ϵ to the right is strictly positive. Therefore,

$$u_\epsilon(x) > 0,$$

for every $x \neq 0$.

Step 3. In this one dimensional case it is easy to identify the optimal strategies for players I and II: to jump ϵ to the right for Player I and to jump ϵ to the left for Player II. That is, if we are at x, the optimal strategies lead to

$$x \to \min\{x+\epsilon, 1\}$$

for Player I, and to

$$x \to \max\{x-\epsilon, 0\}$$

for Player II.

This follows from step 2, where we have proved that the function u_ϵ is increasing in x. As a consequence, the optimal strategies follow: for instance, Player I will choose the point where the expected payoff is maximized and this is given by $\min\{x+\epsilon, 1\}$,

$$\sup_{z\in[x-\epsilon,x+\epsilon]\cap[0,1]} u_\epsilon(z) = \max_{z\in[x-\epsilon,x+\epsilon]\cap[0,1]} u_\epsilon(z) = u_\epsilon(\min\{x+\epsilon, 1\}),$$

since u_ϵ is increasing.

This is also clear from the following intuitive fact: player I wants to maximize the payoff (reaching $x = 1$) and player II wants to minimize the payoff (hence pointing to 0).

Step 4. u_ϵ is constant in every interval of the form $(k\epsilon, (k+1)\epsilon)$ for $k = 1, ..., N$ (we denote by N the total number of such intervals in $(0, 1]$).

Indeed, from step 3 we know what are the optimal strategies for both players, and hence the result follows noticing that the number of steps that one has to advance to reach $x = 0$ (or $x = 1$) is the same for every point in $(k\epsilon, (k+1)\epsilon)$.

Remark 4.3. *Note that u_ϵ is necessarily discontinuos at every point of the form $y_k = k\epsilon \in (0, 1)$.*

Step 5. Let us call $a_k := u_\epsilon \mid_{(k\epsilon,(k+1)\epsilon)}$. Then we have

$$a_0 = 0, \qquad a_k = \frac{1}{2}(a_{k-1} + a_{k+1}),$$

for every $i = 2, ..., n-1$, and

$$a_n = 1.$$

Notice that these identities follow from the Dynamic Programming Principle, using that from step 3 we know the optimal strategies, that from step 4 u_ϵ is constant in every subinterval of the form $(k\epsilon, (k+1)\epsilon)$, we immediately get the conclusion.

Remark 4.4. *Note the similarity with a finite difference scheme used to solve $u_{xx} = 0$ in $(0, 1)$ with boundary conditions $u(0) = 0$ and $u(1) = 1$. In fact, a discretization of this problem in a uniform mesh of size ϵ leads to the same formulas obtained in step 5.*

Step 6. We have

$$u_\epsilon(x) = \epsilon k, \qquad x \in (k\epsilon, (k+1)\epsilon). \tag{33}$$

Indeed, the constants

$$a_k = \epsilon k$$

are the unique solution to the formulas obtained in step 5.

Remark 4.5. *Since formula (33) is in fact valid for u_I^ϵ and u_{II}^ϵ, this proves that the game has a value.*

Remark 4.6. *Note that u_ϵ verifies that*

$$0 \leq u_\epsilon(x) - u_\epsilon(y) \leq 2(x - y)$$

for every $x > y$ with $x - y > \epsilon$. This is a sort of equicontinuity valid for "far apart points".

In this one dimensional case, we can pass to the limit directly, by using the explicit formula for u_ϵ (see Step 7 below). However, in the N-dimensional case there is no explicit formula, and then we will need a compactness result (a sort of Arzela-Ascoli lemma).

Step 7.

$$\lim_{\epsilon \to 0} u_\epsilon(x) = x,$$

uniformly in $[0, 1]$.

Indeed, this follows from the explicit formula for u_ϵ in every interval of the form $(k\epsilon, (k+1)\epsilon)$ found in step 6 and from the monotonicity stated in step 2 (to take care of the values of u_ϵ at points of the form $k\epsilon$, we have $a_{k-1} \leq u_\epsilon(k\epsilon) \leq a_k$).

Remark 4.7. *Note that the limit function*

$$u(x) = x$$

is the unique viscosity (and classical) solution to

$$\Delta_\infty u(x) = (u_{xx}(u_x)^2)(x) = 0 \qquad x \in (0, 1),$$

with boundary conditions

$$u(0) = 0, \qquad u(1) = 1.$$

Remark 4.8. *Notice that an alternative approach to the previous analysis can be done by using the theory of Markov chains.*

4.3. **Mixed boundary conditions for Δ_∞.** Now we continue the analysis of the Tug-of-War game described previously. As before we assume that we are in the general case of a bounded domain Ω in $\mathbb{R}^N$. The game ends when the position reaches one part of the boundary Γ_D (where there is a specified final payoff F) and look for the condition that the limit must verify on the rest of it, $\partial\Omega \setminus \Gamma_D$.

All these $\epsilon-$values are Lipschitz functions with respect to the discrete distance d^ϵ defined in (32), see [51] (but in general they are not continuous as the one-dimensional example shows), which converge uniformly when $\epsilon \to 0$. The uniform limit as $\epsilon \to 0$ of the game values u_ϵ is called *the continuous value* of the game that we will denote by u and it can be seen (see below) that u is a viscosity solution to the problem

$$\left\{ \begin{array}{ll} -\Delta_\infty u(x) = 0 & x \in \Omega, \\ u(x) = F(x) & x \in \Gamma_D. \end{array} \right. \tag{34}$$

When $\Gamma_D \equiv \partial\Omega$ it is known that this problem has a unique viscosity solution, (as proved in [30]; see also [7], [15], and in a more general framework, [51]).

However, when $\Gamma_D \neq \partial\Omega$ the PDE problem (34) is incomplete, since there is a missing boundary condition on $\Gamma_N = \partial\Omega \setminus \Gamma_D$. Our concern now is to find the boundary condition that completes the problem. Assuming that Γ_N is regular, in the sense that the normal vector field $\vec{n}(x)$ is well defined and continuous for all $x \in \Gamma_N$, it is proved in [12] that it is in fact the homogeneous Neumann boundary condition

$$\frac{\partial u}{\partial n}(x) = 0\,, \quad x \in \Gamma_N.$$

On the other hand, instead of using the beautiful and involved proof based on game theory arguments, written in [51], we give here (based on [12]) an alternative proof of the property $-\Delta_\infty u = 0$ in Ω, by using direct viscosity techniques, perhaps more natural in this context. The key point in our proof is the *Dynamic Programming Principle*, that in our case reads as follows: the value of the game u_ϵ verifies

$$2u_\epsilon(x) = \sup_{y \in \overline{B_\epsilon(x)} \cap \bar{\Omega}} u_\epsilon(y) + \inf_{y \in \overline{B_\epsilon(x)} \cap \bar{\Omega}} u_\epsilon(y) \qquad \forall x \in \bar{\Omega} \setminus \Gamma_D,$$

where $B_\epsilon(x)$ denotes the open ball of radius ϵ centered at x.

This Dynamic Programming Principle, in some sense, plays the role of the mean property for harmonic functions in the infinity-harmonic case. This principle turns out to be an important qualitative property of the approximations of infinity-harmonic functions, and is the main tool to construct convergent numerical methods in this kind of problems; see [48].

We have the following result.

Theorem 4.9. *Let $u(x)$ be the continuous value of the Tug-of-War game described above (as introduced in* [51]*). Assume that $\partial\Omega = \Gamma_N \cup \Gamma_D$, where Γ_N is of class C^1, and F is a Lipschitz function defined on Γ_D. Then,*

i) *$u(x)$ is a viscosity solution to the mixed boundary value problem*

$$\begin{cases} -\Delta_\infty u(x) = 0 & x \in \Omega, \\ \dfrac{\partial u}{\partial n}(x) = 0 & x \in \Gamma_N, \\ u(x) = F(x) & x \in \Gamma_D. \end{cases} \tag{35}$$

ii) *Reciprocally, assume that Ω verifies that for every $z \in \overline{\Omega}$ and every $x^* \in \Gamma_N$ $z \neq x^*$ that*

$$\left\langle \frac{x^* - z}{|x^* - z|}; n(x^*) \right\rangle > 0.$$

Then, if $u(x)$ is a viscosity solution to (35)*, it coincides with the unique continuous value of the game.*

The hypothesis imposed on Ω in part ii) holds whenever Ω is strictly convex. The first part of the theorem comes as a consequence of the Dynamic Programming Principle read in the viscosity sense.

The proof of the second part is not included in this work. We refer to [12] for details and remark that the proof uses that the continuous value of the game is determined by the fact that it enjoys comparison with quadratic functions in the sense described in [51].

We have found a PDE problem, (35), which allows to find both the continuous value of the game and the AMLE of the Dirichlet data F (which is given only on a subset of the boundary) to $\overline{\Omega}$. To summarize, we point out that a complete equivalence holds, in the following sense:

Theorem 4.10. *It holds*

u is AMLE of $F|_{\Gamma_D}$ in $\overline{\Omega} \Leftrightarrow u$ is the limit of the values of the game $\Leftrightarrow u$ solves (35).

The first equivalence was proved in [51] and the second one is just Theorem 4.9.

Another consequence of Theorem 4.9 is the following:

Corollary 4.11. *There exists a unique viscosity solution to* (35).

The existence of a solution is a consequence of the existence of a continuous value for the game together with part i) in the previous theorem, while the uniqueness follows by uniqueness of the value of the game and part ii).

Note that to obtain uniqueness we have to invoke the uniqueness of the game value. It should be desirable to obtain a direct proof (using only PDE methods) of existence and uniqueness for (35) but it is not clear how to find the appropriate perturbations near Γ_N

to obtain uniqueness (existence follows easily by taking the limit as $p \to \infty$ in the mixed boundary value problem problem for the $p-$laplacian).

4.4. **The continuous value of the game is a viscosity solution to the mixed problem.** Our aim in the present section is to prove that u satisfies

$$\begin{cases} -\Delta_\infty u(x) = 0 & x \in \Omega, \\ \dfrac{\partial u}{\partial n}(x) = 0 & x \in \Gamma_N, \\ u(x) = F(x) & x \in \Gamma_D, \end{cases} \tag{36}$$

in the viscosity sense, where

$$\Delta_\infty u(x) = \begin{cases} \left\langle D^2 u(x) \dfrac{\nabla u(x)}{|\nabla u(x)|}, \dfrac{\nabla u(x)}{|\nabla u(x)|} \right\rangle, & \text{if } \nabla u(x) \neq 0, \\ \lim\limits_{y \to x} \dfrac{2\big(u(y) - u(x)\big)}{|y-x|^2}, & \text{otherwise.} \end{cases} \tag{37}$$

In defining $\Delta_\infty u$ we have followed [51]. Let us point out that it is possible to define the infinity laplacian at points with zero gradient in an alternative way, as in [31]. However, it is easy to see that both definitions are equivalent.

To motivate the above definition, notice that $\Delta_\infty u$ is the second derivative of u in the direction of the gradient. In fact, if u is a C^2 function and we take a direction v, then the second derivative of u in the direction of v is

$$D_v^2 u(x) = \frac{d^2}{dt^2}\bigg|_{t=0} u(x+tv) = \sum_{i,j=1}^{n} \frac{\partial^2 u}{\partial x_i \partial x_j}(x) v_i v_j.$$

If $\nabla u(x) \neq 0$, we can take $v = \dfrac{\nabla u(x)}{|\nabla u(x)|}$, and get $\Delta_\infty u(x) = D_v^2 u(x)$.

In points where $\nabla u(x) = 0$, no direction is preferred, and then expression (37) arises from the second-order Taylor's expansion of u at the point x,

$$\frac{2(u(y)-u(x))}{|y-x|^2} = \left\langle D^2 u(x) \frac{y-x}{|y-x|}, \frac{y-x}{|y-x|} \right\rangle + o(1).$$

We say that, at these points, $\Delta_\infty u(x)$ is defined if $D^2 u(x)$ is the same in every direction, that is, if the limit $\dfrac{(u(y)-u(x))}{|y-x|^2}$ exists as $y \to x$.

Because of the singular nature of (37) in points where $\nabla u(x) = 0$, we have to restrict our class of test functions. We will denote

$$S(x) = \left\{ \phi \in \mathcal{C}^2 \text{ near } x \text{ for which } \Delta_\infty \phi(x) \text{ has been defined} \right\},$$

this is, $\phi \in S(x)$ if $\phi \in \mathcal{C}^2$ in a neighborhood of x and either $\nabla\phi(x) \neq 0$ or $\nabla\phi(x) = 0$ and the limit

$$\lim_{y \to x} \frac{2\big(\phi(y) - \phi(x)\big)}{|y-x|^2},$$

exists.

Now, using the above discussion of the infinity laplacian, we give the precise definition of viscosity solution to (36) following [6].

Definition 4.12. *Consider the boundary value problem* (36)*. Then,*

(1) *A lower semi-continuous function u is a viscosity supersolution if for every $\phi \in S(x_0)$ such that $u - \phi$ has a strict minimum at the point $x_0 \in \overline{\Omega}$ with $u(x_0) = \phi(x_0)$ we have: If $x_0 \in \Gamma_D$,*
$$F(x_0) \le \phi(x_0);$$
if $x_0 \in \Gamma_N$, the inequality
$$\max\left\{\langle n(x_0), \nabla\phi(x_0)\rangle,\ -\Delta_\infty\phi(x_0)\right\} \ge 0$$
holds, and if $x_0 \in \Omega$ then we require
$$-\Delta_\infty\phi(x_0) \ge 0,$$
with $\Delta_\infty\phi(x_0)$ given by (37)*.*

(2) *An upper semi-continuous function u is a subsolution if for every $\psi \in S(x_0)$ such that $u - \psi$ has a strict maximum at the point $x_0 \in \overline{\Omega}$ with $u(x_0) = \psi(x_0)$ we have: If $x_0 \in \Gamma_D$,*
$$F(x_0) \ge \psi(x_0);$$
if $x_0 \in \Gamma_N$, the inequality
$$\min\left\{\langle n(x_0), \nabla\psi(x_0)\rangle,\ -\Delta_\infty\psi(x_0)\right\} \le 0$$
holds, and if $x_0 \in \Omega$ then we require
$$-\Delta_\infty\psi(x_0) \le 0,$$
with $\Delta_\infty\psi(x_0)$ given by (37)*.*

(3) *Finally, u is a viscosity solution if it is both a super- and a subsolution.*

Proof of part i) *of Theorem 4.9.* The starting point is the following Dynamic Programming Principle, which is satisfied by the value of the $\epsilon-$game (see [51]):

$$2u_\epsilon(x) = \sup_{y \in \overline{B_\epsilon(x)} \cap \bar{\Omega}} u_\epsilon(y) + \inf_{y \in \overline{B_\epsilon(x)} \cap \bar{\Omega}} u_\epsilon(y) \qquad \forall x \in \bar{\Omega} \setminus \Gamma_D, \tag{38}$$

where $B_\epsilon(x)$ denotes the open ball of radius ϵ centered at x.

Let us check that u (a uniform limit of u_ϵ) is a viscosity supersolution to (36). To this end, consider a function $\phi \in S(x_0)$ such that $u - \phi$ has a strict local minimum at x_0, this is,

$$u(x) - \phi(x) > u(x_0) - \phi(x_0), \quad x \neq x_0.$$

Without loss of generality, we can suppose that $\phi(x_0) = u(x_0)$. Let us see the inequality that these test functions satisfy, as a consequence of the Dynamic Programming Principle.

Let $\eta(\epsilon) > 0$ such that $\eta(\epsilon) = o(\epsilon^2)$. By the uniform convergence of u_ϵ to u, there exist a sequence $x_\epsilon \to x_0$ such that

$$u_\epsilon(x) - \phi(x) \geq u_\epsilon(x_\epsilon) - \phi(x_\epsilon) - \eta(\epsilon), \tag{39}$$

for every x in a fixed neighborhood of x_0.

From (39), we deduce

$$\sup_{y \in \overline{B_\epsilon(x_\epsilon)} \cap \bar\Omega} u_\epsilon(y) \geq \max_{y \in \overline{B_\epsilon(x_\epsilon)} \cap \bar\Omega} \phi(y) + u_\epsilon(x_\epsilon) - \phi(x_\epsilon) - \eta(\epsilon)$$

and

$$\inf_{y \in \overline{B_\epsilon(x_\epsilon)} \cap \bar\Omega} u_\epsilon(y) \geq \min_{y \in \overline{B_\epsilon(x_\epsilon)} \cap \bar\Omega} \phi(y) + u_\epsilon(x_\epsilon) - \phi(x_\epsilon) - \eta(\epsilon).$$

Then, we have from (38)

$$2\phi(x_\epsilon) \geq \max_{y \in \overline{B_\epsilon(x_\epsilon)} \cap \bar\Omega} \phi(y) + \min_{y \in \overline{B_\epsilon(x_\epsilon)} \cap \bar\Omega} \phi(y) - 2\eta(\epsilon). \tag{40}$$

The above expression can be read as a *Dynamic Programming Principle in the viscosity sense.*

It is clear that the uniform limit of u_ϵ, u, verifies

$$u(x) = F(x) \qquad x \in \Gamma_D.$$

In $\overline{\Omega} \setminus \Gamma_D$ there are two possibilities: $x_0 \in \Omega$ and $x_0 \in \Gamma_N$. In the former case we have to check that

$$-\Delta_\infty \phi(x_0) \geq 0, \tag{41}$$

while in the latter, what we have to prove is

$$\max\left\{ \frac{\partial \phi}{\partial n}(x_0), -\Delta_\infty \phi(x_0) \right\} \geq 0. \tag{42}$$

CASE A. First, assume that $x_0 \in \Omega$. Our aim is to prove $-\Delta_\infty \phi(x_0) \geq 0$. Notice that this is a consequence of the results in [51], nevertheless the elementary arguments below provide an alternative proof using only direct viscosity techniques.

If $\nabla \phi(x_0) \neq 0$ we proceed as follows. Since $\nabla \phi(x_0) \neq 0$ we also have $\nabla \phi(x_\epsilon) \neq 0$ for ϵ small enough.

In the sequel, $x_1^\epsilon, x_2^\epsilon \in \bar\Omega$ will be the points such that

$$\phi(x_1^\epsilon) = \max_{y \in \overline{B_\epsilon(x_\epsilon)} \cap \bar\Omega} \phi(y) \qquad \text{and} \quad \phi(x_2^\epsilon) = \min_{y \in \overline{B_\epsilon(x_\epsilon)} \cap \bar\Omega} \phi(y).$$

We remark that $x_1^\epsilon, x_2^\epsilon \in \partial B_\epsilon(x_\epsilon)$. Suppose to the contrary that there exists a subsequence $x_1^{\epsilon_j} \in B_{\epsilon_j}(x_{\epsilon_j})$ of maximum points of ϕ. Then, $\nabla \phi(x_1^{\epsilon_j}) = 0$ and, since $x_1^{\epsilon_j} \to x_0$ as $\epsilon_j \to 0$, we have by continuity that $\nabla \phi(x_0) = 0$, a contradiction. The argument for x_2^ϵ is similar.

Hence, since $\overline{B_\epsilon(x_\epsilon)} \cap \partial\Omega = \emptyset$, we have

$$x_1^\epsilon = x_\epsilon + \epsilon \left[\frac{\nabla \phi(x_\epsilon)}{|\nabla \phi(x_\epsilon)|} + o(1) \right], \quad \text{and} \quad x_2^\epsilon = x_\epsilon - \epsilon \left[\frac{\nabla \phi(x_\epsilon)}{|\nabla \phi(x_\epsilon)|} + o(1) \right] \tag{43}$$

as $\epsilon \to 0$. This can be deduced from the fact that, for ϵ small enough ϕ is approximately the same as its tangent plane. In fact, if we write $x_1^\epsilon = x_\epsilon + \epsilon v^\epsilon$ with $|v^\epsilon| = 1$, and we fix any direction w, then the Taylor expansion of ϕ gives

$$\phi(x_\epsilon) + \langle \nabla \phi(x_\epsilon), \epsilon v^\epsilon \rangle + o(\epsilon) = \phi(x_1^\epsilon) \geq \phi(x_\epsilon + \epsilon w)$$

and hence

$$\langle \nabla \phi(x_\epsilon), v^\epsilon \rangle + o(1) \geq \frac{\phi(x_\epsilon + \epsilon w) - \phi(x_\epsilon)}{\epsilon} = \langle \nabla \phi(x_\epsilon), w \rangle + o(1)$$

for any direction w. This implies

$$v^\epsilon = \frac{\nabla \phi(x_\epsilon)}{|\nabla \phi(x_\epsilon)|} + o(1).$$

Now, consider the Taylor expansion of second order of ϕ

$$\phi(y) = \phi(x_\epsilon) + \nabla \phi(x_\epsilon) \cdot (y - x_\epsilon) + \frac{1}{2} \langle D^2 \phi(x_\epsilon)(y - x_\epsilon), (y - x_\epsilon) \rangle + o(|y - x_\epsilon|^2)$$

as $|y - x_\epsilon| \to 0$. Evaluating the above expansion at the point at which ϕ attains its minimum in $\overline{B_\epsilon(x_\epsilon)}$, x_2^ϵ, we get

$$\phi(x_2^\epsilon) = \phi(x_\epsilon) + \nabla \phi(x_\epsilon)(x_2^\epsilon - x_\epsilon) + \frac{1}{2} \langle D^2 \phi(x_\epsilon)(x_2^\epsilon - x_\epsilon), (x_2^\epsilon - x_\epsilon) \rangle + o(\epsilon^2),$$

as $\epsilon \to 0$. Evaluating at its symmetric point in the ball $\overline{B_\epsilon(x_\epsilon)}$, that is given by

$$\tilde{x}_2^\epsilon = 2x_\epsilon - x_2^\epsilon \tag{44}$$

we get

$$\phi(\tilde{x}_2^\epsilon) = \phi(x_\epsilon) - \nabla \phi(x_\epsilon)(x_2^\epsilon - x_\epsilon) + \frac{1}{2} \langle D^2 \phi(x_\epsilon)(x_2^\epsilon - x_\epsilon), (x_2^\epsilon - x_\epsilon) \rangle + o(\epsilon^2).$$

Adding both expressions we obtain

$$\phi(\tilde{x}_2^\epsilon) + \phi(x_2^\epsilon) - 2\phi(x_\epsilon) = \langle D^2 \phi(x_\epsilon)(x_2^\epsilon - x_\epsilon), (x_2^\epsilon - x_\epsilon) \rangle + o(\epsilon^2).$$

We observe that, by our choice of x_2^ϵ as the point where the minimum is attained,

$$\phi(\tilde{x}_2^\epsilon) + \phi(x_2^\epsilon) - 2\phi(x_\epsilon) \leq \max_{y \in \overline{B_\epsilon(x)} \cap \bar{\Omega}} \phi(y) + \min_{y \in \overline{B_\epsilon(x)} \cap \bar{\Omega}} \phi(y) - 2\phi(x_\epsilon) \leq \eta(\epsilon).$$

Therefore

$$0 \geq \langle D^2 \phi(x_\epsilon)(x_2^\epsilon - x_\epsilon), (x_2^\epsilon - x_\epsilon) \rangle + o(\epsilon^2).$$

Note that from (43) we get

$$\lim_{\epsilon \to 0} \frac{x_2^\epsilon - x_\epsilon}{\epsilon} = -\frac{\nabla \phi}{|\nabla \phi|}(x_0).$$

Then we get, dividing by ϵ^2 and passing to the limit,

$$0 \leq -\Delta_\infty \phi(x_0).$$

Now, if $\nabla\phi(x_0) = 0$ we can argue exactly as above and moreover, we can suppose (considering a subsequence) that

$$\frac{(x_2^\epsilon - x_\epsilon)}{\epsilon} \to v_2 \qquad \text{as } \epsilon \to 0,$$

for some $v_2 \in \mathbb{R}^n$. Thus

$$0 \le -\left\langle D^2\phi(x_0)v_2, v_2\right\rangle = -\Delta_\infty\phi(x_0)$$

by definition, since $\phi \in S(x_0)$.

CASE B. Suppose that $x_0 \in \Gamma_N$. There are four sub-cases to be considered depending on the direction of the gradient $\nabla\phi(x_0)$ and the distance of the points x_ϵ to the boundary.

CASE 1: If either $\nabla\phi(x_0) = 0$, or $\nabla\phi(x_0) \neq 0$ and $\nabla\phi(x_0) \perp n(x_0)$, then

$$\frac{\partial\phi}{\partial n}(x_0) = 0 \quad \Rightarrow \quad \max\left\{\frac{\partial\phi}{\partial n}(x_0), -\Delta_\infty\phi(x_0)\right\} \ge 0, \tag{45}$$

where

$$\Delta_\infty\phi(x_0) = \lim_{y\to x_0} \frac{2\big(\phi(y) - \phi(x_0)\big)}{|y - x_0|^2}$$

is well defined since $\phi \in S(x_0)$.

CASE 2: $\displaystyle\liminf_{\epsilon\to 0} \frac{\operatorname{dist}(x_\epsilon, \partial\Omega)}{\epsilon} > 1$, and $\nabla\phi(x_0) \neq 0$.

Since $\nabla\phi(x_0) \neq 0$ we also have $\nabla\phi(x_\epsilon) \neq 0$ for ϵ small enough. Hence, since $B_\epsilon(x_\epsilon) \cap \partial\Omega = \emptyset$, we have, as before,

$$x_1^\epsilon = x_\epsilon + \epsilon\left[\frac{\nabla\phi(x_\epsilon)}{|\nabla\phi(x_\epsilon)|} + o(1)\right], \quad \text{and} \quad x_2^\epsilon = x_\epsilon - \epsilon\left[\frac{\nabla\phi(x_\epsilon)}{|\nabla\phi(x_\epsilon)|} + o(1)\right]$$

as $\epsilon \to 0$. Notice that both $x_1^\epsilon, x_2^\epsilon \to \partial B_\epsilon(x_\epsilon)$. This can be deduced from the fact that, for ϵ small enough ϕ is approximately the same as its tangent plane.

Then we can argue exactly as before (when $x_0 \in \Omega$) to obtain that

$$0 \le -\Delta_\infty\phi(x_0).$$

CASE 3: $\displaystyle\limsup_{\epsilon\to 0} \frac{\operatorname{dist}(x_\epsilon, \partial\Omega)}{\epsilon} \le 1$, and $\nabla\phi(x_0) \neq 0$ points inwards Ω.

In this case, for ϵ small enough we have that $\nabla\phi(x_\epsilon) \neq 0$ points inwards as well. Thus,

$$x_1^\epsilon = x_\epsilon + \epsilon\left[\frac{\nabla\phi(x_\epsilon)}{|\nabla\phi(x_\epsilon)|} + o(1)\right] \in \Omega,$$

while $x_2^\epsilon \in \overline{\Omega} \cap \overline{B_\epsilon}(x_\epsilon)$. Indeed,

$$\frac{|x_2^\epsilon - x_\epsilon|}{\epsilon} = \delta_\epsilon \le 1.$$

We have the following first-order Taylor's expansions,

$$\phi(x_1^\epsilon) = \phi(x_\epsilon) + \epsilon|\nabla\phi(x_\epsilon)| + o(\epsilon),$$

and

$$\phi(x_2^\epsilon) = \phi(x_\epsilon) + \nabla\phi(x_\epsilon)\cdot(x_2^\epsilon - x_\epsilon) + o(\epsilon),$$

as $\epsilon \to 0$. Adding both expressions, we arrive at

$$\phi(x_1^\epsilon) + \phi(x_2^\epsilon) - 2\phi(x_\epsilon) = \epsilon|\nabla\phi(x_\epsilon)| + \nabla\phi(x_\epsilon)\cdot(x_2^\epsilon - x_\epsilon) + o(\epsilon).$$

Using (40) and dividing by $\epsilon > 0$,

$$0 \geq |\nabla\phi(x_\epsilon)| + \nabla\phi(x_\epsilon)\cdot\frac{(x_2^\epsilon - x_\epsilon)}{\epsilon} + o(1)$$

as $\epsilon \to 0$. We can write

$$0 \geq |\nabla\phi(x_\epsilon)|\cdot(1 + \delta_\epsilon\cos\theta_\epsilon) + o(1)$$

where

$$\theta_\epsilon = \text{angle}\left(\nabla\phi(x_\epsilon), \frac{(x_2^\epsilon - x_\epsilon)}{\epsilon}\right).$$

Letting $\epsilon \to 0$ we get

$$0 \geq |\nabla\phi(x_0)|\cdot(1 + \delta_0\cos\theta_0),$$

where $\delta_0 \leq 1$, and

$$\theta_0 = \lim_{\epsilon\to 0}\theta_\epsilon = \text{angle}\left(\nabla\phi(x_0), v(x_0)\right),$$

with

$$v(x_0) = \lim_{\epsilon\to 0}\frac{x_2^\epsilon - x_\epsilon}{\epsilon}.$$

Since $|\nabla\phi(x_0)| \neq 0$, we find out $(1 + \delta_0\cos\theta_0) \leq 0$, and then $\theta_0 = \pi$ and $\delta_0 = 1$. Hence

$$\lim_{\epsilon\to 0}\frac{x_2^\epsilon - x_\epsilon}{\epsilon} = -\frac{\nabla\phi}{|\nabla\phi|}(x_0), \tag{46}$$

or what is equivalent,

$$x_2^\epsilon = x_\epsilon - \epsilon\left[\frac{\nabla\phi(x_\epsilon)}{|\nabla\phi(x_\epsilon)|} + o(1)\right].$$

Now, consider $\tilde{x}_2^\epsilon = 2x_\epsilon - x_2^\epsilon$ the symmetric point of x_2^ϵ with respect to x_ϵ. We go back to (40) and use the Taylor expansions of second order,

$$\phi(x_2^\epsilon) = \phi(x_\epsilon) + \nabla\phi(x_\epsilon)(x_2^\epsilon - x_\epsilon) + \frac{1}{2}\langle D^2\phi(x_\epsilon)(x_2^\epsilon - x_\epsilon), (x_2^\epsilon - x_\epsilon)\rangle + o(\epsilon^2),$$

and

$$\phi(\tilde{x}_2^\epsilon) = \phi(x_\epsilon) + \nabla\phi(x_\epsilon)(\tilde{x}_2^\epsilon - x_\epsilon) + \frac{1}{2}\langle D^2\phi(x_\epsilon)(\tilde{x}_2^\epsilon - x_\epsilon), (\tilde{x}_2^\epsilon - x_\epsilon)\rangle + o(\epsilon^2),$$

to get

$$\begin{aligned} 0 &\geq \min_{y\in \overline{B_\epsilon(x_\epsilon)\cap\bar\Omega}} \phi(y) + \max_{y\in \overline{B_\epsilon(x_\epsilon)\cap\bar\Omega}} \phi(y) - 2\phi(x_\epsilon) \\ &\geq \phi(x_2^\epsilon) + \phi(\tilde{x}_2^\epsilon) - 2\phi(x_\epsilon) \\ &= \nabla\phi(x_\epsilon)(x_2^\epsilon - x_\epsilon) + \nabla\phi(x_\epsilon)(\tilde{x}_2^\epsilon - x_\epsilon) + \frac{1}{2}\langle D^2\phi(x_\epsilon)(x_2^\epsilon - x_\epsilon), (x_2^\epsilon - x_\epsilon)\rangle \\ &\quad + \frac{1}{2}\langle D^2\phi(x_\epsilon)(\tilde{x}_2^\epsilon - x_\epsilon), (\tilde{x}_2^\epsilon - x_\epsilon)\rangle + o(\epsilon^2), \\ &= \langle D^2\phi(x_\epsilon)(x_2^\epsilon - x_\epsilon), (x_2^\epsilon - x_\epsilon)\rangle + o(\epsilon^2), \end{aligned}$$

by the definition of $\tilde{x}_2^\epsilon$. Then, we can divide by ϵ^2 and use (46) to obtain

$$-\Delta_\infty\phi(x_0) \geq 0.$$

CASE 4: $\limsup_{\epsilon\to 0} \dfrac{\operatorname{dist}(x_\epsilon, \partial\Omega)}{\epsilon} \leq 1$, and $\nabla\phi(x_0) \neq 0$ points outwards Ω.

In this case we have

$$\frac{\partial\phi}{\partial n}(x_0) = \nabla\phi(x_0)\cdot n(x_0) \geq 0,$$

since $n(x_0)$ is the exterior normal at x_0 and $\nabla\phi(x_0)$ points outwards Ω. Thus

$$\max\left\{\frac{\partial\phi}{\partial n}(x_0), -\Delta_\infty\phi(x_0)\right\} \geq 0,$$

and we conclude that u is a viscosity supersolution of (36).

It remains to check that u is a viscosity subsolution of (36). This fact can be proved in an analogous way, taking some care in the choice of the points where we perform Taylor expansions. In fact, instead of taking (44) we have to choose

$$\tilde{x}_1^\epsilon = 2x_\epsilon - x_1^\epsilon,$$

that is, the reflection of the point where the maximum in the ball $\overline{B_\epsilon(x_\epsilon)}$ of the test function is attained.

This ends the proof. □

4.5. **Comments.**

(1) When we add a running cost of the form $\epsilon^2 g(x)$, with g nonnegative (without this condition the game may not have a value, see [51]) we obtain a solution to

$$\begin{cases} -\Delta_\infty u(x) = g(x) & x\in\Omega, \\ \dfrac{\partial u}{\partial n}(x) = 0 & x\in\Gamma_N, \\ u(x) = F(x) & x\in\Gamma_D. \end{cases}$$

(2) Many of the results presented here are valid in general length spaces (see [51]) and on graphs. In case of a graph the next positions that can be selected by both players are the points that are connected with the actual position of the game. This leads to consider the infinity laplacian on a graph.

(3) One can consider games in which the coin is biased, that is the probability of getting a head is p (with $p \neq 1/2$). In this case the limit as $\epsilon \to 0$ (with $p = p(\epsilon) \to 1/2$) was analyzed in [50] and the PDE that appears reads as

$$\Delta_\infty u + \beta |\nabla u| = 0.$$

5. $p-$harmonious functions

The aim of this section is to describe games that approximate the $p-$Laplacian that is give by

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u).$$

We assume that $2 \leq p < \infty$. The case $p = \infty$ was considered in the previous section.

5.1. **Description of the game.** Consider a two-player zero-sum-game in a domain Ω described as follows: starting from a point $x_0 \in \Omega$, Players I and II play the original tug-of-war game described in [51] (see the previous section for details) with probability α, and with probability β (recall that $\alpha + \beta = 1$), a random point in $B_\varepsilon(x_0)$ is chosen. Once the game position reaches a strip near the boundary of width ϵ, Player II pays Player I the amount given by a pay-off function. Naturally, Player I tries to maximize and Player II to minimize the payoff. Hence, the equation

$$u_\varepsilon(x) = \frac{\alpha}{2} \left\{ \sup_{\overline{B}_\varepsilon(x)} u_\varepsilon + \inf_{\overline{B}_\varepsilon(x)} u_\varepsilon \right\} + \beta \fint_{B_\varepsilon(x)} u_\varepsilon \, dy,$$

describes the expected payoff of the above game. Intuitively, the expected payoff at the point can be calculated by summing up all the three cases: Player I moves, Player II moves, or a random point is chosen, with their corresponding probabilities.

In this variant of tug-of-war with noise the noise is distributed uniformly on $B_\varepsilon(x)$. This approach allows us to use the dynamic programming principle in the form

$$u_\varepsilon(x) = \frac{\alpha}{2} \left\{ \sup_{\overline{B}_\varepsilon(x)} u_\varepsilon + \inf_{\overline{B}_\varepsilon(x)} u_\varepsilon \right\} + \beta \fint_{B_\varepsilon(x)} u_\varepsilon \, dy, \tag{47}$$

to conclude that our game has a value and that the value is p-harmonious. There are indications, see Barles-Souganidis [8] and Oberman [48], that our results based on the mean value approach are likely to be useful in applications for example to numerical methods as well as in problems of analysis, cf. Armstrong-Smart [4]. For further results on games see [9] and [39].

5.2. $p-$**harmonious functions.** The goal of this section is to study functions that satisfy (47) with fixed $\varepsilon > 0$ and suitable nonnegative α and β, with $\alpha + \beta = 1$ (we will call such functions $p-$harmonious functions).

Consider a bounded domain $\Omega \subset \mathbb{R}^N$ and fix $\varepsilon > 0$. To prescribe boundary values for $p-$harmo-nious functions, let us denote the compact boundary strip of width ε by

$$\Gamma_\varepsilon = \{x \in \mathbb{R}^N \setminus \Omega \, : \, \operatorname{dist}(x, \partial\Omega) \le \varepsilon\}.$$

Definition 5.1. *The function u_ε is p-harmonious in Ω with boundary values a bounded Borel function $F : \Gamma_\varepsilon \to \mathbb{R}$ if*

$$u_\varepsilon(x) = \frac{\alpha}{2} \left\{ \sup_{\overline{B}_\varepsilon(x)} u_\varepsilon + \inf_{\overline{B}_\varepsilon(x)} u_\varepsilon \right\} + \beta \fint_{B_\varepsilon(x)} u_\varepsilon \, dy \qquad \textit{for every} \quad x \in \Omega,$$

where α, β are defined in (54)*, and*

$$u_\varepsilon(x) = F(x), \qquad \textit{for every} \quad x \in \Gamma_\varepsilon.$$

The reason for using the boundary strip Γ_ε instead of simply using the boundary $\partial\Omega$ is the fact that for $x \in \Omega$ the ball $\overline{B}_\varepsilon(x)$ is not necessarily contained in Ω.

Let us first explain the name p-harmonious. When u is harmonic, then it satisfies the well known mean value property

$$u(x) = \fint_{B_\varepsilon(x)} u \, dy, \tag{48}$$

that is (47) with $\alpha = 0$ and $\beta = 1$. On the other hand, functions satisfying (47) with $\alpha = 1$ and $\beta = 0$

$$u_\varepsilon(x) = \frac{1}{2} \left\{ \sup_{\overline{B}_\varepsilon(x)} u_\varepsilon + \inf_{\overline{B}_\varepsilon(x)} u_\varepsilon \right\} \tag{49}$$

are called *harmonious* functions in [28] and [29] and are values of Tug-of-War games like the ones described in the previous section. As we have seen, as ε goes to zero, they approximate solutions to the infinity Laplacian.

Now, recall that the p-Laplacian is given by

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u) = |\nabla u|^{p-2} \left\{ (p-2)\Delta_\infty u + \Delta u \right\}. \tag{50}$$

Since the p-Laplace operator can be written as a combination of the Laplace operator and the infinity Laplacian, it seems reasonable to expect that the combination (47) of the averages in (48) and (49) give an approximation to a solution to the p-Laplacian. We will show that this is indeed the case. To be more precise, we prove that p-harmonious functions are uniquely determined by their boundary values and that they converge uniformly to the p-harmonic function with the given boundary data. Furthermore, we show that p-harmonious functions satisfy the strong maximum and comparison principles. Observe that the validity of the strong comparison principle is an open problem for the solutions of the p-Laplace equation in $\mathbb{R}^N$, $N \ge 3$.

5.3. **A heuristic argument.** It follows from expansion (50) that u is a solution to $\Delta_p u = 0$ if and only if

$$(p-2)\Delta_\infty u + \Delta u = 0, \tag{51}$$

because this equivalence can be justified in the viscosity sense even when $\nabla u = 0$ as shown in [36]. Averaging the classical Taylor expansion

$$u(y) = u(x) + \nabla u(x) \cdot (y-x) + \frac{1}{2}\langle D^2 u(x)(y-x), (y-x)\rangle + O(|y-x|^3),$$

over $B_\varepsilon(x)$, we obtain

$$u(x) - \fint_{B_\varepsilon(x)} u\, dy = -\frac{\varepsilon^2}{2(n+2)}\Delta u(x) + O(\varepsilon^3), \tag{52}$$

when u is smooth. Here we used the shorthand notation

$$\fint_{B_\varepsilon(x)} u\, dy = \frac{1}{|B_\varepsilon(x)|}\int_{B_\varepsilon(x)} u\, dy.$$

Then observe that gradient direction is almost the maximizing direction. Thus, summing up the two Taylor expansions roughly gives us

$$\begin{aligned} & u(x) - \frac{1}{2}\left\{ \sup_{\overline{B}_\varepsilon(x)} u + \inf_{\overline{B}_\varepsilon(x)} u \right\} \\ & \approx u(x) - \frac{1}{2}\left\{ u\left(x + \varepsilon\frac{\nabla u(x)}{|\nabla u(x)|}\right) + u\left(x - \varepsilon\frac{\nabla u(x)}{|\nabla u(x)|}\right)\right\} \\ & = -\frac{\varepsilon^2}{2}\Delta_\infty u(x) + O(\varepsilon^3). \end{aligned} \tag{53}$$

Next we multiply (52) and (53) by suitable constants α and β, $\alpha+\beta=1$, and add up the formulas to obtain

$$u(x) - \frac{\alpha}{2}\left\{ \sup_{\overline{B}_\varepsilon(x)} u - \inf_{\overline{B}_\varepsilon(x)} u \right\} + \beta \fint_{B_\varepsilon(x)} u\, dy = -\alpha\frac{\varepsilon^2}{2}\Delta_\infty u(x) - \beta\frac{\varepsilon^2}{2(n+2)}\Delta u(x) + O(\varepsilon^3)$$

Next, we choose α and β so that we have the operator in (51) on the right hand side. This process gives us the choices of the constants

$$\alpha = \frac{p-2}{p+N}, \quad \text{and} \quad \beta = \frac{2+N}{p+N}. \tag{54}$$

and we deduce

$$u(x) = \frac{\alpha}{2}\left\{ \sup_{\overline{B}_\varepsilon(x)} u + \inf_{\overline{B}_\varepsilon(x)} u \right\} + \beta \fint_{B_\varepsilon(x)} u\, dy + O(\varepsilon^3)$$

as $\varepsilon \to 0$.

5.4. **p-harmonious functions and Tug-of-War games.** In this section, we describe the connection between p-harmonious functions and tug-of-war games. Fix $\varepsilon > 0$ and consider the two-player zero-sum-game described before. At the beginning, a token is placed at a point $x_0 \in \Omega$ and the players toss a biased coin with probabilities α and β, $\alpha + \beta = 1$. If they get heads (probability α), they play a tug-of-war, that is, a fair coin is tossed and the winner of the toss is allowed to move the game position to any $x_1 \in \overline{B}_\varepsilon(x_0)$. On the other hand, if they get tails (probability β), the game state moves according to the uniform probability to a random point in the ball $B_\varepsilon(x_0)$. Then they continue playing the same game from x_1.

This procedure yields a possibly infinite sequence of game states $x_0, x_1, \dots$ where every x_k is a random variable. We denote by $x_\tau \in \Gamma_\varepsilon$ the first point in Γ_ε in the sequence, where τ refers to the first time we hit Γ_ε. The payoff is $F(x_\tau)$, where $F : \Gamma_\varepsilon \to \mathbb{R}$ is a given *payoff function.* Player I earns $F(x_\tau)$ while Player II earns $-F(x_\tau)$.

Note that, due to the fact that $\beta > 0$, or equivalently $p < \infty$, the game ends almost surely

$$\mathbb{P}^{x_0}_{S_{\mathrm{I}},S_{\mathrm{II}}}(\{\omega \in H^\infty \colon \tau(\omega) < \infty\}) = 1$$

for any choice of strategies.

The *value of the game for Player I* is given by

$$u^\varepsilon_{\mathrm{I}}(x_0) = \sup_{S_{\mathrm{I}}} \inf_{S_{\mathrm{II}}} \mathbb{E}^{x_0}_{S_{\mathrm{I}},S_{\mathrm{II}}}[F(x_\tau)]$$

while the *value of the game for Player II* is given by

$$u^\varepsilon_{\mathrm{II}}(x_0) = \inf_{S_{\mathrm{II}}} \sup_{S_{\mathrm{I}}} \mathbb{E}^{x_0}_{S_{\mathrm{I}},S_{\mathrm{II}}}[F(x_\tau)].$$

The values $u^\varepsilon_{\mathrm{I}}(x_0)$ and $u^\varepsilon_{\mathrm{II}}(x_0)$ are the best expected outcomes each player can guarantee when the game starts at x_0.

We start by the statement of the *Dynamic Programming Principle* (DPP) applied to our game.

Lemma 5.2 (DPP)**.** *The value function for Player I satisfies*

$$\begin{aligned} u^\varepsilon_I(x_0) &= \frac{\alpha}{2}\left\{ \sup_{\overline{B}_\varepsilon(x_0)} u^\varepsilon_I + \inf_{\overline{B}_\varepsilon(x_0)} u^\varepsilon_I \right\} + \beta \fint_{B_\varepsilon(x_0)} u^\varepsilon_I \, dy, \qquad x_0 \in \Omega, \\ u^\varepsilon_I(x_0) &= F(x_0), \qquad x_0 \in \Gamma_\varepsilon. \end{aligned} \tag{55}$$

The value function for Player II, u^ε_{II}, satisfies the same equation.

Formulas similar to (55) can be found in Chapter 7 of [41]. A detailed proof adapted to our case can also be found in [44].

Let us explain intuitively why the DPP holds by considering the expectation of the payoff at x_0. Whenever the players get heads (probability α) in the first coin toss, they toss a fair coin and play the tug-of-war. If Player I wins the fair coin toss in the tug-of-war (probability 1/2), she steps to a point maximizing the expectation and if Player II wins, he steps to a point minimizing the expectation. Whenever they get tails (probability β) in the first coin toss, the game state moves to a random point according to a uniform probability on $B_\varepsilon(x_0)$. The expectation at x_0 can be obtained by summing up these different alternatives.

We warn the reader that, as happens for the tug-of-war game without noise described previously, the value functions are discontinuous in general as the next example shows.

Example 5.3. Consider the case $\Omega = (0,1)$ and

$$F(x) = u_{\rm I}^\varepsilon(x) = \begin{cases} 1, & x \geq 1 \\ 0, & x \leq 0. \end{cases}$$

In this case the optimal strategies for both players are clear: Player I moves ε to the right and Player II moves ε to the left. Now, there is a positive probability of reaching $x \geq 1$ that can be uniformly bounded from below in $(0,1)$ by $C = (2/\alpha)^{-(1/\varepsilon+1)}$. This can be seen by considering the probability of Player I winning all the time until the game ends with $x \geq 1$. Therefore $u_{\rm I}^\varepsilon > C > 0$ in the whole $(0,1)$. This implies a discontinuity at $x = 0$ and hence a discontinuity at $x = \varepsilon$. Indeed, first, note that u_ε is nondecreasing and hence

$$u_{\rm I}^\varepsilon(\varepsilon-) = \lim_{x \nearrow \varepsilon} \frac{\alpha}{2} \sup_{|x-y|\leq\varepsilon} u_{\rm I}^\varepsilon(y) + \frac{\beta}{2\varepsilon}\int_0^{2\varepsilon} u_{\rm I}^\varepsilon\, dy = \frac{\alpha}{2} u_{\rm I}^\varepsilon(2\varepsilon-) + \frac{\beta}{2\varepsilon}\int_0^{2\varepsilon} u_{\rm I}^\varepsilon\, dy,$$

because $\sup_{|x-y|\leq\varepsilon} u_{\rm I}^\varepsilon(y) = u_{\rm I}^\varepsilon(x+\varepsilon)$ and $\inf_{|x-\varepsilon|\leq\varepsilon} u_{\rm I}^\varepsilon$ is zero for $x \in (0,\varepsilon)$. However,

$$u_{\rm I}^\varepsilon(\varepsilon+) \geq \frac{\alpha}{2} C + \lim_{x \searrow \varepsilon} \frac{\alpha}{2} \sup_{|x-y|\leq\varepsilon} u_{\rm I}^\varepsilon(y) + \frac{\beta}{2\varepsilon}\int_0^{2\varepsilon} u_{\rm I}^\varepsilon\, dy \geq \frac{\alpha}{2} C + u_{\rm I}^\varepsilon(\varepsilon-)$$

because $\sup_{|x-y|\leq\varepsilon} u_{\rm I}^\varepsilon(y) = u_{\rm I}^\varepsilon(x+\varepsilon) \geq u_{\rm I}^\varepsilon(2\varepsilon-)$ and $\inf_{|x-\varepsilon|\leq\varepsilon} u_{\rm I}^\varepsilon \geq C$ for $x > \varepsilon$.

By adapting the martingale methods used in [51], we prove a comparison principle. This also implies that $u_{\rm I}^\varepsilon$ and $u_{\rm II}^\varepsilon$ are respectively the smallest and the largest p-harmonious function.

Theorem 5.4. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set. If v_ε is a p-harmonious function with boundary values F_v in Γ_ε such that $F_v \geq F_{u_I^\varepsilon}$, then $v \geq u_I^\varepsilon$.*

Proof. We show that by choosing a strategy according to the minimal values of v, Player II can make the process a supermartingale. The optional stopping theorem then implies that the expectation of the process under this strategy is bounded by v. Moreover, this process provides an upper bound for $u_{\rm I}^\varepsilon$.

Player I follows any strategy and Player II follows a strategy $S_{\rm II}^0$ such that at $x_{k-1} \in \Omega$ he chooses to step to a point that almost minimizes v, that is, to a point $x_k \in \overline{B}_\varepsilon(x_{k-1})$ such that

$$v(x_k) \leq \inf_{\overline{B}_\varepsilon(x_{k-1})} v + \eta 2^{-k}$$

for some fixed $\eta > 0$. We start from the point x_0. It follows that

$$\begin{aligned}
&\mathbb{E}^{x_0}_{S_{\mathrm{I}},S^0_{\mathrm{II}}}[v(x_k)+\eta 2^{-k}\,|\,x_0,\dots,x_{k-1}]\\
&\le \frac{\alpha}{2}\left\{\inf_{\overline{B}_\varepsilon(x_{k-1})} v+\eta 2^{-k}+\sup_{\overline{B}_\varepsilon(x_{k-1})} v\right\}+\beta\fint_{B_\varepsilon(x_{k-1})} v\,dy+\eta 2^{-k}\\
&\le v(x_{k-1})+\eta 2^{-(k-1)},
\end{aligned}$$

where we have estimated the strategy of Player I by sup and used the fact that v is p-harmonious. Thus

$$M_k = v(x_k)+\eta 2^{-k}$$

is a supermartingale. Since $F_v \ge F_{u^\varepsilon_{\mathrm{I}}}$ at Γ_ε, we deduce

$$\begin{aligned}
u^\varepsilon_{\mathrm{I}}(x_0) &= \sup_{S_{\mathrm{I}}}\inf_{S_{\mathrm{II}}}\mathbb{E}^{x_0}_{S_{\mathrm{I}},S_{\mathrm{II}}}[F_{u^\varepsilon_{\mathrm{I}}}(x_\tau)] \le \sup_{S_{\mathrm{I}}}\mathbb{E}^{x_0}_{S_{\mathrm{I}},S^0_{\mathrm{II}}}[F_v(x_\tau)+\eta 2^{-\tau}]\\
&\le \sup_{S_{\mathrm{I}}}\liminf_{k\to\infty}\mathbb{E}^{x_0}_{S_{\mathrm{I}},S^0_{\mathrm{II}}}[v(x_{\tau\wedge k})+\eta 2^{-(\tau\wedge k)}]\\
&\le \sup_{S_{\mathrm{I}}}\mathbb{E}_{S_{\mathrm{I}},S^0_{\mathrm{II}}}[M_0] = v(x_0)+\eta,
\end{aligned}$$

where $\tau\wedge k = \min(\tau,k)$, and we used Fatou's lemma as well as the optional stopping theorem for M_k. Since η was arbitrary this proves the claim. □

Similarly, we can prove that $u^\varepsilon_{\mathrm{II}}$ is the largest p-harmonious function: Player II follows any strategy and Player I always chooses to step to the point where v is almost maximized. This implies that $v(x_k)-\eta 2^{-k}$ is a submartingale. Fatou's lemma and the optional stopping theorem then prove the claim.

Next we show that the game has a value. This together with the previous comparison principle proves the uniqueness of p-harmonious functions with given boundary values.

Theorem 5.5. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set, and F a given boundary data in Γ_ε. Then $u^\varepsilon_I = u^\varepsilon_{II}$, that is, the game has a value.*

Proof. Clearly, $u^\varepsilon_{\mathrm{I}} \le u^\varepsilon_{\mathrm{II}}$ always holds, so we are left with the task of showing that $u^\varepsilon_{\mathrm{II}} \le u^\varepsilon_{\mathrm{I}}$. To see this we use the same method as in the proof of the previous theorem: Player II follows a strategy S^0_{II} such that at $x_{k-1}\in\Omega$, he always chooses to step to a point that almost minimizes $u^\varepsilon_{\mathrm{I}}$, that is, to a point x_k such that

$$u^\varepsilon_{\mathrm{I}}(x_k) \le \inf_{\overline{B}_\varepsilon(x_{k-1})} u^\varepsilon_{\mathrm{I}}+\eta 2^{-k},$$

for a fixed $\eta > 0$. We start from the point x_0. It follows that from the choice of strategies and the dynamic programming principle for $u^\varepsilon_{\mathrm{I}}$ that

$$\begin{aligned}
&\mathbb{E}^{x_0}_{S_{\mathrm{I}},S^0_{\mathrm{II}}}[u^\varepsilon_{\mathrm{I}}(x_k)+\eta 2^{-k}\,|\,x_0,\dots,x_{k-1}]\\
&\qquad\le \frac{\alpha}{2}\left\{\sup_{\overline{B}_\varepsilon(x_{k-1})} u^\varepsilon_{\mathrm{I}}+\inf_{\overline{B}_\varepsilon(x_{k-1})} u^\varepsilon_{\mathrm{I}}+\eta 2^{-k}\right\}+\beta\fint_{B_\varepsilon(x_{k-1})} u^\varepsilon_{\mathrm{I}}\,dy+\eta 2^{-k}\\
&\qquad= u^\varepsilon_{\mathrm{I}}(x_{k-1})+\eta 2^{-(k-1)}.
\end{aligned}$$

Thus

$$M_k = u_{\mathrm{I}}^{\varepsilon}(x_k) + \eta 2^{-k}$$

is a supermartingale. We get by Fatou's lemma and the optional stopping theorem that

$$\begin{aligned} u_{\mathrm{II}}^{\varepsilon}(x_0) &= \inf_{S_{\mathrm{II}}} \sup_{S_{\mathrm{I}}} \mathbb{E}_{S_{\mathrm{I}},S_{\mathrm{II}}}^{x_0}[F(x_\tau)] \le \sup_{S_{\mathrm{I}}} \mathbb{E}_{S_{\mathrm{I}},S_{\mathrm{II}}^0}^{x_0}[F(x_\tau) + \eta 2^{-\tau}] \\ &\le \sup_{S_{\mathrm{I}}} \liminf_{k\to\infty} \mathbb{E}_{S_{\mathrm{I}},S_{\mathrm{II}}^0}^{x_0}[u_{\mathrm{I}}^{\varepsilon}(x_{\tau\wedge k}) + \eta 2^{-(\tau\wedge k)}] \\ &\le \sup_{S_{\mathrm{I}}} \mathbb{E}_{S_{\mathrm{I}},S_{\mathrm{II}}^0}[u_{\mathrm{I}}^{\varepsilon}(x_0) + \eta] = u_{\mathrm{I}}^{\varepsilon}(x_0) + \eta. \end{aligned}$$

Similarly to the previous theorem, we also used the fact that the game ends almost surely. Since $\eta > 0$ is arbitrary, this completes the proof. □

Theorems 5.4 and 5.5 imply that with a fixed boundary data there exists a unique p-harmonious function.

Theorem 5.6. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Then there exists a unique p-harmonious function in Ω with given boundary values F.*

Proof of Theorem 5.6. Due to the dynamic programming principle, the value functions of the games are p-harmonious functions. This proves the existence part of Theorem 5.6.

Theorems 5.4 and 5.5 imply the uniqueness part of Theorem 5.6. □

Corollary 5.7. *The value of the game with pay-off function F coincides with the p-harmonious function with boundary values F.*

5.5. **Maximum principles for p-harmonious functions.**

In this section, we show that the strong maximum and strong comparison principles hold for p-harmonious functions.

First let us state that p-harmonious functions satisfy the *strong maximum principle.*

Theorem 5.8. *Let $\Omega \subset \mathbb{R}^N$ be a bounded, open, and connected set. If u_ε is p-harmonious in Ω with boundary values F, then*

$$\sup_{\Gamma_\varepsilon} F \ge \sup_{\Omega} u_\varepsilon.$$

Moreover, if there is a point $x_0 \in \Omega$ such that $u_\varepsilon(x_0) = \sup_{\Gamma_\varepsilon} F$, then u_ε is constant in Ω.

Proof of Theorem 5.8. The proof uses the fact that if the maximum is attained inside the domain then all the quantities in the definition of a p-harmonious function must be equal to the maximum. This is possible in a connected domain only if the function is constant.

We begin by observing that a p-harmonious function u_ε with a boundary data F satisfies

$$\sup_{\Omega} |u_\varepsilon| \le \sup_{\Gamma_\varepsilon} |F|,$$

see Lemma 5.14 below. Assume then that there exists a point $x_0 \in \Omega$ such that

$$u_\varepsilon(x_0) = \sup_{\Omega} u_\varepsilon = \sup_{\Gamma_\varepsilon} F.$$

Then we employ the definition of a p-harmonious function, Definition 5.1, and obtain

$$u_\varepsilon(x_0) = \frac{\alpha}{2}\left\{\sup_{\overline{B}_\varepsilon(x_0)} u_\varepsilon + \inf_{\overline{B}_\varepsilon(x_0)} u_\varepsilon\right\} + \beta \fint_{B_\varepsilon(x_0)} u_\varepsilon \, dy.$$

Since $u_\varepsilon(x_0)$ is the maximum, the terms

$$\sup_{\overline{B}_\varepsilon(x_0)} u_\varepsilon, \quad \inf_{\overline{B}_\varepsilon(x_0)} u_\varepsilon, \quad \text{and} \quad \fint_{B_\varepsilon(x_0)} u_\varepsilon \, dy$$

on the right hand side must be smaller than or equal to $u_\varepsilon(x_0)$. On the other hand, when $p > 2$, it follows that $\alpha, \beta > 0$ and thus the terms must equal to $u_\varepsilon(x_0)$. Therefore,

$$u_\varepsilon(x) = u_\varepsilon(x_0) = \sup_\Omega u_\varepsilon \tag{56}$$

for every $x \in B_\varepsilon(x_0)$ when $p > 2$. Now we can repeat the argument for each $x \in B_\varepsilon(x_0)$ and by continuing in this way, we can extend the result to the whole domain because Ω is connected. This implies that u is constant everywhere when $p > 2$.

Finally, if $p = 2$, then (56) holds for almost every $x \in B_\varepsilon(x_0)$ and consequently for almost every x in the whole domain. Then since

$$u(x) = \fint_{B_\varepsilon(x)} u \, dy$$

holds at every point in Ω and u is constant almost everywhere, it follows that u is constant everywhere. □

In addition, p-harmonious functions with *continuous* boundary values satisfy the *strong comparison principle*. Note that the validity of the strong comparison principle is not known for the p-harmonic functions in $\mathbb{R}^N$, $N \geq 3$.

Theorem 5.9. *Let $\Omega \subset \mathbb{R}^N$ be a bounded, open and connected set, and let u_ε and v_ε be p-harmonious functions with continuous boundary values $F_u \geq F_v$ in Γ_ε. Then if there exists a point $x_0 \in \Omega$ such that $u_\varepsilon(x_0) = v_\varepsilon(x_0)$, it follows that*

$$u_\varepsilon = v_\varepsilon \quad \text{in} \quad \Omega,$$

and, moreover, the boundary values satisfy

$$F_u = F_v \quad \text{in} \quad \Gamma_\varepsilon.$$

The proof heavily uses the fact that $p < \infty$. Note that it is known that the strong comparison principle does not hold for infinity harmonic functions.

Proof of Theorem 5.9. According to Corollary 5.7 and Theorem 5.4, $F_u \geq F_v$ implies $u_\varepsilon \geq v_\varepsilon$. By the definition of a p-harmonious function, we have

$$u_\varepsilon(x_0) = \frac{\alpha}{2}\left\{\sup_{\overline{B}_\varepsilon(x_0)} u_\varepsilon + \inf_{\overline{B}_\varepsilon(x_0)} u_\varepsilon\right\} + \beta \fint_{B_\varepsilon(x_0)} u_\varepsilon \, dy$$

and

$$v_\varepsilon(x_0) = \frac{\alpha}{2}\left\{ \sup_{\overline{B}_\varepsilon(x_0)} v_\varepsilon + \inf_{\overline{B}_\varepsilon(x_0)} v_\varepsilon \right\} + \beta \fint_{B_\varepsilon(x_0)} v_\varepsilon \, dy.$$

Next we compare the right hand sides. Because $u_\varepsilon \geq v_\varepsilon$, it follows that

$$\begin{aligned} \sup_{\overline{B}_\varepsilon(x_0)} u_\varepsilon &\leq \sup_{\overline{B}_\varepsilon(x_0)} v_\varepsilon, \\ \inf_{\overline{B}_\varepsilon(x_0)} u_\varepsilon &\leq \inf_{\overline{B}_\varepsilon(x_0)} v_\varepsilon, \quad \text{and} \\ \fint_{B_\varepsilon(x_0)} u_\varepsilon \, dy &\leq \fint_{B_\varepsilon(x_0)} v_\varepsilon \, dy. \end{aligned} \tag{57}$$

Since

$$u_\varepsilon(x_0) = v_\varepsilon(x_0),$$

we must have equalities in (57). In particular, we have equality in the third inequality in (57), and thus

$$u_\varepsilon = v_\varepsilon \qquad \text{almost everywhere in} \quad B_\varepsilon(x_0).$$

Again, the connectedness of Ω immediately implies that

$$u_\varepsilon = v_\varepsilon \qquad \text{almost everywhere in} \quad \Omega \cup \Gamma_\varepsilon.$$

In particular,

$$F_u = F_v \qquad \text{everywhere in} \quad \Gamma_\varepsilon$$

since F_u and F_v are continuous. Because the boundary values coincide, the uniqueness result, Theorem 5.6, shows that $u_\varepsilon = v_\varepsilon$ everywhere in Ω. □

5.6. **Convergence as $\varepsilon \to 0$.** In this section, we prove that p-harmonious functions with a fixed boundary datum converge to the unique p-harmonic function.

To prove that p-harmonious functions converge to the unique solution of the Dirichlet problem for the p-Laplacian in Ω with fixed continuous boundary values, we assume that the bounded domain Ω satisfies the following boundary regularity condition:

Boundary Regularity Condition 5.10. *There exists $\delta' > 0$ and $\mu \in (0,1)$ such that for every $\delta \in (0, \delta']$ and $y \in \partial\Omega$ there exists a ball*

$$B_{\mu\delta}(z) \subset B_\delta(y) \setminus \Omega.$$

For example, when Ω satisfies the exterior cone condition it satisfies this requirement. This is indeed the case when Ω is a Lipschitz domain.

Theorem 5.11. *Let Ω be a bounded domain satisfying Condition 5.10 and F be a continuous function. Consider the unique viscosity solution u to*

$$\begin{cases} \operatorname{div}(|\nabla u|^{p-2}\nabla u)(x) = 0, & x \in \Omega \\ u(x) = F(x), & x \in \partial\Omega, \end{cases} \tag{58}$$

and let u_ε be the unique p-harmonious function with boundary values F. Then

$$u_\varepsilon \to u \qquad \textit{uniformly in} \quad \overline{\Omega}$$

as $\varepsilon \to 0$.

The above limit only depends on the values of F on $\partial\Omega$, and therefore any continuous extension of $F|_{\partial\Omega}$ to Γ_{ε_0} gives the same limit.

First, we prove a convergence result under additional assumptions by employing game theoretic arguments from [51] and [52]. Then we present a different proof that avoids the technical restrictions. The second proof uses a fact that although p-harmonious functions are, in general, discontinuous, they are, in a certain sense, asymptotically uniformly continuous.

Let Ω be a bounded open set. We assume below that u is p-harmonic in an open set Ω' such that $\Omega \cup \Gamma_\varepsilon \subset \Omega'$. In addition, we assume that $\nabla u \neq 0$ in Ω'. This assumption guarantees that u is real analytic according to a classical theorem of Hopf [27], and thus equation (59) below holds with a uniform error term in Ω. Later we show how to deal directly with the Dirichlet problem without this extra assumption.

Theorem 5.12. *Let u be p-harmonic with nonvanishing gradient $\nabla u \neq 0$ in Ω' as above and let u_ε be the p-harmonious function in Ω with the boundary values u in Γ_ε. Then*

$$u_\varepsilon \to u \qquad \textit{uniformly in } \ \overline{\Omega}$$

as $\varepsilon \to 0$.

Proof. The proof uses some ideas from the proof of Theorem 2.4 in [52]. As noticed provi-ously, the p-harmonious function with boundary values coincides with the value of the game and thus we can use a game theoretic approach.

Recall from the introduction (see also [43]) that u satisfies

$$u(x) = \frac{\alpha}{2}\left\{ \sup_{\overline{B}_\varepsilon(x)} u + \inf_{\overline{B}_\varepsilon(x)} u \right\} + \beta \fint_{B_\varepsilon(x)} u\,dy + O(\varepsilon^3) \tag{59}$$

with a uniform error term for $x \in \overline{\Omega}$ as $\varepsilon \to 0$. The error term is uniform due to our assumptions on u.

Assume, for the moment, that $p > 2$ implying $\alpha > 0$ so that the strategies are relevant. Now, Player II follows a strategy S^0_{II} such that at a point x_{k-1} he chooses to step to a point that minimizes u, that is, to a point $x_k \in B_\varepsilon(x_{k-1})$ such that

$$u(x_k) = \inf_{\overline{B}_\varepsilon(x_{k-1})} u(y).$$

Choose $C_1 > 0$ such that $|O(\varepsilon^3)| \le C_1\varepsilon^3$. Under the strategy S^0_{II}

$$M_k = u(x_k) - C_1 k \varepsilon^3$$

is a supermartingale. Indeed,

$$\begin{aligned} &\mathbb{E}_{S_{\rm I},S^0_{\rm II}}(u(x_k)-C_1k\varepsilon^3 \,|\, x_0,\dots,x_{k-1}) \\ &\le \frac{\alpha}{2}\left\{\sup_{\overline{B}_\varepsilon(x_{k-1})} u + \inf_{\overline{B}_\varepsilon(x_{k-1})} u\right\} + \beta \fint_{B_\varepsilon(x_{k-1})} u\,dy - C_1k\varepsilon^3 \\ &\le u(x_{k-1}) - C_1(k-1)\varepsilon^3. \end{aligned} \tag{60}$$

The first inequality follows from the choice of the strategy and the second from (59). Now we can estimate $u^\varepsilon_{\rm II}(x_0)$ by using Fatou's lemma and the optional stopping theorem for supermartingales. We have

$$\begin{aligned} u^\varepsilon_{\rm II}(x_0) &= \inf_{S_{\rm II}} \sup_{S_{\rm I}} \mathbb{E}^{x_0}_{S_{\rm I},S_{\rm II}}[F(x_\tau)] \\ &\le \sup_{S_{\rm I}} \mathbb{E}^{x_0}_{S_{\rm I},S^0_{\rm II}}[u(x_\tau)] \\ &\le \sup_{S_{\rm I}} \mathbb{E}^{x_0}_{S_{\rm I},S^0_{\rm II}}[u(x_\tau)+C_1\tau\varepsilon^3 - C_1\tau\varepsilon^3] \\ &\le \sup_{S_{\rm I}} \Big(\liminf_{k\to\infty} \mathbb{E}^{x_0}_{S_{\rm I},S^0_{\rm II}}[u(x_{\tau\wedge k}) - C_1(\tau\wedge k)\varepsilon^3] + C_1\varepsilon^3 \mathbb{E}^{x_0}_{S_{\rm I},S^0_{\rm II}}[\tau]\Big) \\ &\le u(x_0) + C_1\varepsilon^3 \sup_{S_{\rm I}} \mathbb{E}^{x_0}_{S_{\rm I},S^0_{\rm II}}[\tau]. \end{aligned}$$

This inequality and the analogous argument for Player I implies for $u_\varepsilon = u^\varepsilon_{\rm II} = u^\varepsilon_I$ that

$$u(x_0) - C_1\varepsilon^3 \inf_{S_{\rm II}} \mathbb{E}^{x_0}_{S^0_{\rm I},S_{II}}[\tau] \le u_\varepsilon(x_0) \le u(x_0) + C_1\varepsilon^3 \sup_{S_{\rm I}} \mathbb{E}^{x_0}_{S_{\rm I},S^0_{\rm II}}[\tau]. \tag{61}$$

Letting $\varepsilon \to 0$ the proof is completed if we prove that there exists C such that

$$\mathbb{E}^{x_0}_{S_{\rm I},S^0_{\rm II}}[\tau] \le C\varepsilon^{-2}. \tag{62}$$

To establish this bound, we show that

$$\tilde{M}_k = -u(x_k)^2 + u(x_0)^2 + C_2\varepsilon^2 k$$

is a supermartingale for small enough $\varepsilon > 0$. If Player II wins the toss, we have

$$u(x_k) - u(x_{k-1}) \le -C_3\varepsilon$$

because $\nabla u \neq 0$, as we can choose for example $C_3 = \inf_{x\in\Omega} |\nabla u|$. It follows that

$$\begin{aligned} &\mathbb{E}_{S_{\rm I},S^0_{\rm II}}[\big((u(x_k)-u(x_{k-1})\big)^2 \,|\, x_0,\dots,x_{k-1}] \\ &\qquad \ge \frac{\alpha}{2}\big((-C_3\varepsilon)^2 + 0\big) + \beta\cdot 0 = \frac{\alpha {C_3}^2}{2}\varepsilon^2. \end{aligned} \tag{63}$$

We have

$$\begin{aligned} &\mathbb{E}_{S_{\rm I},S^0_{\rm II}}[\tilde{M}_k - \tilde{M}_{k-1} \,|\, x_0,\dots,x_{k-1}] \\ &= \mathbb{E}_{S_{\rm I},S^0_{\rm II}}[-u(x_k)^2 + u(x_{k-1})^2 + C_2\varepsilon^2 \,|\, x_0,\dots,x_{k-1}] \\ &= \mathbb{E}_{S_{\rm I},S^0_{\rm II}}[-\big(u(x_k)-u(x_{k-1})\big)^2 \,|\, x_0,\dots,x_{k-1}] \\ &\qquad - \mathbb{E}_{S_{\rm I},S^0_{\rm II}}[2\big(u(x_k)-u(x_{k-1})\big)u(x_{k-1}) \,|\, x_0,\dots,x_{k-1}] + C_2\varepsilon^2. \end{aligned} \tag{64}$$

By subtracting a constant if necessary, we may assume that $u < 0$. Moreover, $u(x_{k-1})$ is determined by the point x_{k-1}, and thus, we can estimate the second term on the right hand side as

$$\begin{aligned}
&-\mathbb{E}_{S_{\mathrm{I}},S^0_{\mathrm{II}}}[2\big(u(x_k)-u(x_{k-1})\big)u(x_{k-1})\,|\,x_0,\dots,x_{k-1}] \\
&= -2u(x_{k-1})\Big(\mathbb{E}_{S_{\mathrm{I}},S^0_{\mathrm{II}}}[u(x_k)\,|\,x_0,\dots,x_{k-1}]-u(x_{k-1})\Big) \\
&\le 2\,||u||_\infty\, C_1\varepsilon^3.
\end{aligned}$$

The last inequality follows from (59) similarly as estimate (60). This together with (63) and (64) implies

$$\mathbb{E}_{S_{\mathrm{I}},S^0_{\mathrm{II}}}[\tilde{M}_k-\tilde{M}_{k-1}\,|\,x_0,\dots,x_{k-1}]\le 0,$$

when

$$-\varepsilon^2\alpha {C_3}^2/2+2\,||u||_\infty\, C_1\varepsilon^3+C_2\varepsilon^2\le 0.$$

This holds if we choose, for example, C_2 such that $C_3 \ge 2\sqrt{C_2/\alpha}$ and take $\varepsilon < C_2/(2\,||u||_\infty\, C_1)$. Thus, $\tilde{M}_k$ is a supermartingale. Recall that we assumed that $p > 2$ implying $\alpha > 0$.

According to the optional stopping theorem for supermartingales

$$\mathbb{E}^{x_0}_{S_{\mathrm{I}},S^0_{\mathrm{II}}}[\tilde{M}_{\tau\wedge k}]\le \tilde{M}_0=0,$$

and thus

$$C_2\varepsilon^2\mathbb{E}^{x_0}_{S_{\mathrm{I}},S^0_{\mathrm{II}}}[\tau\wedge k]\le \mathbb{E}^{x_0}_{S_{\mathrm{I}},S^0_{\mathrm{II}}}[u(x_{\tau\wedge k})^2-u(x_0)^2].$$

The result follows by passing to the limit with k since u is bounded in Ω.

Finally, if $p = 2$, then the mean value property holds without a correction for u due to the classical mean value property for harmonic functions and the claim immediately follows by repeating the beginning of the proof till (61) without the correction term. □

Above we obtained the convergence result for p-harmonious functions under the extra assumption that $\nabla u \neq 0$. Now we show how to deal directly with the Dirichlet problem and give a different proof for the uniform convergence without using this hypothesis. The proof is based on a variant of the classical Arzela-Ascoli's compactness lemma, Lemma 5.13. The functions u_ε are not continuous, in general, as shown in Example 5.3. Nonetheless, the jumps can be controlled and we will show that the p-harmonious functions are asymptotically uniformly continuous as shown in Theorem 5.17.

Lemma 5.13. *Let $\{u_\varepsilon : \overline{\Omega} \to \mathbb{R},\ \varepsilon > 0\}$ be a set of functions such that*

(1) *there exists $C > 0$ so that $|u_\varepsilon(x)| < C$ for every $\varepsilon > 0$ and every $x \in \overline{\Omega}$,*
(2) *given $\eta > 0$ there are constants r_0 and ε_0 such that for every $\varepsilon < \varepsilon_0$ and any $x, y \in \overline{\Omega}$ with $|x - y| < r_0$ it holds*

$$|u_\varepsilon(x)-u_\varepsilon(y)|<\eta.$$

Then, there exists a uniformly continuous function $u : \overline{\Omega} \to \mathbb{R}$ *and a subsequence still denoted by* $\{u_\varepsilon\}$ *such that*

$$u_\varepsilon \to u \qquad \text{uniformly in} \quad \overline{\Omega},$$

as $\varepsilon \to 0$.

Proof. First, we find a candidate to be the uniform limit u. Let $X \subset \overline{\Omega}$ be a dense countable set. Because functions are uniformly bounded, a diagonal procedure provides a subsequence still denoted by $\{u_\varepsilon\}$ that converges for all $x \in X$. Let $u(x)$ denote this limit. Note that at this point u is defined only for $x \in X$.

By assumption, given $\eta > 0$, there exists r_0 such that for any $x, y \in X$ with $|x-y| < r_0$ it holds

$$|u(x) - u(y)| < \eta.$$

Hence, we can extend u to the whole $\overline{\Omega}$ continuously by setting

$$u(z) = \lim_{X \ni x \to z} u(x).$$

Our next step is to prove that $\{u_\varepsilon\}$ converges to u uniformly. We choose a finite covering

$$\overline{\Omega} \subset \bigcup_{i=1}^{N} B_r(x_i)$$

and $\varepsilon_0 > 0$ such that

$$|u_\varepsilon(x) - u_\varepsilon(x_i)|, |u(x) - u(x_i)| < \eta/3$$

for every $x \in B_r(x_i)$ and $\varepsilon < \varepsilon_0$ as well as

$$|u_\varepsilon(x_i) - u(x_i)| < \eta/3,$$

for every x_i and $\varepsilon < \varepsilon_0$. To obtain the last inequality, we used the fact that $N < \infty$. Thus for any $x \in \overline{\Omega}$, we can find x_i so that $x \in B_r(x_i)$ and

$$\begin{aligned} &|u_\varepsilon(x) - u(x)| \\ &\qquad \le |u_\varepsilon(x) - u_\varepsilon(x_i)| + |u_\varepsilon(x_i) - u(x_i)| + |u(x_i) - u(x)| \\ &\qquad < \eta, \end{aligned}$$

for every $\varepsilon < \varepsilon_0$, where ε_0 is independent of x. □

Next we show that for fixed F, a family of p-harmonious functions, with ε as the parameter, satisfies the conditions of Lemma 5.13. First observe that p-harmonious functions are bounded since

$$\min_{y \in \Gamma_\varepsilon} F(y) \le F(x_\tau) \le \max_{y \in \Gamma_\varepsilon} F(y)$$

for any $x_\tau \in \Gamma_\varepsilon$ implies:

Lemma 5.14. *A p-harmonious function u_ε with boundary values F satisfies*

$$\min_{y \in \Gamma_\varepsilon} F(y) \le u_\varepsilon(x) \le \max_{y \in \Gamma_\varepsilon} F(y). \tag{65}$$

Next we will show that p-harmonious functions are asymptotically uniformly continuous. We give two proofs for this result. The first proof applies Theorem 5.4 and a comparison with solutions for the p-Dirichlet problem in annular domains. We also use Theorem 5.12 for these solutions, which satisfy the conditions of the theorem. The proof utilizes some ideas from [52] but does not explicitly employ probabilistic tools.

Lemma 5.15. *Let $\{u_\varepsilon\}$ be a family of p-harmonious functions in Ω with a fixed continuous boundary data F. Then this family satisfies condition* (2) *in Lemma 5.13.*

Proof. Observe that the case $x, y \in \Gamma_\varepsilon$ readily follows from the continuity of F, and thus we can concentrate on the cases $x \in \Omega,\ y \in \Gamma_\varepsilon$, and $x, y \in \Omega$.

We divide the proof into three steps: First for $x \in \Omega,\ y \in \Gamma_\varepsilon$, we employ comparison with a p-harmonious function close to a solution for the p-Dirichlet problem in an annular domain. It follows that the p-harmonious function with the boundary data F is bounded close to $y \in \Gamma_\varepsilon$ by a slightly smaller constant than the maximum of the boundary values. Second, we iterate this argument to show that the p-harmonious function is close to the boundary values near $y \in \Gamma_\varepsilon$ when ε is small. Third, we extend this result to the case $x, y \in \Omega$ by translation, by taking the boundary values from the strip already controlled during the previous steps.

To start, we choose $B_{\mu\delta}(z) \subset B_\delta(y) \setminus \Omega$, $\delta < \delta'$, by Condition (5.10), and consider a problem

$$\begin{cases} \operatorname{div}(|\nabla u|^{p-2}\nabla u)(x) = 0, & x \in B_{4\delta}(z) \setminus \overline{B}_{\mu\delta}(z), \\ u(x) = \sup_{B_{5\delta}(y)\cap\Gamma_\varepsilon} F, & x \in \partial B_{\mu\delta}(z), \\ u(x) = \sup_{\Gamma_\varepsilon} F, & x \in \partial B_{4\delta}(z). \end{cases} \tag{66}$$

We denote $r = |x - z|$. This problem has an explicit, radially symmetric solution of the form

$$u(r) = a r^{-(n-p)/(p-1)} + b \tag{67}$$

when $p \neq n$ and

$$u(r) = a \log(r) + b, \tag{68}$$

when $p = n$. We extend the solutions to $B_{4\delta+2\varepsilon}(z) \setminus \overline{B}_{\mu\delta-2\varepsilon}(z)$ and use the same notation for the extensions. Now because $\nabla u \neq 0$, Theorem 5.12 shows that for the p-harmonious functions $\{u^\varepsilon_{\text{fund}}\}$ in $B_{4\delta+\varepsilon}(z) \setminus \overline{B}_{\mu\delta-\varepsilon}(z)$ with boundary values u, it holds that

$$u^\varepsilon_{\text{fund}} \to u, \qquad \text{uniformly in} \quad \overline{B}_{4\delta+\varepsilon}(z) \setminus B_{\mu\delta-\varepsilon}(z)$$

as $\varepsilon \to 0$.

It follows that

$$|u^\varepsilon_{\text{fund}} - u| = o(1) \qquad \text{in} \quad \overline{B}_{4\delta+\varepsilon}(z) \setminus B_{\mu\delta-\varepsilon}(z),$$

where $o(1) \to 0$ as $\varepsilon \to 0$. For small enough ε, the comparison principle, Theorem 5.4, implies that in $B_\delta(y) \cap \Omega \subset B_{2\delta}(z) \cap \Omega$ there is $\theta \in (0,1)$ such that

$$u^\varepsilon \leq u^\varepsilon_{\text{fund}} + o(1) \leq u + o(1) \leq \sup_{B_{5\delta}(y)\cap\Gamma_\varepsilon} F + \theta(\sup_{\Gamma_\varepsilon} F - \sup_{B_{5\delta}(y)\cap\Gamma_\varepsilon} F).$$

Observe that by solving a, b in (67) or (68) above, we see that $0 < \theta < 1$ does not depend on δ.

To prove the second step, we solve the p-harmonic function in $B_\delta(z) \setminus \overline{B}_{\mu\delta/4}(z)$ with boundary values $\sup_{B_{5\delta}(y)\cap\Gamma_\varepsilon} F$ at $\partial B_{\mu\delta/4}(z)$ and from the previous step

$$\sup_{B_{5\delta}(y)\cap\Gamma_\varepsilon} F + \theta(\sup_{\Gamma_\varepsilon} F - \sup_{B_{5\delta}(y)\cap\Gamma_\varepsilon} F)$$

at $\partial B_\delta(z)$. Again the explicit solution and the comparison principle implies for small enough $\varepsilon > 0$ that

$$u_\varepsilon \leq \sup_{B_{5\delta}(y)\cap\Gamma_\varepsilon} F + \theta^2(\sup_{\Gamma_\varepsilon} F - \sup_{B_{5\delta}(y)\cap\Gamma_\varepsilon} F) \quad \text{in} \quad B_{\delta/4}(y) \cap \Omega.$$

Continuing in this way, we see that for small enough $\varepsilon > 0$ that

$$u_\varepsilon \leq \sup_{B_{5\delta}(y)\cap\Gamma_\varepsilon} F + \theta^k(\sup_{\Gamma_\varepsilon} F - \sup_{B_{5\delta}(y)\cap\Gamma_\varepsilon} F) \quad \text{in} \quad B_{\delta/4^k}(y) \cap \Omega.$$

This gives an upper bound for u^ε. The argument for the lower bound is similar. We have shown that for any $\eta > 0$, we can choose small enough $\delta > 0$, large enough k, and small enough $\varepsilon > 0$ above so that for $x \in \Omega, y \in \Gamma_\varepsilon$ with $|x - y| < \delta/4^k$ it holds

$$|u_\varepsilon(x) - F(y)| < \eta. \tag{69}$$

This shows that the second condition in Theorem 5.13 holds when $y \in \Gamma_\varepsilon$.

Next we extend the estimate to the interior of the domain. First choose small enough δ and large enough k so that

$$|F(x') - F(y')| < \eta \tag{70}$$

whenever $|x' - y'| < \delta/4^k$, and $\varepsilon > 0$ small enough so that (69) holds.

Next we consider a slightly smaller domain

$$\tilde{\Omega} = \{z \in \Omega \,:\, \mathrm{dist}(z, \partial\Omega) > \delta/4^{k+2}\}$$

with the boundary strip

$$\tilde{\Gamma} = \{z \in \overline{\Omega} \,:\, \mathrm{dist}(z, \partial\Omega) \leq \delta/4^{k+2}\}.$$

Suppose that $x, y \in \Omega$ with $|x - y| < \delta/4^{k+2}$. First, if $x, y \in \tilde{\Gamma}$, then we can estimate

$$|u_\varepsilon(x) - u_\varepsilon(y)| \leq 3\eta \tag{71}$$

by comparing the values at x and y to the nearby boundary values and using (69). Finally, let $x, y \in \tilde{\Omega}$ and define

$$\tilde{F}(z) = u_\varepsilon(z - x + y) + 3\eta \quad \text{in} \quad \tilde{\Gamma}.$$

We have

$$\tilde{F}(z) \geq u_\varepsilon(z) \quad \text{in} \quad \tilde{\Gamma}$$

by (69), (70), and (71). Solve the p-harmonious function $\tilde{u}_\varepsilon$ in $\tilde{\Omega}$ with the boundary values $\tilde{F}$ in $\tilde{\Gamma}$. By the comparison principle and the uniqueness, we deduce

$$u_\varepsilon(x) \leq \tilde{u}_\varepsilon(x) = u_\varepsilon(x - x + y) + 3\eta = u_\varepsilon(y) + 3\eta \quad \text{in} \quad \tilde{\Omega}.$$

The lower bound follows by a similar argument. □

The second proof for Lemma 5.15 is based on the connection to games and a choice of a strategy. In Lemma 5.17, we prove slightly stronger estimate that implies Lemma 5.15. The proof of this lemma avoids the use of Theorem 5.12 but we assume a stronger boundary regularity condition instead.

At each step, we make a small correction in order to show that the process is a supermartingale. To show that the effect of the correction is small also in the long run, we need to estimate the expectation of the stopping time τ. We bound τ by the exit time τ^* for a random walk in a larger annular domain with a reflecting condition on the outer boundary.

Lemma 5.16. *Let us consider an annular domain $B_R(y) \setminus \overline{B}_\delta(y)$ and a random walk such that when at x_{k-1}, the next point x_k is chosen according to a uniform distribution at $B_\varepsilon(x_{k-1}) \cap B_R(y)$. Let*

$$\tau^* = \inf\{k \,:\, x_k \in \overline{B}_\delta(y)\}.$$

Then

$$\mathbb{E}^{x_0}(\tau^*) \le \frac{C(R/\delta)\,\mathrm{dist}(\partial B_\delta(y), x_0) + o(1)}{\varepsilon^2}, \tag{72}$$

for $x_0 \in B_R(y) \setminus \overline{B}_\delta(y)$. Here $o(1) \to 0$ as $\varepsilon \to 0$.

Proof. We will use a solution to a corresponding Poisson problem to prove the result. Let us denote

$$g_\varepsilon(x) = \mathbb{E}^x(\tau^*).$$

The function g_ε satisfies a dynamic programming principle

$$g_\varepsilon(x) = \fint_{B_\varepsilon(x)\cap B_R(y)} g_\varepsilon \, dz + 1$$

because the number of steps always increases by one when making a step to one of the neighboring points at random. Further, we denote $v_\varepsilon(x) = \varepsilon^2 g_\varepsilon(x)$ and obtain

$$v_\varepsilon(x) = \fint_{B_\varepsilon(x)\cap B_R(y)} v_\varepsilon \, dz + \varepsilon^2.$$

This formula suggests a connection to the problem

$$\begin{cases} \Delta v(x) = -2(n+2), & x \in B_{R+\varepsilon}(y) \setminus \overline{B}_\delta(y), \\ v(x) = 0, & x \in \partial B_\delta(y), \\ \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial B_{R+\varepsilon}(y), \end{cases} \tag{73}$$

where $\frac{\partial u}{\partial \nu}$ refers to the normal derivative. Indeed, when $B_\varepsilon(x) \subset B_{R+\varepsilon}(y) \setminus \overline{B}_\delta(y)$, the classical calculation shows that the solution of this problem satisfies the mean value property

$$v(x) = \fint_{B_\varepsilon(x)} v \, dz + \varepsilon^2. \tag{74}$$

The solution of problem (73) is positive, radially symmetric, and strictly increasing in $r = |x-y|$. It takes the form $v(r) = -ar^2 - br^{2-n} + c$, if $n > 2$ and $v(r) = -ar^2 - b\log(r) + c$, if $n = 2$.

We extend this function as a solution to the same equation to $\overline{B}_\delta(y)\setminus\overline{B}_{\delta-\varepsilon}(y)$ and use the same notation for the extension. Thus, v satisfies (74) for each $B_\varepsilon(x)\subset B_{R+\varepsilon}(y)\setminus\overline{B}_{\delta-\varepsilon}(y)$. In addition, because v is increasing in r, it holds for each $x\in B_R(y)\setminus\overline{B}_\delta(y)$ that

$$\fint_{B_\varepsilon(x)\cap B_R(y)} v\,dz \le \fint_{B_\varepsilon(x)} v\,dz = v(x)-\varepsilon^2.$$

It follows that

$$\mathbb{E}[v(x_k)+k\varepsilon^2\,|\,x_0,\dots,x_{k-1}] = \fint_{B_\varepsilon(x_{k-1})} v\,dz + k\varepsilon^2 = v(x_{k-1}) + (k-1)\varepsilon^2,$$

if $B_\varepsilon(x_{k-1})\subset B_R(y)\setminus\overline{B}_{\delta-\varepsilon}(y)$, and if $B_\varepsilon(x_{k-1})\setminus B_R(y)\neq\emptyset$, then

$$\begin{aligned}\mathbb{E}[v(x_k)+k\varepsilon^2\,|\,x_0,\dots,x_{k-1}] &= \fint_{B_\varepsilon(x_{k-1})\cap B_R(y)} v\,dz + k\varepsilon^2\\ &\le \fint_{B_\varepsilon(x_{k-1})} v\,dz = v(x_{k-1}) + (k-1)\varepsilon^2.\end{aligned}$$

Thus $v(x_k)+k\varepsilon^2$ is a supermartingale, and the optional stopping theorem yields

$$\mathbb{E}^{x_0}[v(x_{\tau^*\wedge k}) + (\tau^*\wedge k)\varepsilon^2] \le v(x_0). \tag{75}$$

Because $x_{\tau^*}\in\overline{B}_\delta(y)\setminus\overline{B}_{\delta-\varepsilon}(y)$, we have

$$0\le -\mathbb{E}^{x_0}[v(x_{\tau^*})]\le o(1).$$

Furthermore, the estimate

$$0\le v(x_0)\le C(R/\delta)\,\mathrm{dist}(\partial B_\delta(y),x_0)$$

holds for the solutions of (73). Thus, by passing to a limit with k in (75), we obtain

$$\varepsilon^2\mathbb{E}^{x_0}[\tau^*]\le v(x_0)-\mathbb{E}[u(x_{\tau^*})]\le C(R/\delta)(\mathrm{dist}(\partial B_\delta(y),x_0)+o(1)).$$

This completes the proof. □

By estimating the dominating terms br^{-N+2} or $b\log(r)$ of explicit solutions to (73) close to $r=\delta$, we see that

$$|\mathbb{E}^{x_0}[v(x_{\tau^*})]|\le C\log(1+\varepsilon). \tag{76}$$

Thus the error term $o(1)$ could be improved to $C\log(1+\varepsilon)$.

Next we derive an estimate for the asymptotic uniform continuity of the family of p-harmonious functions which implies Lemma 5.15. For simplicity, we assume that Ω satisfies an exterior sphere condition: For each $y\in\partial\Omega$, there exists $B_\delta(z)\subset\mathbb{R}^n\setminus\Omega$ such that $y\in\partial B_\delta(z)$. With this assumption, the iteration used in the first proof of Lemma 5.15 can be avoided. To simplify the notation and to obtain an explicit estimate, we also assume that F is Lipschitz continuous in Γ_ε.

Lemma 5.17. *Let F and Ω be as above. The p-harmonious function u_ε with the boundary data F satisfies*

$$|u_\varepsilon(x) - u_\varepsilon(y)| \le \operatorname{Lip}(F)\delta + C(R/\delta)(|x-y| + o(1)), \tag{77}$$

for every small enough $\delta > 0$ and for every two points $x, y \in \Omega \cup \Gamma_\varepsilon$.

Proof. As in the first proof of Lemma 5.15, the case $x, y \in \Gamma_\varepsilon$ is clear. Thus, we can concentrate on the cases $x \in \Omega$ and $y \in \Gamma_\varepsilon$ as well as $x, y \in \Omega$.

We utilize the connection to games. Suppose first that $x \in \Omega$ and $y \in \Gamma_\varepsilon$. By the exterior sphere condition, there exists $B_\delta(z) \subset \mathbb{R}^n \setminus \Omega$ such that $y \in \partial B_\delta(z)$. Player I chooses a strategy of pulling towards z, denoted by S_I^z. Then

$$M_k = |x_k - z| - C\varepsilon^2 k$$

is a supermartingale for a constant C large enough independent of ε. Indeed,

$$\begin{aligned}
&\mathbb{E}^{x_0}_{S_{\mathrm{I}}^z, S_{\mathrm{II}}}[|x_k - z| \,|\, x_0, \dots, x_{k-1}] \\
&\qquad \le \frac{\alpha}{2}\{|x_{k-1} - z| + \varepsilon + |x_{k-1} - z| - \varepsilon\} + \beta \fint_{B_\varepsilon(x_{k-1})} |x - z|\, dx \\
&\qquad \le |x_{k-1} - z| + C\varepsilon^2.
\end{aligned}$$

The first inequality follows from the choice of the strategy, and the second from the estimate

$$\fint_{B_\varepsilon(x_{k-1})} |x - z|\, dx \le |x_{k-1} - z| + C\varepsilon^2.$$

By the optional stopping theorem, this implies that

$$\mathbb{E}^{x_0}_{S_{\mathrm{I}}^z, S_{\mathrm{II}}}[|x_\tau - z|] \le |x_0 - z| + C\varepsilon^2 \mathbb{E}^{x_0}_{S_{\mathrm{I}}^z, S_{\mathrm{II}}}[\tau]. \tag{78}$$

Next we estimate $\mathbb{E}^{x_0}_{S_{\mathrm{I}}^z, S_{\mathrm{II}}}[\tau]$ by the stopping time of Lemma 5.16. Player I pulls towards z and Player II uses any strategy. The expectation of $|x_k - z|$ when at x_{k-1} is at the most $|x_{k-1} - z|$ when we know that the tug-of-war occurs. On the other hand, if the random walk occurs, then we know that the expectation of $|x_k - z|$ is greater than or equal to $|x_{k-1} - z|$. Therefore we can bound the expectation of the original process by considering a suitable random walk in $B_R(z) \setminus B_\delta(z)$ for $B_R(z)$ such that $\Omega \subset B_{R/2}(z)$. When $x_k \in B_R(z) \setminus \overline{B}_\delta(z)$, the successor x_{k+1} is chosen according to a uniform probability in $B_\varepsilon(x) \cap B_R(z)$. The process stops when it hits $\overline{B}_\delta(z)$. Thus, by (72),

$$\varepsilon^2 \mathbb{E}^{x_0}_{S_{\mathrm{I}}^z, S_{\mathrm{II}}}[\tau] \le \varepsilon^2 \mathbb{E}^{x_0}_{S_{\mathrm{I}}^z, S_{\mathrm{II}}}[\tau^*] \le C(R/\delta)(\operatorname{dist}(\partial B_\delta(z), x_0) + o(1)).$$

Since $y \in \partial B_\delta(z)$,

$$\operatorname{dist}(\partial B_\delta(z), x_0) \le |y - x_0|,$$

and thus, (78) implies

$$\mathbb{E}^{x_0}_{S_{\mathrm{I}}^z, S_{\mathrm{II}}}[|x_\tau - z|] \le C(R/\delta)(|x_0 - y| + o(1)).$$

We get

$$F(z) - C(R/\delta)(|x-y| + o(1)) \le \mathbb{E}^{x_0}_{S^z_{\rm I}, S_{\rm II}}[F(x_\tau)] \le F(z) + C(R/\delta)(|x-y| + o(1)).$$

Thus, we obtain

$$\begin{aligned} \sup_{S_{\rm I}} \inf_{S_{\rm II}} \mathbb{E}^{x_0}_{S_{\rm I}, S_{\rm II}}[F(x_\tau)] &\ge \inf_{S_{\rm II}} \mathbb{E}^{x_0}_{S^z_{\rm I}, S_{\rm II}}[F(x_\tau)] \\ &\ge F(z) - C(R/\delta)(|x_0 - y| + o(1)) \\ &\ge F(y) - \mathrm{Lip}(F)\delta - C(R/\delta)(|x_0 - y| + o(1)). \end{aligned}$$

The upper bound can be obtained by choosing for Player II a strategy where he points to z, and thus, (77) follows.

Finally, let $x, y \in \Omega$ and fix the strategies $S_{\rm I}, S_{\rm II}$ for the game starting at x. We define a virtual game starting at y: we use the same coin tosses and random steps as the usual game starting at x. Furthermore, the players adopt their strategies $S^v_{\rm I}, S^v_{\rm II}$ from the game starting at x, that is, when the game position is y_{k-1} a player chooses the step that would be taken at x_{k-1} in the game starting at x. We proceed in this way until for the first time $x_k \in \Gamma_\varepsilon$ or $y_k \in \Gamma_\varepsilon$. At that point we have $|x_k - y_k| = |x - y|$, and we may apply the previous steps that work for $x_k \in \Omega$, $y_k \in \Gamma_\varepsilon$ or for $x_k, y_k \in \Gamma_\varepsilon$. □

Note that, thanks to Lemmas 5.14 and 5.15 (or alternatively Lemma 5.17), the family u_ε satisfies the hypothesis of the compactness Lemma 5.13.

Corollary 5.18. *Let $\{u_\varepsilon\}$ be a family of p-harmonious functions with a fixed continuous boundary data F. Then there exists a uniformly continuous u and a subsequence still denoted by $\{u_\varepsilon\}$ such that*

$$u_\varepsilon \to u \qquad \textit{uniformly in} \quad \overline{\Omega}.$$

Next we prove that the limit u in Corollary 5.18 is a solution to (58). The idea is to work in the viscosity setting and to show that the limit is a viscosity sub- and supersolution. To accomplish this, we utilize some ideas from [43], where p-harmonic functions were characterized in terms of asymptotic expansions. We start by recalling the viscosity characterization of p-harmonic functions, see [36].

Definition 5.19. *For $1 < p < \infty$ consider the equation*

$$-\mathrm{div}\left(|\nabla u|^{p-2}\nabla u\right) = 0.$$

(1) *A lower semi-continuous function u is a viscosity supersolution if for every $\phi \in C^2$ such that ϕ touches u at $x \in \Omega$ strictly from below with $\nabla\phi(x) \ne 0$, we have*
$$-(p-2)\Delta_\infty \phi(x) - \Delta\phi(x) \ge 0.$$
(2) *An upper semi-continuous function u is a subsolution if for every $\phi \in C^2$ such that ϕ touches u at $x \in \Omega$ strictly from above with $\nabla\phi(x) \ne 0$, we have*
$$-(p-2)\Delta_\infty \phi(x) - \Delta\phi(x) \le 0.$$
(3) *Finally, u is a viscosity solution if it is both a sub- and supersolution.*

Theorem 5.20. *Let F and Ω be as in Theorem 5.11. Then the uniform limit u of p-harmonious functions $\{u_\varepsilon\}$ is a viscosity solution to* (58).

Proof. First, $u = F$ on $\partial\Omega$ due to Lemma 5.15, and we can focus attention on showing that u is p-harmonic in Ω in the viscosity sense. To this end, we recall from [43] an estimate that involves the regular Laplacian ($p = 2$) and an approximation for the infinity Laplacian ($p = \infty$). Choose a point $x \in \Omega$ and a C^2-function ϕ defined in a neighborhood of x. Let x_1^ε be the point at which ϕ attains its minimum in $\overline{B}_\varepsilon(x)$

$$\phi(x_1^\varepsilon) = \min_{y\in\overline{B}_\varepsilon(x)} \phi(y).$$

It follows from the Taylor expansions in [43] that

$$\begin{aligned} &\frac{\alpha}{2}\left\{\max_{y\in\overline{B}_\varepsilon(x)}\phi(y) + \min_{y\in\overline{B}_\varepsilon(x)}\phi(y)\right\} + \beta \fint_{B_\varepsilon(x)} \phi(y)\,dy - \phi(x) \\ &\geq \frac{\beta\varepsilon^2}{2(n+2)}\left((p-2)\left\langle D^2\phi(x)\left(\frac{x_1^\varepsilon - x}{\varepsilon}\right), \left(\frac{x_1^\varepsilon - x}{\varepsilon}\right)\right\rangle + \Delta\phi(x)\right) \\ &\quad + o(\varepsilon^2). \end{aligned} \tag{79}$$

Suppose that ϕ touches u at x strictly from below and that $\nabla\phi(x) \neq 0$. Observe that according to Definition 5.19, it is enough to test with such functions. By the uniform convergence, there exists sequence $\{x_\varepsilon\}$ converging to x such that $u_\varepsilon - \phi$ has an approximate minimum at x_ε, that is, for $\eta_\varepsilon > 0$, there exists x_ε such that

$$u_\varepsilon(x) - \phi(x) \geq u_\varepsilon(x_\varepsilon) - \phi(x_\varepsilon) - \eta_\varepsilon.$$

Moreover, considering $\tilde{\phi} = \phi - u_\varepsilon(x_\varepsilon) - \phi(x_\varepsilon)$, we can assume that $\phi(x_\varepsilon) = u_\varepsilon(x_\varepsilon)$. Thus, by recalling the fact that u_ε is p-harmonious, we obtain

$$\eta_\varepsilon \geq -\phi(x_\varepsilon) + \frac{\alpha}{2}\left\{\max_{\overline{B}_\varepsilon(x_\varepsilon)}\phi + \min_{\overline{B}_\varepsilon(x_\varepsilon)}\phi\right\} + \beta\fint_{B_\varepsilon(x_\varepsilon)}\phi(y)\,dy,$$

and thus, by (79), and choosing $\eta_\varepsilon = o(\varepsilon^2)$, we have

$$\begin{aligned} 0 \geq \frac{\beta\varepsilon^2}{2(n+2)}&((p-2)\left\langle D^2\phi(x_\varepsilon)\left(\frac{x_1^\varepsilon - x_\varepsilon}{\varepsilon}\right), \left(\frac{x_1^\varepsilon - x_\varepsilon}{\varepsilon}\right)\right\rangle + \Delta\phi(x_\varepsilon)) \\ &+ o(\varepsilon^2). \end{aligned}$$

Since $\nabla\phi(x) \neq 0$, letting $\varepsilon \to 0$, we get

$$0 \geq \frac{\beta}{2(n+2)}\left((p-2)\Delta_\infty\phi(x) + \Delta\phi(x)\right).$$

Therefore u is a viscosity supersolution.

To prove that u is a viscosity subsolution, we use a reverse inequality to (79) by considering the maximum point of the test function and choose a function ϕ that touches u from above. □

End of the proof of Theorem 5.11. We just have to observe that since the viscosity solution of (58) is unique, then we have convergence for the whole family $\{u_\varepsilon\}$ as $\varepsilon \to 0$. □

5.7. **Comments.**

(1) When we add a running cost of the form $\epsilon^2 g(x)$ we obtain a solution to the following inhomogeneous problem that involves the 1−homogeneous p−Laplacian,

$$\begin{cases} -|\nabla u|^{2-p}\Delta_p u(x) = kg(x) & x \in \Omega, \\ u(x) = F(x) & x \in \partial\Omega. \end{cases}$$

Here k is a constant that depends only on p and N.

Note that the 1−homogeneous p−Laplacian is not variational (it is not in divergence form).

6. A mean value property that characterizes p−harmonic functions

Inspired by the analysis performed in the previous section we can guess a mean value formula for p−harmonic functions. In fact, we have proved that p−harmonious functions (that can be viewed as solutions to a mean value property) approximate p−harmonic functions (solutions to $\Delta_p u = 0$ as $\epsilon \to 0$, hence one may expect that p−harmonic functions verify the mean value formula given by the DPP but for a small error. It turns out that this intuitive fact can be proved rigorously, and moreover, it characterizes the fact of being a solution to $\Delta_p u = 0$.

A well known fact that one can find in any elementary PDE textbook states that u is harmonic in a domain $\Omega \subset \mathbb{R}^N$ (that is u satisfies $\Delta u = 0$ in Ω) if and only if it satisfies the mean value property

$$u(x) = \frac{1}{|B_\varepsilon(x)|} \int_{B_\varepsilon(x)} u(y)\, dy,$$

whenever $B_\varepsilon(x) \subset \Omega$. In fact, we can relax this condition by requiring that it holds asymptotically

$$u(x) = \frac{1}{|B_\varepsilon(x)|} \int_{B_\varepsilon(x)} u(y)\, dy + o(\varepsilon^2),$$

as $\varepsilon \to 0$. This follows easily for C^2 functions by using the Taylor expansion and for continuous functions by using the theory of viscosity solutions. Interestingly, a weak asymptotic mean value formula holds in some nonlinear cases as well. Our goal in this paper is to characterize p-harmonic functions, $1 < p \leq \infty$, by means of this type of asymptotic mean value properties.

We begin by stating what we mean by weak asymptotic expansions and why is it reasonable to say that our asymptotic expansions hold in "a viscosity sense". As is the case in the theory of viscosity solutions, we test the expansions of a function u against test functions ϕ that touch u from below or above at a particular point.

Select α and β determined by the conditions $\alpha + \beta = 1$ and $\alpha/\beta = (p-2)/(N+2)$. That is, we have

$$\alpha = \frac{p-2}{p+N}, \quad \text{and} \quad \beta = \frac{2+N}{p+N}. \tag{80}$$

Observe that if $p = 2$ above, then $\alpha = 0$ and $\beta = 1$, and if $p = \infty$, then $\alpha = 1$ and $\beta = 0$.

As before we follow the usual convention to denote the mean value of a function

$$\fint_B f(y)\,dy = \frac{1}{|B|}\int_B f(y)\,dy.$$

Definition 6.1. *A continuous function u satisfies*

$$u(x) = \frac{\alpha}{2}\left\{\max_{B_\varepsilon(x)} u + \min_{B_\varepsilon(x)} u\right\} + \beta \fint_{B_\varepsilon(x)} u(y)\,dy + o(\varepsilon^2), \quad \textit{as } \varepsilon \to 0, \tag{81}$$

in the viscosity sense if

(1) *for every $\phi \in C^2$ such that $u - \phi$ has a strict minimum at the point $x \in \overline{\Omega}$ with $u(x) = \phi(x)$, we have*

$$0 \geq -\phi(x) + \frac{\alpha}{2}\left\{\max_{B_\varepsilon(x)} \phi + \min_{B_\varepsilon(x)} \phi\right\} + \beta \fint_{B_\varepsilon(x)} \phi(y)\,dy + o(\varepsilon^2).$$

(2) *for every $\phi \in C^2$ such that $u - \phi$ has a strict maximum at the point $x \in \overline{\Omega}$ with $u(x) = \phi(x)$, we have*

$$0 \leq -\phi(x) + \frac{\alpha}{2}\left\{\max_{B_\varepsilon(x)} \phi + \min_{B_\varepsilon(x)} \phi\right\} + \beta \fint_{B_\varepsilon(x)} \phi(y)\,dy + o(\varepsilon^2).$$

Observe that a C^2-function u satisfies (81) in the classical sense if and only if it satisfies it in the viscosity sense. However, the viscosity sense is actually weaker than the classical sense for non C^2-functions as the following example shows.

Example: Set $p = \infty$ and consider Aronsson's function

$$u(x, y) = |x|^{4/3} - |y|^{4/3}$$

near the point $(x, y) = (1, 0)$. Aronsson's function is ∞-harmonic in the viscosity sense but it is not of class C^2, see [5]. It will follow from Theorem 6.2 below that u satisfies

$$u(x) = \frac{1}{2}\left\{\max_{B_\varepsilon(x)} u + \min_{B_\varepsilon(x)} u\right\} + o(\varepsilon^2) \quad \text{as } \varepsilon \to 0,$$

in the viscosity sense of Definition 6.1. However, let us verify that the expansion does not hold in the classical sense.

Clearly, we have

$$\max_{B_\varepsilon(1,0)} u = u(1+\varepsilon, 0) = (1+\varepsilon)^{4/3}.$$

To find the minimum, we set $x = \varepsilon\cos(\theta)$, $y = \varepsilon\sin(\theta)$ and solve the equation

$$\frac{d}{d\theta} u(1+\varepsilon\cos(\theta), \varepsilon\sin(\theta)) = -\frac{4}{3}(1+\varepsilon\cos(\theta))^{1/3}\varepsilon\sin(\theta) - \frac{4}{3}(\varepsilon\sin(\theta))^{1/3}\varepsilon\cos(\theta) = 0.$$

By symmetry, we can focus our attention on the solution

$$\theta_\varepsilon = \arccos\left(\frac{\varepsilon - \sqrt{4+\varepsilon^2}}{2}\right).$$

Hence, we obtain

$$\begin{aligned}\min_{B_\varepsilon(1,0)} u &= u(1+\varepsilon\cos(\theta_\varepsilon), \varepsilon\sin(\theta_\varepsilon)) \\ &= \left(1+\frac{1}{2}\varepsilon\left(\varepsilon-\sqrt{4+\varepsilon^2}\right)\right)^{4/3} - \left(\varepsilon\sqrt{1-\frac{1}{4}\left(\varepsilon-\sqrt{4+\varepsilon^2}\right)^2}\right)^{4/3}.\end{aligned}$$

We are ready to compute

$$\lim_{\varepsilon\to 0+} \frac{\frac{1}{2}\left\{\max\limits_{B_\varepsilon(1,0)} u + \min\limits_{B_\varepsilon(1,0)} u\right\} - u(1,0)}{\varepsilon^2} = \frac{1}{18}.$$

But if an asymptotic expansion held in the classical sense, this limit would have to be zero.

The following theorem states our main result and provides a characterization to the p-harmonic functions.

Theorem 6.2. *Let $1 < p \le \infty$ and let u be a continuous function in a domain $\Omega \subset \mathbb{R}^N$. The asymptotic expansion*

$$u(x) = \frac{\alpha}{2}\left\{\max_{B_\varepsilon(x)} u + \min_{B_\varepsilon(x)} u\right\} + \beta \fint_{B_\varepsilon(x)} u(y)\,dy + o(\varepsilon^2), \quad \text{as } \varepsilon \to 0,$$

holds for all $x \in \Omega$ in the viscosity sense if and only if

$$\Delta_p u(x) = 0$$

in the viscosity sense. Here α and β are determined by (80).

We use the notation

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \Delta_p u$$

for the regular p-Laplacian and

$$\Delta_\infty u = |\nabla u|^{-2}\left\langle D^2 u\,\nabla u, \nabla u\right\rangle$$

for the 1-homogeneous infinity Laplacian.

We observe that the notions of a viscosity solution and a Sobolev weak solution for the p-Laplace equation agree for $1 < p < \infty$, see Juutinen-Lindqvist-Manfredi [36]. Therefore, Theorem 6.2 characterizes weak solutions when $1 < p < \infty$.

Also, we note that Wang [56] has also used Taylor series to give sufficient conditions for p-subharmonicity in terms of asymptotic mean values of $(u(x) - u(0))^p$.

6.1. **Proof of Theorem 6.2.** As we did before, to gain some intuition on why such asymptotic mean value formula might be true, let us formally expand the p-Laplacian as follows

$$\Delta_p u = (p-2)|\nabla u|^{p-4} \langle D^2 u \, \nabla u, \nabla u \rangle + |\nabla u|^{p-2} \Delta u. \tag{82}$$

This formal expansion was used by Peres and Sheffield in [52] (see also Peres et. al. [51]) to find $p-$harmonic functions as limits of values of Tug-of-War games.

Suppose that u is a smooth function with $\nabla u \neq 0$. We see from (82), that u is a solution to $\Delta_p u = 0$ if and only if

$$(p-2)\Delta_\infty u + \Delta u = 0. \tag{83}$$

It follows from the classical Taylor expansion that

$$u(x) - \fint_{B_\varepsilon(x)} u \, dy = -\varepsilon^2 \Delta u(x) \frac{1}{2N} \fint_{B(0,1)} |z|^2 \, dz + o(\varepsilon^2) \tag{84}$$

and

$$\begin{aligned} & u(x) - \frac{1}{2} \left\{ \max_{B_\varepsilon(x)} u + \min_{B_\varepsilon(x)} u \right\} \\ & \approx u(x) - \frac{1}{2} \left\{ u\left(x + \varepsilon \frac{\nabla u(x)}{|\nabla u(x)|} \right) + u\left(x - \varepsilon \frac{\nabla u(x)}{|\nabla u(x)|} \right) \right\} \\ & = -\frac{\varepsilon^2}{2} \Delta_\infty u(x) + o(\varepsilon^2). \end{aligned} \tag{85}$$

The volume of the unit ball in $\mathbb{R}^N$ will be denoted by ω_N and the $N-1$ dimensional area of the unit sphere will be denoted by σ_{N-1}. Observe that since $\sigma_{N-1}/\omega_N = N$ we have

$$\frac{1}{N} \fint_{B(0,1)} |z|^2 \, dz = \frac{1}{N+2}.$$

Multiply (84) and (85) by suitable constants and add up the formulas so that we have the operator from (83) on the right hand side. This process gives us the choices of the constants α and β in (80) needed to obtain the asymptotic expansion of Theorem 6.2.

The main idea of the proof of Theorem 6.2 is just to work in the viscosity setting and use the expansions (84) and (85). The derivation of (85) also needs some care. We start by recalling the viscosity characterization of p-harmonic functions for $p < \infty$, see [36].

Definition 6.3. *For $1 < p < \infty$ consider the equation*

$$-div\left(|\nabla u|^{p-2} \nabla u \right) = 0.$$

(1) *A lower semi-continuous function u is a viscosity supersolution if for every $\phi \in C^2$ such that $u - \phi$ has a strict minimum at the point $x \in \Omega$ with $\nabla \phi(x) \neq 0$ we have*

$$-(p-2)\Delta_\infty \phi(x) - \Delta \phi(x) \geq 0.$$

(2) *An upper semi-continuous function u is a subsolution if for every $\phi \in C^2$ such that $u-\phi$ has a strict maximum at the point $x \in \Omega$ with $\nabla\phi(x) \neq 0$ we have*
$$-(p-2)\Delta_\infty \phi(x) - \Delta\phi(x) \leq 0.$$
(3) *Finally, u is a viscosity solution if it is both a supersolution and a subsolution.*

For the case $p=\infty$ we must restrict the class of test functions as in [51]. Let $S(x)$ denote the class of C^2 functions ϕ such that either $\nabla\phi(x) \neq 0$ or $\nabla\phi(x) = 0$ and the limit
$$\lim_{y\to x} \frac{2(\phi(y)-\phi(x))}{|y-x|^2} = \Delta_\infty \phi(x)$$
exists.

Definition 6.4. *Consider the equation $-\Delta_\infty u = 0$.*

(1) *A lower semi-continuous function u is a viscosity supersolution if for every $\phi \in S(x)$ such that $u-\phi$ has a strict minimum at the point $x \in \Omega$ we have*
$$-\Delta_\infty \phi(x) \geq 0.$$
(2) *An upper semi-continuous function u is a subsolution if for every $\phi \in S(x)$ such that $u-\phi$ has a strict maximum at the point $x \in \Omega$ we have*
$$-\Delta_\infty \phi(x) \leq 0.$$
(3) *Finally, u is a viscosity solution if it is both a supersolution and a subsolution.*

Proof of Theorem 6.2. We first consider asymptotic expansions for smooth functions that involve the infinity Laplacian ($p=\infty$) and the regular Laplacian ($p=2$).

Choose a point $x \in \Omega$ and a C^2-function ϕ defined in a neighborhood of x. Let x_1^ε and x_2^ε be the point at which ϕ attains its minimum and maximum in $\overline{B_\varepsilon(x)}$ respectively; that is,
$$\phi(x_1^\varepsilon) = \min_{y\in\overline{B_\varepsilon(x)}} \phi(y) \qquad \text{and} \qquad \phi(x_2^\varepsilon) = \max_{y\in\overline{B_\varepsilon(x)}} \phi(y).$$

Next, we use some ideas from [12]. Consider the Taylor expansion of the second order of ϕ
$$\phi(y) = \phi(x) + \nabla\phi(x)\cdot(y-x) + \frac{1}{2}\langle D^2\phi(x)(y-x),(y-x)\rangle + o(|y-x|^2)$$
as $|y-x| \to 0$. Evaluating this Taylor expansion of ϕ at the point x with $y = x_1^\varepsilon$ and $y = 2x - x_1^\varepsilon = \tilde{x}_1^\varepsilon$, we get
$$\phi(x_1^\varepsilon) = \phi(x) + \nabla\phi(x)(x_1^\varepsilon - x) + \frac{1}{2}\langle D^2\phi(x)(x_1^\varepsilon - x),(x_1^\varepsilon - x)\rangle + o(\varepsilon^2)$$
and
$$\phi(\tilde{x}_1^\varepsilon) = \phi(x) - \nabla\phi(x)(x_1^\varepsilon - x) + \frac{1}{2}\langle D^2\phi(x)(x_1^\varepsilon - x),(x_1^\varepsilon - x)\rangle + o(\varepsilon^2)$$
as $\varepsilon \to 0$. Adding the expressions, we obtain
$$\phi(\tilde{x}_1^\varepsilon) + \phi(x_1^\varepsilon) - 2\phi(x) = \langle D^2\phi(x)(x_1^\varepsilon - x),(x_1^\varepsilon - x)\rangle + o(\varepsilon^2).$$

Since x_1^ε is the point where the minimum of ϕ is attained, it follows that

$$\phi(\tilde{x}_1^\varepsilon) + \phi(x_1^\varepsilon) - 2\phi(x) \le \max_{y \in \overline{B_\varepsilon(x)}} \phi(y) + \min_{y \in \overline{B_\varepsilon(x)}} \phi(y) - 2\phi(x),$$

and thus

$$\frac{1}{2}\left\{ \max_{y \in \overline{B_\varepsilon(x)}} \phi(y) + \min_{y \in \overline{B_\varepsilon(x)}} \phi(y) \right\} - \phi(x) \ge \frac{1}{2}\langle D^2\phi(x)(x_1^\varepsilon - x), (x_1^\varepsilon - x)\rangle + o(\varepsilon^2). \tag{86}$$

Repeating the same process at the point x_2^ε we get instead

$$\frac{1}{2}\left\{ \max_{y \in \overline{B_\varepsilon(x)}} \phi(y) + \min_{y \in \overline{B_\varepsilon(x)}} \phi(y) \right\} - \phi(x) \le \frac{1}{2}\langle D^2\phi(x)(x_2^\varepsilon - x), (x_2^\varepsilon - x)\rangle + o(\varepsilon^2). \tag{87}$$

Next we derive a counterpart for the expansion with the usual Laplacian ($p = 2$). Averaging both sides of the classical Taylor expansion of ϕ at x we get

$$\fint_{B_\varepsilon(x)} \phi(y)\, dy = \phi(x) + \sum_{i,j=1}^{N} \frac{\partial^2 \phi}{\partial x_i^2}(x) \fint_{B_\varepsilon(0)} \frac{1}{2} z_i z_j\, dz + o(\varepsilon^2).$$

The values of the integrals in the sum above are zero when $i \ne j$. Using symmetry, we compute

$$\fint_{B_\varepsilon(0)} z_i^2\, dz = \frac{1}{N}\fint_{B_\varepsilon(0)} |z|^2\, dz = \frac{1}{N\omega_N \varepsilon^N}\int_0^\varepsilon \int_{\partial B_\rho} \rho^2\, dS\, d\rho = \frac{\sigma_{N-1}\varepsilon^2}{N(N+2)\omega_N} = \frac{\varepsilon^2}{(N+2)},$$

with the notation introduced after (85). We end up with

$$\fint_{B_\varepsilon(x)} \phi(y)\, dy - \phi(x) = \frac{\varepsilon^2}{2(N+2)}\Delta\phi(x) + o(\varepsilon^2). \tag{88}$$

Assume for the moment that $p \ge 2$ so that $\alpha \ge 0$. Multiply (86) by α and (88) by β and add. We arrive at the expansion valid for any smooth function ϕ:

$$\begin{aligned} &\frac{\alpha}{2}\left\{ \max_{y \in \overline{B_\varepsilon(x)}} \phi(y) + \min_{y \in \overline{B_\varepsilon(x)}} \phi(y) \right\} + \beta \fint_{B_\varepsilon(x)} \phi(y)\, dy - \phi(x) \\ &\ge \frac{\beta\varepsilon^2}{2(N+2)}\left((p-2)\left\langle D^2\phi(x)\left(\frac{x_1^\varepsilon - x}{\varepsilon}\right), \left(\frac{x_1^\varepsilon - x}{\varepsilon}\right)\right\rangle + \Delta\phi(x)\right) \\ &\quad + o(\varepsilon^2). \end{aligned} \tag{89}$$

We remark that $x_1^\varepsilon \in \partial B_\varepsilon(x)$ for $\varepsilon > 0$ small enough whenever $\nabla\phi(x) \ne 0$. In fact, suppose, on the contrary, that there exists a subsequence $x_1^{\varepsilon_j} \in B_{\varepsilon_j}(x)$ of minimum points of ϕ. Then, $\nabla\phi(x_1^{\varepsilon_j}) = 0$ and, since $x_1^{\varepsilon_j} \to x$ as $\varepsilon_j \to 0$, we have by continuity that $\nabla\phi(x) = 0$. A simple argument based on Lagrange multipliers then shows that

$$\lim_{\varepsilon \to 0} \frac{x_1^\varepsilon - x}{\varepsilon} = -\frac{\nabla\phi}{|\nabla\phi|}(x). \tag{90}$$

We are ready to prove that if the asymptotic mean value formula holds for u, then u is a viscosity solution. Suppose that function u satisfies the asymptotic expansion in the viscosity sense according to Definition 6.1. Consider a smooth ϕ such that $u-\phi$ has a strict minimum at x and $\phi \in S(x)$ if $p=\infty$. We obtain

$$0 \geq -\phi(x) + \frac{\alpha}{2}\left\{\max_{B_\varepsilon(x)} \phi + \min_{B_\varepsilon(x)} \phi\right\} + \beta \fint_{B_\varepsilon(x)} \phi(y)\,dy + o(\varepsilon^2),$$

and thus, by (89),

$$0 \geq \frac{\beta\varepsilon^2}{2(N+2)}\left((p-2)\left\langle D^2\phi(x)\left(\frac{x_1^\varepsilon - x}{\varepsilon}\right), \left(\frac{x_1^\varepsilon - x}{\varepsilon}\right)\right\rangle + \Delta\phi(x)\right) + o(\varepsilon^2).$$

If $\nabla\phi(x) \neq 0$ we take limits as $\varepsilon \to 0$. Taking into consideration (90) we get

$$0 \geq \frac{\beta}{2(N+2)}\left((p-2)\Delta_\infty\phi(x) + \Delta\phi(x)\right).$$

Suppose now that $p=\infty$ and that the limit

$$\lim_{y\to x} \frac{\phi(y)-\phi(x)}{|y-x|^2} = L$$

exists. We need to deduce that $L \leq 0$ from

$$0 \geq \frac{1}{2}\left\{\max_{B_\varepsilon(x)} \phi + \min_{B_\varepsilon(x)} \phi\right\} - \phi(x).$$

Let us argue by contradiction. Suppose that $L>0$ and choose $\eta>0$ small enough so that $L-\eta>0$. Use the limit condition to obtain the inequalities

$$(L-\eta)|x-y|^2 \leq \phi(x)-\phi(y) \leq (L+\eta)|x-y|^2,$$

for small $|x-y|$. Therefore, we get

$$\begin{aligned} 0 \geq\ & \frac{1}{2}\max_{B_\varepsilon(x)}(\phi-\phi(x)) + \frac{1}{2}\min_{B_\varepsilon(x)}(\phi-\phi(x)) \\ \geq\ & \frac{1}{2}\max_{B_\varepsilon(x)}(\phi-\phi(x)) \geq (\frac{L-\eta}{2})\varepsilon^2, \end{aligned}$$

which is a contradiction. Thus, we have proved that $L \geq 0$.

To prove that u is a viscosity subsolution, we first derive a reverse inequality to (89) by considering the maximum point of the test function, that is, using (87) and (88), and then choose a function ϕ that touches u from above. We omit the details.

To prove the converse implication, assume that u is a viscosity solution. In particular u is a subsolution. Let ϕ be a smooth test function such that $u-\phi$ has a strict local maximum at $x\in\Omega$. If $p=\infty$, we also assume $\phi\in S(x)$. If $\nabla\phi(x)\neq 0$, we get

$$-(p-2)\Delta_\infty\phi(x) - \Delta\phi(x) \leq 0. \qquad (91)$$

The statement to be proven is

$$\liminf_{\varepsilon\to 0+}\frac{1}{\varepsilon^2}\left(-\phi(x)+\frac{\alpha}{2}\left\{\max_{\overline{B_\varepsilon(x)}}\phi+\min_{\overline{B_\varepsilon(x)}}\phi\right\}+\beta\fint_{B_\varepsilon(x)}\phi(y)\,dy\right)\geq 0.$$

This again follows from (89). Indeed, divide (89) by ε^2, use (90), and deduce from (91) that the limit on the right hand side is bounded from below by zero.

For the case $p=\infty$ with $\nabla\phi(x)=0$ we assume the existence of the limit

$$\lim_{y\to x}\frac{\phi(y)-\phi(x)}{|y-x|^2}=L\geq 0$$

and observe that

$$\liminf_{\varepsilon\to 0+}\frac{1}{\varepsilon^2}\left(-\phi(x)+\frac{1}{2}\left\{\max_{\overline{B_\varepsilon(x)}}\phi+\min_{\overline{B_\varepsilon(x)}}\phi\right\}\right)\geq 0.$$

The argument for the case of supersolutions is analogous.

Finally, we need to address the case $1<p<2$. Since $\alpha\leq 0$ we use (87) instead of (86) to get a version of (89) with x_2^ε in place of x_1^ε. The argument then continues in the same way as before. □

7. Appendix. Probability results

One of the key tools in this paper is the optional stopping theorem for supermartingales. A remarkable fact is that the theory of martingales is efficient also in our nonlinear setting.

Definition 7.1. *Let $(\mathcal{O},\mathcal{F},\mathbb{P})$ be a probability space. A sequence of random variables*

$$\{M_k(\omega)\}_{k=1}^\infty,\quad \omega\in\mathcal{O}$$

is a martingale *with respect to the sub-σ-fields $\mathcal{F}_k\subset\mathcal{F}$, $k=1,2,\dots$, if the following conditions hold.*

1. *Each random variable M_k is measurable with respect to the corresponding σ-field $\mathcal{F}_k$, and $\mathbb{E}(|M_k|)<\infty$.*
2. *The σ-fields increase, that is, $\mathcal{F}_k\subset\mathcal{F}_{k+1}$.*
3. *For each random variable, we have the relation*

$$\mathbb{E}[M_k\,|\,\mathcal{F}_{k-1}]=M_{k-1}\ \textit{almost surely w.r.t. } \mathbb{P}.$$

Furthermore, the sequence is a supermartingale *if instead*

$$\mathbb{E}[M_k\,|\,\mathcal{F}_{k-1}]\leq M_{k-1}\ \textit{almost surely w.r.t. } \mathbb{P},$$

and a submartingale *if*

$$\mathbb{E}[M_k\,|\,\mathcal{F}_{k-1}]\geq M_{k-1}\ \textit{almost surely w.r.t. } \mathbb{P}.$$

When we specify strategies and a sequence of game positions when calculating expectations we fix the underlying σ-fields.

Next we recall the optional stopping theorem.

Theorem 7.2 (Optional Stopping). *Let* $\{M_k\}_{k=1}^{\infty}$ *be a martingale and let* τ *be a bounded stopping time. Then it holds that*

$$\mathbb{E}[M_\tau] = \mathbb{E}[M_0].$$

Furthermore, for a supermartingale it holds that

$$\mathbb{E}[M_\tau] \leq \mathbb{E}[M_0],$$

and for a submartingale that

$$\mathbb{E}[M_\tau] \geq \mathbb{E}[M_0].$$

For further details on martingales, the reader can consult, for example, Varadhan [54].

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