Tug-of-War games. Games that PDE people like to play

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Pisa, 2012

Main goal

Our aim is to explain through elementary examples a way in which elliptic PDEs arise in Probability.

1. First we show how simple is the relation between random walks and the Laplace operator.
2. Next, we will enter in what is the core of this course, the approximation by means of values of games of solutions to nonlinear problems like $p-$harmonic functions, that is, solutions to the PDE, $\mathrm{div}(|\nabla u|^{p-2}\nabla u) = 0$ (including the case $p = \infty$, of course).

Part 1: The Laplacian, Δ

Let us being by considering a bounded and smooth two-dimensional domain $\Omega \subset \mathbb{R}^2$ and assume that the boundary, $\partial\Omega$ is decomposed in two parts, Γ_1 and Γ_2 (that is, $\Gamma_1 \cup \Gamma_2 = \partial\Omega$ with $\Gamma_1 \cap \Gamma_2 = \emptyset$).

Let $(x, y) \in \Omega$ and ask the following question: assume that you move completely at random beginning at (x, y) what is the probability $u(x, y)$ of hitting the first part of the boundary Γ_1 the first time that the particle hits the boundary ?.

We assume that Ω is homogeneous. In addition, we assume that every time the movement is independent of its past history.

Part 1: The Laplacian, Δ

We will call Γ_1 the "open part" of the boundary and think that when we hit this part we can "exit" the domain, while we will call Γ_2 the "closed part" of the boundary, when we hit it we are dead.

A clever and simple way to solve the question runs as follows: First, we simplify the problem and approximate the movement by random increments of step h in each of the axes directions, with $h > 0$ small. From (x, y) the particle can move to $(x + h, y)$, $(x - h, y)$, $(x, y + h)$, or $(x, y - h)$, each movement being chosen at random with probability $1/4$.

Part 1: The Laplacian, Δ

Starting at (x, y), let $u_h(x, y)$ be the probability of hitting the exit part $\Gamma_1 + B_\delta(0)$ the first time that $\partial\Omega + B_\delta(0)$ is hitted when we move on the lattice of side h.

Observe that we need to enlarge a little the boundary to capture points on the lattice of size h (that do not necessarily lie on $\partial\Omega$).

Applying conditional expectations we get

$$u_h(x,y) = \frac{1}{4}u_h(x+h,y)+\frac{1}{4}u_h(x-h,y)+\frac{1}{4}u_h(x,y+h)+\frac{1}{4}u_h(x,y-h).$$

That is,

$$\begin{aligned} 0 = \; & \Big\{ u_h(x+h,y) - 2u_h(x,y) + u_h(x-h,y) \Big\} \\ & + \Big\{ u_h(x,y+h) - 2u_h(x,y) + u_h(x,y-h) \Big\}. \end{aligned}$$

Now, assume that u_h converges as $h \to 0$ to a function u uniformly in $\overline{\Omega}$. Note that this convergence can be proved rigorously.
Let ϕ a smooth function such that $u - \phi$ has a strict minimum at $(x_0, y_0) \in \Omega$. By the uniform convergence of u_h to u there are points (x_h, y_h) such that

$$(u_h - \phi)(x_h, y_h) \le (u_h - \phi)(x, y) + o(h^2) \qquad (x, y) \in \Omega$$

and

$$(x_h, y_h) \to (x_0, y_0) \qquad h \to 0.$$

Note that u_h is not necessarily continuous.

Part 1: The Laplacian, Δ

Hence, using that

$$u_h(x,y) - u_h(x_h,y_h) \geq \phi(x,y) - \phi(x_h,y_h) + o(h^2) \qquad (x,y) \in \Omega,$$

we get

$$0 \geq \Big\{ \phi(x_h + h, y_h) - 2\phi(x_h,y_h) + \phi(x_h - h, y_h) \Big\}$$

$$+ \Big\{ \phi(x_h, y_h + h) - 2\phi(x_h,y_h) + \phi(x_h, y_h - h) \Big\} + o(h^2).$$

Now, we just observe that

$$\phi(x_h + h, y_h) - 2\phi(x_h, y_h) + \phi(x_h - h, y_h) = h^2 \frac{\partial^2 \phi}{\partial x^2}(x_h, y_h) + o(h^2)$$

$$\phi(x_h, y_h + h) - 2\phi(x_h, y_h) + \phi(x_h, y_h - h) = h^2 \frac{\partial^2 \phi}{\partial y^2}(x_h, y_h) + o(h^2).$$

Hence, taking limit as $h \to 0$ we get

$$0 \geq \frac{\partial^2 \phi}{\partial x^2}(x_0, y_0) + \frac{\partial^2 \phi}{\partial y^2}(x_0, y_0).$$

Therefore, a uniform limit of the approximate values u_h, u, has the following property:

Each time that a smooth function ϕ touches u from below at a point (x_0, y_0) the derivatives of ϕ must verify,

$$0 \geq \frac{\partial^2 \phi}{\partial x^2}(x_0, y_0) + \frac{\partial^2 \phi}{\partial y^2}(x_0, y_0).$$

An analogous argument considering ψ a smooth function such that $u - \psi$ has a strict maximum at $(x_0, y_0) \in \Omega$ shows a reverse inequality. Therefore,

Each time that a smooth function ψ touches u from above at a point (x_0, y_0) the derivatives of ψ must verify

$$0 \leq \frac{\partial^2 \psi}{\partial x^2}(x_0, y_0) + \frac{\partial^2 \psi}{\partial y^2}(x_0, y_0).$$

Part 1: The Laplacian, Δ

But at this point we realize that this is exactly the definition of being u a **viscosity solution to the Laplace equation**

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Hence, we obtain that the uniform limit of the sequence of solutions to the approximated problems u_h, u is the unique viscosity solution (that is also a classical solution in this case) to the following boundary value problem

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ u = 1 & \text{on } \Gamma_1, \\ u = 0 & \text{on } \Gamma_2. \end{cases}$$

The boundary conditions can be easily obtained from the fact that $u_h \equiv 1$ in a neighborhood (of width h) of Γ_1 and $u_h \equiv 0$ in a neighborhood of Γ_2.

Note that we have only required *uniform* convergence to get the result, and hence no requirement is made on derivatives of the approximating sequence u_h.

Moreover, we do not assume that u_h is continuous.

Part 2: Δ_∞

Consider the $p-$laplacian: (formally, but we did it !!!)

$$\begin{aligned} \Delta_p u &= \text{div}\left(|\nabla u|^{p-2}\nabla u\right) = \\ &= |\nabla u|^{p-2}\Delta u + (p-2)|\nabla u|^{p-4}\sum_{i,j} u_{x_i}u_{x_j}u_{x_i,x_j} = \\ &= (p-2)|\nabla u|^{p-4}\left\{\frac{1}{p-2}|\nabla u|^2\Delta u + \sum_{i,j} u_{x_i}u_{x_j}u_{x_i,x_j}\right\} \end{aligned}$$

If we pass formally to the limit in the equation $\Delta_p u = 0$, we get the ∞-Laplacian, defined as

$$\Delta_\infty u = \sum_{i,j} u_{x_i}u_{x_j}u_{x_i,x_j} = Du \cdot D^2u \cdot (Du)^t.$$

Remarks

This limit can be justified in the viscosity sense.

This operator is non linear, not in divergence form , elliptic and degenerate.

Aronsson

$u(x,y) = x^{4/3} - y^{4/3}$ is infinity-harmonic in $\mathbb{R}^2$.

Absolutely Minimizing Lipschitz Extensions (AMLE)

Lipschitz extensions

Given a domain Ω and a Lipschitz function F defined on $\partial\Omega$, find a Lipschitz extension u, with the same Lipschitz constant as F: $\quad Lip_{\partial\Omega}\{F\} = Lip_{\Omega}\{u\}$

Many solutions (McShane-Withney extensions, 1934)

$$u_+(x) = inf_{y\in\partial\Omega}\{F(y) + L_F|x-y|\}$$

$$u_-(x) = sup_{y\in\partial\Omega}\{F(y) - L_F|x-y|\}$$

Aronsson: AMLE

Find a best Lipschitz extension u in every subdomain $D \subset \Omega$.

Existence

Absolutely Minimal Lipschitz extensions are viscosity solutions to

$$\begin{cases} \Delta_\infty u = 0 & \Omega, \\ u|_{\partial\Omega} = F & \partial\Omega. \end{cases}$$

Uniqueness

Jensen (1993), Barles and Busca (2001), Crandall, Aronsson and Juutinen (2004).

Tug-of-War games

Rules

- Two-person, zero-sum game: two players are in contest and the total earnings of one are the losses of the other.
- Player I, plays trying to maximize his expected outcome.
- Player II is trying to minimize Player I's outcome.
- $\Omega \subset \mathbb{R}^n$, bounded domain ; $\Gamma_D \subset \partial\Omega$ and $\Gamma_N \equiv \partial\Omega \setminus \Gamma_D$.
- $F : \Gamma_D \to \mathbb{R}$ be a Lipschitz continuous final payoff function.
- Starting point $x_0 \in \overline{\Omega} \setminus \Gamma_D$. A coin is tossed and the winner chooses a new position $x_1 \in \overline{B_\epsilon(x_0)} \cap \overline{\Omega}$.
- At each turn, the coin is tossed again, and the winner chooses a new game state $x_k \in \overline{B_\epsilon(x_{k-1})} \cap \overline{\Omega}$.
- Game ends when $x_\tau \in \Gamma_D$, and Player I earns $F(x_\tau)$ (Player II earns $-F(x_\tau)$)

Remark

Sequence $\{x_0, x_1, \cdots, x_N\}$ has some probability, which depends on

- The starting point x_0.
- The strategies of players, S_I and S_{II}.

Expected result

Taking into account the probability defined by the initial value and the strategies:

$$E^{x_0}_{S_I,S_{II}}(F(x_N))$$

"Perfect" players

- Player I chooses at each step the strategy which maximizes the result.
- Player II chooses at each step the strategy which minimizes the result.

Extremal cases

- $$u_I(x) = \sup_{S_I} \inf_{S_{II}} E^x_{S_I,S_{II}}(F(x_N))$$
- $$u_{II}(x) = \inf_{S_{II}} \sup_{S_I} E^x_{S_I,S_{II}}(F(x_N))$$

Definition

The game has a value $\Leftrightarrow$ $u_I = u_{II}$.

Theorem

Under very general hypotheses, the game has a value.

Reference

Peres-Schram-Sheffield-Wilson (2008).

Dynamic Programming Principle

Main Property (Dynamic Programming Principle)

$$u(x) = \frac{1}{2}\Big\{ \sup_{B_\epsilon(x)} u + \inf_{B_\epsilon(x)} u \Big\}.$$

Denote u the value of the game, and assume that there are x_M, x_m such that:

- $u(x_M) = \max_{|x-y|\le\epsilon} u(y)$.
- $u(x_m) = \min_{|x-y|\le\epsilon} u(y)$.

Dynamic Programming Principle

$$u(x) = \frac{1}{2}\Big\{ \max_{B_\epsilon(x)} u + \min_{B_\epsilon(x)} u \Big\} = \frac{1}{2}\Big\{ u(x_M) + u(x_m) \Big\}$$

that is,

$$0 = \Big\{ u(x_M) + u(x_m) - 2u(x) \Big\}$$

Idea

- $x_M \approx x + \epsilon \dfrac{\nabla u(x)}{|\nabla u(x)|}$
- $x_m \approx x - \epsilon \dfrac{\nabla u(x)}{|\nabla u(x)|}$
- $\dfrac{u(x + \epsilon \vec{v}) + u(x - \epsilon \vec{v}) - 2u(x)}{\epsilon^2} \equiv$ discretization of the second derivative in the direction of $\vec{v}$

Therefore

Dynamic programming principle $\approx$ discretization of the second derivative in the direction of the gradient.

Again the infinity-Laplacian

Remark

Second derivative in the direction of the gradient
$\equiv$ ∞-Laplacian.

Theorem by Peres-Schramm-Sheffield-Wilson.

Existence and uniqueness of the limit of the values of ϵ-Tug-of-war games as $\epsilon \to 0$

$\Rightarrow$

Alternative proof of existence and uniqueness for the problem

$$\begin{cases} \Delta_\infty u = 0, & \Omega, \\ u|_{\partial\Omega} = F & \partial\Omega. \end{cases}$$

Remark

The existence and uniqueness result for the limit of the values of ϵ-Tug-of-war games holds true even if the final payoff function F is defined only on a subset of the boundary $\Gamma_D \subset \partial\Omega$

Theorem (Garcia Azorero-Charro-R.)

The limit is a viscosity solution to the mixed boundary value problem:

$$\begin{cases} \Delta_\infty u = 0 & \Omega, \\ u = F & \Gamma_D, \\ \frac{\partial u}{\partial \nu} = 0 & \partial\Omega \setminus \Gamma_D. \end{cases}$$

Consequence

Uniqueness of viscosity solutions for the mixed boundary value problem.

Definition of viscosity solutions.

G. Barles, (1993). Let

$$\begin{cases} F(x, \nabla u, D^2 u) = 0 & \Omega, \\ B(x, u, \nabla u) = 0 & \partial\Omega. \end{cases}$$

Model

$$F(x, \nabla u, D^2 u) = -\Delta u,$$

$$B(x, u, \nabla u) = \frac{\partial u}{\partial \eta}.$$

Definition of viscosity solutions.

Definition 1

A lower semicontinuous function u is a **supersolution** in the sense of viscosity, if for any test function $\phi \in C^2(\overline{\Omega})$ such that $u - \phi$ has a strict minimum at $x_0 \in \overline{\Omega}$ with $u(x_0) = \phi(x_0)$ (that is, the graph of ϕ touches the graph of u **from below** at x_0), it holds:

i) If $x_0 \in \Omega$, $F(x_0, \nabla\phi(x_0), D^2\phi(x_0)) \geq 0$.

ii) If $x_0 \in \partial\Omega$,
$\max\{B(x_0, \phi(x_0), \nabla\phi(x_0)),\ F(x_0, \nabla\phi(x_0), D^2\phi(x_0))\} \geq 0$.

Definition 2

An upper semicontinuous function u is a **subsolution** in the sense of viscosity if for each test function $\psi \in C^2(\overline{\Omega})$ such that $u - \psi$ has a strict maximum at $x_0 \in \overline{\Omega}$ with $u(x_0) = \psi(x_0)$ (that is, the graph of ψ touches the graph of u **from above** at x_0), it holds:

i) If $x_0 \in \Omega$, $F(x_0, \nabla \psi(x_0), D^2\psi(x_0)) \leq 0$.

ii) If $x_0 \in \partial\Omega$,

$\min\{B(x_0, \psi(x_0), \nabla\psi(x_0)),\ F(x_0, \nabla\psi(x_0), D^2\psi(x_0))\} \leq 0$.

Definition 3

u is a solution in the sense of viscosity if it is a supersolution and a subsolution.

Proof of the result in Ω

Assume

- $\sup_{B_\epsilon(x)} u^\epsilon + \inf_{B_\epsilon(x)} u^\epsilon(x_m^\epsilon) - 2u^\epsilon = 0.$
- $u^\epsilon \to u(x)$ uniformly.

$x_0 \in \Omega$. We want to show:

- If ϕ touches the graph of u from below at x_0, then $-\Delta_\infty \phi(x_0) \geq 0$.
- If ψ touches the graph of u from above at x_0, then $-\Delta_\infty \psi(x_0) \leq 0$.

- Suppose ϕ touches the graph of u from below at x_0. Then $u - \phi$ has a strict local minimum at x_0.
- Then $u^\epsilon - \phi$ has an approximate local minimum at some x_ϵ, with $x_\epsilon \to x_0$, that is,

$$(u^\epsilon - \phi)(x_\epsilon) \leq (u^\epsilon - \phi)(x) + o(\epsilon^2) \qquad x \in \Omega.$$

- Discrete programming principle for u^ϵ gives:

$$\phi(\tilde{x}^\epsilon_M) + \phi(\tilde{x}^\epsilon_m) - 2\phi(x_\epsilon) \geq o(\epsilon^2).$$

- Lemma

$$\tilde{x}^\epsilon_{M,m} = x_\epsilon \pm \epsilon\{v_\epsilon + o(1)\}$$

  where

$$v_\epsilon = \frac{D\phi(x_\epsilon)}{|D\phi(x_\epsilon)|}$$

Now, consider the Taylor expansion of second order of ϕ

$$\phi(y) = \phi(x_\epsilon) + \nabla\phi(x_\epsilon)\cdot(y - x_\epsilon) + \frac{1}{2}\langle D^2\phi(x_\epsilon)(y - x_\epsilon), (y - x_\epsilon)\rangle + o(|y - x_\epsilon|^2)$$

as $|y - x_\epsilon| \to 0$. Evaluating the above expansion at the point at which ϕ attains its minimum in $\overline{B_\epsilon(x_\epsilon)}$, x_2^ϵ, we get

$$\phi(x_2^\epsilon) = \phi(x_\epsilon) + \nabla\phi(x_\epsilon)(x_2^\epsilon - x_\epsilon) + \frac{1}{2}\langle D^2\phi(x_\epsilon)(x_2^\epsilon - x_\epsilon), (x_2^\epsilon - x_\epsilon)\rangle + o(\epsilon^2),$$

as $\epsilon \to 0$. Evaluating at its symmetric point in the ball $\overline{B_\epsilon(x_\epsilon)}$, that is given by

$$\tilde{x}_2^\epsilon = 2x_\epsilon - x_2^\epsilon$$

we get

$$\phi(\tilde{x}_2^\epsilon) = \phi(x_\epsilon) - \nabla\phi(x_\epsilon)(x_2^\epsilon - x_\epsilon) + \frac{1}{2}\langle D^2\phi(x_\epsilon)(x_2^\epsilon - x_\epsilon), (x_2^\epsilon - x_\epsilon)\rangle + o(\epsilon^2).$$

Adding both expressions we obtain

$$\phi(\tilde{x}_2^\epsilon) + \phi(x_2^\epsilon) - 2\phi(x_\epsilon) = \langle D^2\phi(x_\epsilon)(x_2^\epsilon - x_\epsilon), (x_2^\epsilon - x_\epsilon)\rangle + o(\epsilon^2).$$

We observe that, by our choice of x_2^ϵ as the point where the minimum is attained,

$$\phi(\tilde{x}_2^\epsilon)+\phi(x_2^\epsilon)-2\phi(x_\epsilon) \leq \max_{y\in \overline{B_\epsilon(x)}\cap\Omega} \phi(y)+ \min_{y\in \overline{B_\epsilon(x)}\cap\Omega} \phi(y)-2\phi(x_\epsilon) \leq o(\epsilon^2).$$

Therefore

$$\langle D^2\phi(x_\epsilon)(x_2^\epsilon - x_\epsilon), (x_2^\epsilon - x_\epsilon)\rangle \leq o(\epsilon^2).$$

Note that we have

$$\lim_{\epsilon\to 0} \frac{x_2^\epsilon - x_\epsilon}{\epsilon} = -\frac{\nabla\phi}{|\nabla\phi|}(x_0).$$

Then we get, dividing by ϵ^2 and passing to the limit,

$$0 \leq -\Delta_\infty\phi(x_0).$$

1-d

Let us analyze in detail the one-dimensional game and its limit as $\epsilon \to 0$.
We set $\Omega = (0, 1)$ and play the $\epsilon-$game. To simplify we assume that $\epsilon = 1/2^n$.
As final payoff, we take

$$F(-\infty, 0) = 0, \qquad F(1, +\infty) = 1.$$

Let us assume that there exists a value that we call u_ϵ and proceed, in several steps, with the analysis of this sequence of functions u_ϵ for ϵ small. All the calculations below hold both for u_I^ϵ and for u_{II}^ϵ.

Step 1.

$u_\epsilon(0) = 0$ and $u_\epsilon(1) = 1$. Moreover, $0 \leq u_\epsilon(x) \leq 1$ (the value functions are uniformly bounded).

Step 2. u_ϵ is increasing in x and strictly positive in $(0, 1]$. Indeed, if $x < y$ then for every pair of strategies S_I, S_{II} for Player I and II beginning at x we can construct strategies beginning at y in such a way that

$$x_{i,x} \leq x_{i,y}$$

(here $x_{i,x}$ and $x_{i,y}$ are the positions of the game after i movements beginning at x and y respectively). It follows that

$$u_\epsilon(x) \leq u_\epsilon(y).$$

Now, we just observe that there is a positive probability of obtaining a sequence of $1/\epsilon$ consecutive heads (exactly $2^{-1/\epsilon}$), hence the probability of reaching $x = 1$ when the first player uses the strategy that points ϵ to the right is strictly positive. Therefore,

$$u_\epsilon(x) > 0,$$

for every $x \neq 0$.

Step 3.

In this one dimensional case it is easy to identify the optimal strategies for players I and II: to jump ϵ to the right for Player I and to jump ϵ to the left for Player II. That is, if we are at x, the optimal strategies lead to

$$x \to \min\{x + \epsilon, 1\}$$

for Player I, and to

$$x \to \max\{x - \epsilon, 0\}$$

for Player II.

Step 4.

u_ϵ is constant in every interval of the form $(k\epsilon, (k+1)\epsilon)$ for $k = 1, ..., N$ (we denote by N the total number of such intervals in $(0, 1]$).
Indeed, from step 3 we know what are the optimal strategies for both players, and hence the result follows noticing that the number of steps that one has to advance to reach $x = 0$ (or $x = 1$) is the same for every point in $(k\epsilon, (k+1)\epsilon)$.

Remark Note that u_ϵ is necessarily discontinuos at every point of the form $y_k = k\epsilon \in (0, 1)$.

Step 5.

Let us call $a_k := u_\epsilon \,|_{(k\epsilon,(k+1)\epsilon)}$. Then we have

$$a_0 = 0, \qquad a_k = \frac{1}{2}\Big\{a_{k-1} + a_{k+1}\Big\},$$

for every $i = 2, ..., n-1$, and

$$a_n = 1.$$

Notice that these identities follow from the Dynamic Programming Principle.

Note the similarity with a finite difference scheme used to solve $u_{xx} = 0$ in $(0,1)$ with boundary conditions $u(0) = 0$ and $u(1) = 1$.

Step 6.

We have

$$u_\epsilon(x) = \epsilon k, \qquad x \in (k\epsilon, (k+1)\epsilon).$$

Indeed, taking

$$a_k = \epsilon k$$

we obtain the unique solution to the formulas obtained in step 5.

Step 7.

$$\lim_{\epsilon\to 0} u_\epsilon(x) = x,$$

uniformly in $[0,1]$.

Remark Note that the limit function

$$u(x) = x$$

is the unique viscosity (and classical) solution to

$$\Delta_\infty u(x) = (u_{xx}(u_x)^2)(x) = 0 \qquad x \in (0,1),$$

with boundary conditions

$$u(0) = 0, \qquad u(1) = 1.$$

The problem that solves the value function of a tug-of-war game is given by:

$$\begin{cases} -\Delta_\infty^\varepsilon u = 0 & \text{in } \Omega, \\ u = f & \text{on } \Omega_\varepsilon \setminus \Omega, \end{cases}$$

here

$$-\Delta_\infty^\varepsilon u = \sup_{B_\epsilon(x)} u + \inf_{B_\epsilon(x)} u - 2u(x)$$

is the discrete infinity Laplacian.
It holds

$$\lim_{\varepsilon \to 0} \text{Tug-of-War solutions} = \text{AMLE of } f.$$

QUESTIONS

1) Is the discrete infinity Laplacian the problem associated to an optimal extension problem ??.

2) Is there a $p-$Laplacian approach to the discrete infinity Laplacian ??.

Given $u : \Omega_\varepsilon \to \mathbb{R}$ and $D \subset \Omega$,

$$S_\varepsilon(u, D) \quad := \quad \sup_{\substack{x \in D,\ y \in D_\varepsilon \\ |x-y| \le \varepsilon}} \frac{|u(x) - u(y)|}{\varepsilon}$$

$$(D \text{ convex}) \quad = \quad \sup_{x \in D,\, y \in D_\varepsilon,\, x \neq y} \frac{|u(x) - u(y)|}{d_\varepsilon(x, y)} \ge L_{d_\varepsilon}(u, D)$$

$$d_\varepsilon(x,y) = \begin{cases} 0 & \text{if } x = y, \\ \varepsilon & \text{if } 0 < |x-y| \le \varepsilon, \\ 2\varepsilon & \text{if } \varepsilon < |x-y| \le 2\varepsilon, \\ \vdots & \end{cases}$$

Note that in $(\Omega_\varepsilon, d_\varepsilon)$ the boundary of any subset is empty.

THEOREM (Mazon, R., Toledo)

$u : \Omega_\varepsilon \to \mathbb{R}$ is AMLE of $f|_{\Omega_\epsilon \setminus \Omega}$ with S_ϵ

if and only if

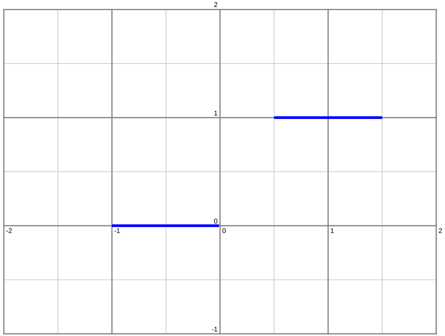
u is the unique solution to

$$\begin{cases} -\Delta^\varepsilon_\infty u = 0 & \text{in } \Omega, \\ u = f & \text{on } \Omega_\varepsilon \setminus \Omega, \end{cases}$$

This is the right extension problem to deal with when $\Delta^\varepsilon_\infty$ is considered.

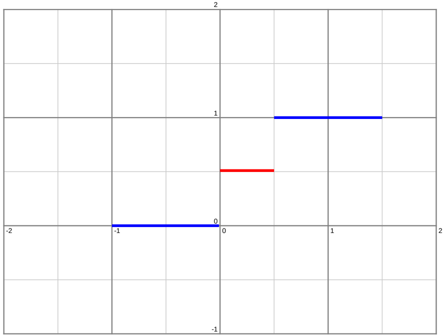
Example

For $\varepsilon = 1$, $\Omega =]0, \frac{1}{2}[$ and $f = 0\chi_{]-1,0]} + 1\chi_{[\frac{1}{2},\frac{3}{2}[}$.



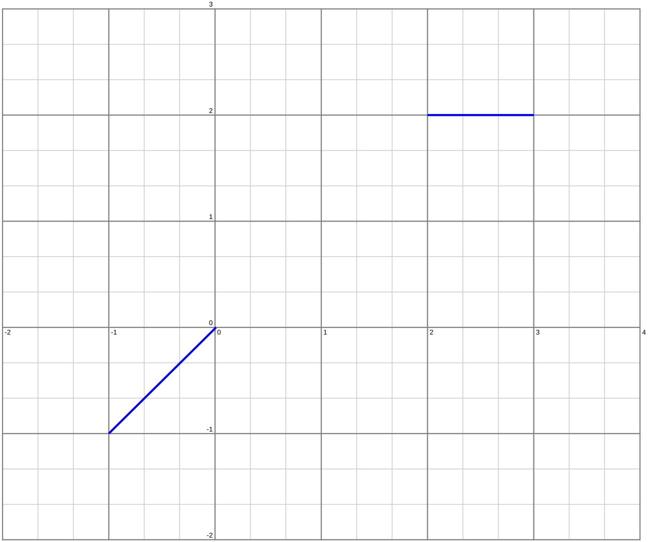
Example

For $\varepsilon = 1$, $\Omega =]0, \frac{1}{2}[$ and $f = 0\chi_{]-1,0]} + 1\chi_{[\frac{1}{2},\frac{3}{2}[}$.



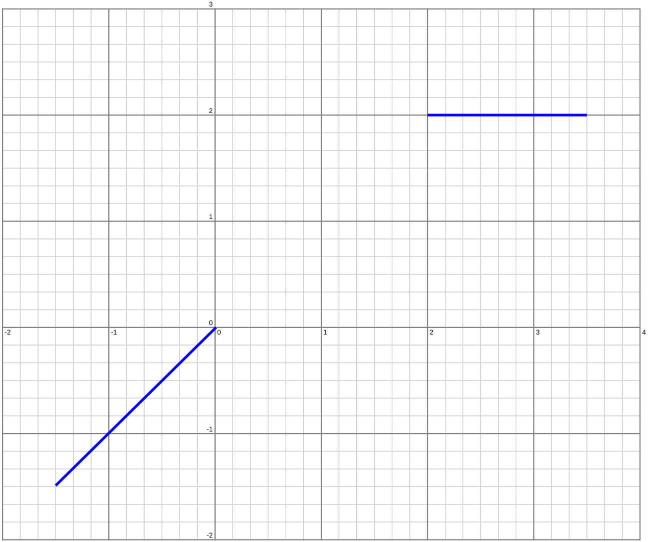
is the unique $\text{AMLE}_1(f, \Omega)$.

Extending $x\chi_{]-1,0]}(x) + 2\chi_{[2,3[}(x)$ to $]0,2[$ for $\varepsilon = 1$



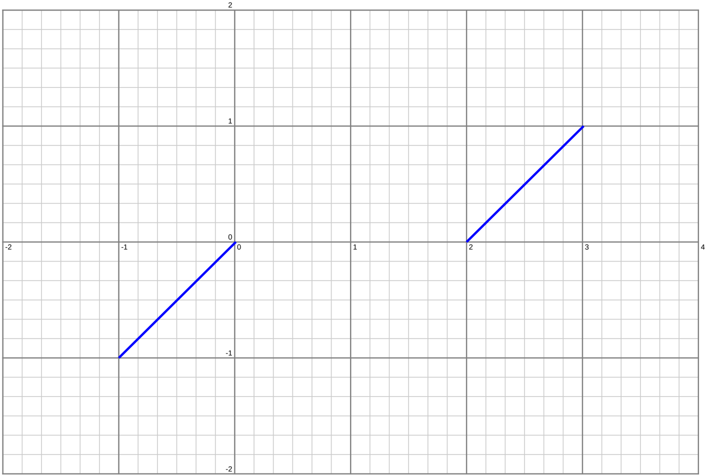
Extending $x\chi_{]-1,0]}(x) + 2\chi_{[2,3[}(x)$ to $]0,2[$ for $\varepsilon = 1$

Extending $x\chi_{]-\frac{3}{2},0]}(x) + 2\chi_{[2,\frac{7}{2}[}(x)$ to $]0,2[$ for $\varepsilon = \frac{3}{2}$



Extending $x\chi_{]-\frac{3}{2},0]}(x) + 2\chi_{[2,\frac{7}{2}[}(x)$ to $]0,2[$ for $\varepsilon = \frac{3}{2}$

Extending $x\chi_{]-1,0]}(x) + (x-2)\chi_{[2,3[}(x)$ to $]0,2[$ for $\varepsilon = 1$



Extending $x\chi_{]-1,0]}(x)+(x-2)\chi_{[2,3[}(x)$ to $]0,2[$ for $\varepsilon = 1$

Similarly to what happens in the Euclidean case, the $\text{AMLE}_\varepsilon(f, \Omega)$ can be obtained as the limit, as $p \to \infty$, of solutions u_p of p–Laplacian problems but of nonlocal nature.

Let $J : \mathbb{R}^N \to \mathbb{R}$ be a nonnegative, radial, continuous function, strictly positive in $B(0,1)$, vanishing in $\mathbb{R}^N \setminus B(0,1)$ and such that $\int_{\mathbb{R}^N} J(z)\,dz = 1$. Let $J_\varepsilon(z) = \frac{1}{\varepsilon^N} J\left(\frac{z}{\varepsilon}\right)$.

For $1 < p < +\infty$ and $f \in L^\infty(\Omega_\varepsilon \setminus \overline{\Omega})$, consider the energy functional

$$G^{J_\varepsilon}_{p,f}(u) = \frac{1}{2p}\int_\Omega \int_\Omega J_\varepsilon(x-y)|u(y)-u(x)|^p\,dy\,dx + \frac{1}{p}\int_\Omega \int_{\Omega_\varepsilon \setminus \Omega} J_\varepsilon(x-y)|f(y)-u(x)|^p\,dy\,dx,$$

and the operator

$$\mathrm{B}^{J_\varepsilon}_{p,f}(u)(x) = -\int_{\Omega_\varepsilon} J_\varepsilon(x-y)|u_f(y)-u(x)|^{p-2}(u_f(y)-u(x))\,dy, \quad x \in \Omega,$$

where

$$u_f(x) := \begin{cases} u(x) & \text{if } x \in \Omega, \\ f(x) & \text{if } x \in \Omega_\varepsilon \setminus \Omega. \end{cases}$$

Theorem

Assume that $p \geq 2$. *Then, there exists a unique* $u_p^\varepsilon \in L^p(\Omega)$ *such that*

$$G_{p,f}^{J_\varepsilon}(u_p^\varepsilon) = \min\{G_{p,f}^{J}(u) \ : \ u \in L^p(\Omega)\}.$$

Moreover, u_p^ε *is the solution of the nonlocal Euler–Lagrange equation (the nonlocal p–Laplacian problem with Dirichlet boundary conditions)*

$$\mathrm{B}_{p,f}^{J_\varepsilon}(u_p^\varepsilon) = 0.$$

$$\frac{1}{\varepsilon^p}\mathrm{B}_{p,f}^{J_\varepsilon} \to -\Delta_p + \textit{Dirichlet BC}\ (u|_{\partial\Omega} = f) \qquad \textit{as}\ \varepsilon \to 0\,.$$

Theorem (Mazon, R., Toledo)

$$u_p^\varepsilon \to v_\infty \in L^\infty(\Omega) \quad \textit{strongly in any } L^q(\Omega) \textit{ as } p \to +\infty,$$

$$\left(G_{p,f}^{J_\varepsilon}(u_p^\varepsilon)\right)^{1/p} \to \inf_{u \in L^\infty(\Omega)} L_\varepsilon(u_f, \Omega) \quad \textit{as } p \to +\infty,$$

$$\inf_{u \in L^\infty(\Omega)} L_\varepsilon(u_f, \Omega) = L_\varepsilon((v_\infty)_f, \Omega),$$

and

$$(v_\infty)_f \textit{ is } \mathrm{AMLE}_\varepsilon(f, \Omega).$$

The main difficulty is to prove that $(v_\infty)_f$ is $\mathrm{AMLE}_\varepsilon(f, \Omega)$.

Pointwise vs. viscosity solutions for $-\Delta_\infty^\varepsilon$

Since the solutions are discontinuous in general, to work with viscosity solutions we need to consider the upper and lower semi-continuous envelopes of u in Ω_ε

$$u^*(x) := \limsup_{y\in\Omega_\varepsilon, y\to x} u(y) \qquad \text{and} \qquad u_*(x) := \liminf_{y\in\Omega_\varepsilon, y\to x} u(y).$$

• u is a *viscosity subsolution* if $-\Delta_\infty^\varepsilon \phi(x_0) \le 0$ when $\phi \in C(\Omega_\varepsilon)$, $\phi(x_0) = u^*(x_0)$ and $u^* - \phi$ achieves a maximum at $x_0 \in \Omega$.
• u is a *viscosity supersolution* if $-\Delta_\infty^\varepsilon \phi(x_0) \ge 0$ when $\phi \in C(\Omega_\varepsilon)$, $\phi(x_0) = u^*(x_0)$ and $u_* - \phi$ achieves a minimum at $x_0 \in \Omega$.

Theorem If $-\Delta_\infty^\varepsilon u(x) = 0$ for all $x \in \Omega$, then u is a viscosity solution.
The converse is not true.

The $p-$Laplacian

Consider the $p-$laplacian: (formally)

$$\begin{aligned} \Delta_p u &= \text{div}\left(|\nabla u|^{p-2}\nabla u\right) = \\ &= |\nabla u|^{p-2}\Delta u + (p-2)|\nabla u|^{p-2}\frac{\nabla u}{|\nabla u|}D^2 u\frac{\nabla u}{|\nabla u|} = \\ &= |\nabla u|^{p-2}\left\{\Delta u + (p-2)\Delta_\infty u\right\}. \end{aligned}$$

Then, we have that u is a solution to $\Delta_p u = 0$ if and only if

$$(p-2)\Delta_\infty u + \Delta u = 0.$$

Consider solutions to

$$\alpha \left\{ \frac{1}{2} \sup_{B_\varepsilon(x)} u + \frac{1}{2} \inf_{B_\varepsilon(x)} u - u(x) \right\} + \beta \left\{ \frac{1}{|B_\varepsilon(x)|} \int_{B_\varepsilon(x)} u - u(x) \right\} = 0.$$

Here α and β are given by $\alpha + \beta = 1$ and $\alpha/\beta = C_N(p-2)$.

These functions are called *p-harmonious*.

Properties of *p-harmonious* functions

Theorem

There exists a unique p-harmonious function in Ω with given boundary values F (in an appropriate sense).

Furthermore, p-harmonious functions satisfy the *strong maximum principle*.

Theorem

If u_ε is the p-harmonious in Ω with boundary values F, then

$$\sup_{\Gamma_\varepsilon} F \geq \sup_{\Omega} u_\varepsilon.$$

Moreover, if there is a point $x_0 \in \Omega$ such that $u_\varepsilon(x_0) = \sup_{\Gamma_\varepsilon} F$, then u_ε is constant in Ω.

Properties of *p-harmonious* functions

and the *strong comparison principle*,

Theorem

Let u_ε and v_ε be p-harmonious functions with continuous boundary values, $F_u \geq F_v$. Then, if there exists a point $x_0 \in \Omega$ such that $u_\varepsilon(x_0) = v_\varepsilon(x_0)$, it follows that

$$u_\varepsilon = v_\varepsilon \quad \text{in} \quad \Omega.$$

Note that the validity of the strong comparison principle is not known for the p-harmonic functions in $\mathbb{R}^n$, $n \geq 3$.

Proof

The proof heavily uses the fact that $p < \infty$. Note that it is known that the strong comparison principle does not hold for infinity harmonic functions.

First, one has to show that, $F_u \geq F_v$ implies $u_\varepsilon \geq v_\varepsilon$.

By the definition of a p-harmonious function, we have

$$u_\varepsilon(x_0) = \frac{\alpha}{2} \left\{ \sup_{\overline{B_\varepsilon}(x_0)} u_\varepsilon + \inf_{\overline{B_\varepsilon}(x_0)} u_\varepsilon \right\} + \beta \fint_{B_\varepsilon(x_0)} u_\varepsilon dy$$

and

$$v_\varepsilon(x_0) = \frac{\alpha}{2} \left\{ \sup_{\overline{B_\varepsilon}(x_0)} v_\varepsilon + \inf_{\overline{B_\varepsilon}(x_0)} v_\varepsilon \right\} + \beta \fint_{B_\varepsilon(x_0)} v_\varepsilon dy.$$

Next we compare the right hand sides. Since $u_\varepsilon \geq v_\varepsilon$, it follows that

$$\sup_{\overline{B_\varepsilon}(x_0)} u_\varepsilon \leq \sup_{\overline{B_\varepsilon}(x_0)} v_\varepsilon,$$

$$\inf_{\overline{B_\varepsilon}(x_0)} u_\varepsilon \leq \inf_{\overline{B_\varepsilon}(x_0)} v_\varepsilon,$$

$$\fint_{B_\varepsilon(x_0)} u_\varepsilon dy \leq \fint_{B_\varepsilon(x_0)} v_\varepsilon dy.$$

As

$$u_\varepsilon(x_0) = v_\varepsilon(x_0),$$

we must have equalities. In particular, we have equality in the third inequality, and thus

$$u_\varepsilon = v_\varepsilon \qquad \text{almost everywhere in} \quad B_\varepsilon(x_0).$$

The connectedness of Ω implies that

$$u_\varepsilon = v_\varepsilon \qquad \text{almost everywhere in} \quad \Omega \cup \Gamma_\varepsilon.$$

Properties of *p-harmonious* functions

Theorem

Let u_ε be the unique p-harmonious function with boundary values F. Then

$$u_\varepsilon \to u \qquad \text{uniformly in} \quad \overline{\Omega}$$

as $\varepsilon \to 0$.
Here u the unique viscosity solution u to

$$\begin{cases} \Delta_p u(x) = \mathrm{div}(|\nabla u|^{p-2}\nabla u)(x) = 0, & x \in \Omega \\ u(x) = F(x), & x \in \partial\Omega, \end{cases}$$

In terms of games

Rules

- $F : \partial\Omega \to \mathbb{R}$ be a Lipschitz continuous final payoff function.
- Starting point $x_0 \in \Omega$.
- An unfair coin is tossed (with probabilities α and β).
- If there is a head then a fair coin is tossed and the winner (player I or II) chooses a new position $x_1 \in \overline{B_\epsilon(x_0)}$.
- If we have a tail then a point in $x_1 \in \overline{B_\epsilon(x_0)}$ is selected at random (uniform probability).
- At each turn, the game is played again.
- Game ends when $x_\tau \in \partial\Omega + B(0, \varepsilon)$, and Player I earns $F(x_\tau)$ (Player II earns $-F(x_\tau)$)

A proof based on games

Theorem If v is a p-harmonious function with boundary values F_v in Γ such that $F_v \geq F_u$, then

$$v \geq u.$$

A proof based on games

Player I follows any strategy and Player II follows a strategy S_{II}^0 such that at $x_{k-1} \in \Omega$ he chooses to step to a point that almost minimizes v, that is, to a point $x_k \in \overline{B_\epsilon(x_{k-1})}$ such that

$$v(x_k) \leq \inf_{\overline{B_\epsilon(x_{k-1})}} v + \eta 2^{-k}$$

for some fixed $\eta > 0$.

A proof based on games

We start from the point x_0. It follows that

$$
\begin{aligned}
&\mathbb{E}^{x_0}_{S_I,S^0_{II}}[v(x_k)+\eta 2^{-k}\mid x_0,\ldots,x_{k-1}]\\
&\le \frac{\alpha}{2}\left\{\inf_{B_\epsilon(x_{k-1})} v+\eta 2^{-k}+\sup_{B_\epsilon(x_{k-1})} v\right\}+\beta\int_{B_\epsilon(x_{k-1})} v+\eta 2^{-k}\\
&\le v(x_{k-1})+\eta 2^{-(k-1)},
\end{aligned}
$$

where we have estimated the strategy of Player I by sup and used the fact that v is p-harmonious.

A proof based on games

Thus

$$M_k = v(x_k) + \eta 2^{-k}$$

is a supermartingale.
[Optional Stopping] Let $\{M_k\}_{k=1}^{\infty}$ be a martingale and let τ be a bounded stopping time. Then it holds that

$$\mathbb{E}[M_\tau] = \mathbb{E}[M_0].$$

Furthermore, for a supermartingale it holds that

$$\mathbb{E}[M_\tau] \leq \mathbb{E}[M_0],$$

A proof based on games

Since $F_v \geq F_u$ at Γ, we deduce

$$
\begin{aligned}
& u(x_0) = \sup_{S_I} \inf_{S_{II}} \mathbb{E}^{x_0}_{S_I,S_{II}}[F_u(x_\tau)] \leq \sup_{S_I} \mathbb{E}^{x_0}_{S_I,S^0_{II}}[F_v(x_\tau) + \eta 2^{-\tau}] \\
& \leq \sup_{S_I} \liminf_{k\to\infty} \mathbb{E}^{x_0}_{S_I,S^0_{II}}[v(x_{\tau\wedge k}) + \eta 2^{-(\tau\wedge k)}] \\
& \leq \sup_{S_I} \mathbb{E}_{S_I,S^0_{II}}[M_0] = v(x_0) + \eta,
\end{aligned}
$$

where $\tau \wedge k = \min(\tau, k)$, and we used Fatou's lemma as well as the optional stopping theorem for M_k. Since η was arbitrary this proves the claim.

A mean value property for the $p-$Laplacian

It is a very well known fact that one can find in any elementary textbook of PDEs that u is harmonic, that is $\Delta u = 0$, if and only if it verifies the mean value property

$$u(x) = \frac{1}{|B_\varepsilon(x)|} \int_{B_\varepsilon(x)} u.$$

We have the following *asymptotic mean value property* characterization for $p-$harmonic functions, in the viscosity sense,

u verifies the *mean value property*

$$u(x) = \frac{\alpha}{2}\left\{\max_{\overline{B}_\varepsilon(x)} u + \min_{\overline{B}_\varepsilon(x)} u\right\} + \beta\left\{\frac{1}{|B_\varepsilon(x)|}\int_{B_\varepsilon(x)} u\right\} + o(\varepsilon^2),$$

as $\varepsilon \to 0$, **if and only if**

u is $p-$harmonic

$$\mathrm{div}(|\nabla u|^{p-2}\nabla u)(x) = 0.$$

Here α and β are given by $\alpha + \beta = 1$ and $\alpha/\beta = C_N(p-2)$, with $C_N = \frac{1}{2|B(0,1)|}\int_{B(0,1)} z_N^2\, dz$.

Some intuition

Let us formally expand the p-Laplacian as before

$$\Delta_p u = (p-2)|\nabla u|^{p-4} \langle D^2 u \nabla u, \nabla u \rangle + |\nabla u|^{p-2} \Delta u.$$

Suppose that u is a smooth function with $\nabla u \neq 0$. Then u is a solution to $\Delta_p u = 0$ if and only if

$$(p-2)\Delta_\infty u + \Delta u = 0.$$

Now, classical Taylor expansions give

$$u(x) - \fint_{B_\varepsilon(x)} u = -\varepsilon^2 \Delta u(x) \frac{1}{N} \int_{B(0,1)} |z|^2 + o(\varepsilon^2)$$

Some intuition

and, as before,

$$
\begin{aligned}
u(x) & -\frac{1}{2}\left\{\max_{B_\varepsilon(x)} u + \min_{B_\varepsilon(x)} u\right\} \\
& \approx u(x) - \frac{1}{2}\left\{u\left(x + \varepsilon\frac{\nabla u(x)}{|\nabla u(x)|}\right) + u\left(x - \varepsilon\frac{\nabla u(x)}{|\nabla u(x)|}\right)\right\} \\
& = -\varepsilon^2 \Delta_\infty u(x) + o(\varepsilon^2).
\end{aligned}
$$

Now, multiply by suitable constants and add up the formulas so that we reconstruct the $p-$Laplacian. This process gives us the choices of the constants α and β.