Tug-of-War games and the ∞-Laplacian.

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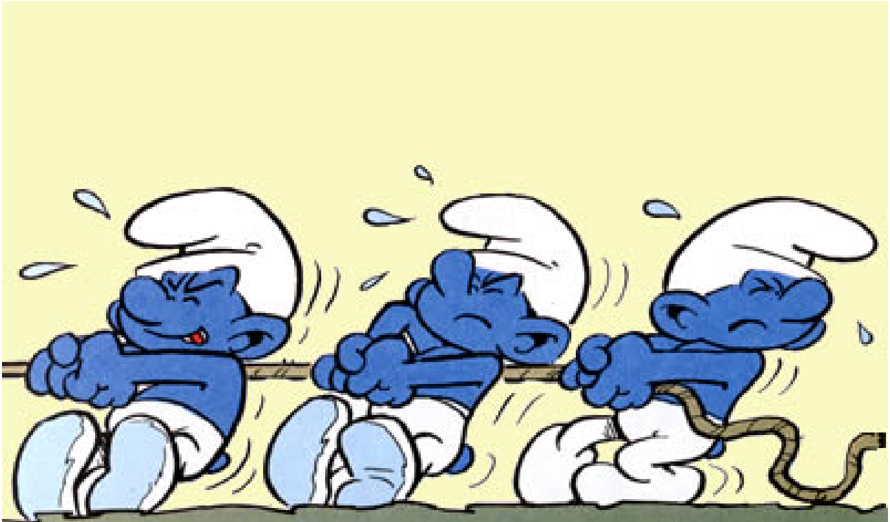
tug-of-war



Plan

1. Introduce the ∞-Laplacian. AMLE. Existence (easy).
2. Uniqueness (difficult).
3. Games (funny).
4. More games and mean value properties (demagogic).

Introduction

Our main goal is to introduce and study the infinity Laplacian that is the second order elliptic operator given by

$$\Delta_\infty u(x) := \left(D^2 u \, \nabla u \right) \cdot \nabla u(x) = \sum_{i,j=1}^{N} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_i x_j}(x).$$

Introduction

There are two excellent surveys concerning the infinity Laplacian operator

G. Aronsson, M.G. Crandall and P. Juutinen, *A tour of the theory of absolutely minimizing functions*. Bull. Amer. Math. Soc., 41 (2004), 439–505.

M.G. Crandall. *A visit with the* $\infty-$*Laplace equation*. CALCULUS OF VARIATIONS AND NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS Lecture Notes in Mathematics, 2008, Volume 1927/2008, 75-122.

Introduction

Some history. It all began in 1967 with Gunnar Aronsson's paper

G. Aronsson. *Extensions of functions satisfying Lipschitz conditions*. Ark. Mat. 6 (1967), 551–561.

Aronsson looked for optimal Lipschitz extensions of a given datum. Recall that a function $u : \Omega \mapsto \mathbb{R}$ is Lipschitz if

$$\mathrm{Lip}(u, \Omega) = \inf\{L : |u(x) - u(y)| \leq L|x - y|, \forall x, y \in \Omega\}$$

is finite.

Introduction

The problem of minimizing the Lipschitz constant subject to a Dirichlet condition was known to have a largest and a smallest solution, given by explicit formulas, from the works of McShane and Whitney

Introduction

In fact,

$$u^*(x) = \min_{y \in \partial\Omega} F(y) + \mathrm{Lip}(F, \partial\Omega)|x - y|$$

and

$$u_*(x) = \max_{y \in \partial\Omega} F(y) - \mathrm{Lip}(F, \partial\Omega)|x - y|$$

are optimal Lipschitz extensions.

Introduction

The following question naturally arose: is it possible to find a canonical Lipschitz constant extension of F into Ω that would enjoy comparison and stability properties? Furthermore, could this special extension be unique once the boundary data is fixed? The point of view was that the problem was an extension problem.

Aronsson's clever proposal in this regard was to introduce the class of absolutely minimizing functions for the Lipschitz constant.

Introduction

During these research Aronsson was led to the now famous pde $\Delta_\infty u = 0$.

Jensen proves uniqueness of viscosity solutions and the validity of a comparison principle using approximations to the equation (and variants of it) by $p-$Laplacian type problems as $p \to \infty$.

R. Jensen, *Uniqueness of Lipschitz extensions: minimizing the sup norm of the gradient*. Arch. Rational Mech. Anal. 123 (1993), 51–74.

Introduction

After existence and uniqueness, one wants to know about regularity. One of the key ideas to prove regularity results was the fact that solutions have the property of comparison with cones.

Introduction

The best known explicit irregular absolutely minimizing function (outside of the relatively regular solutions of eikonal equations) was exhibited again by Aronsson, who showed in 1984, that

$$u(x, y) = x^{4/3} - y^{4/3}$$

is absolutely minimizing in $\mathbb{R}^2$ for the Lipschitz constant and for the L^∞-norm of the gradient.

Introduction

One of the challenging open problems in the subject is concerned with regularity: are ∞-harmonic functions C^1?

Note that the explicit solution $u(x, y) = x^{4/3} - y^{4/3}$ prevents for general C^2 regularity results.

Savin (and also Evans-Savin) proved that they are C^1 in the case $N = 2$. Differentiability in any dimension was recently proved by Evans and Smart.

Introduction

The second main goal of these notes is to introduce the reader (expert or not) to some important techniques and results in the theory of second order elliptic PDEs and their connections with game theory.

The fundamental works of Doob, Hunt, Kakutani, Kolmogorov and many others have shown the profound and powerful connection between the classical linear potential theory and the corresponding probability theory. The idea behind the classical interplay is that harmonic functions and martingales share a common origin in mean value properties. This approach turns out to be useful in the nonlinear theory as well.

Introduction

Next, we will enter in what is the core of this course, the approximation by means of values of games of solutions to nonlinear problems like $p-$harmonic functions, that is, solutions to the PDE,

$$\mathrm{div}(|\nabla u|^{p-2}\nabla u) = 0$$

including of course the case $p = \infty$.

Existence

As we mentioned in the introduction, the infinity Laplacian is given by

$$\Delta_\infty u(x) := \left(D^2 u \, \nabla u\right) \cdot \nabla u(x) = \sum_{i,j=1}^{N} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_i x_j}(x).$$

Existence

Passing to the limit as $p \to \infty$ in the equation $\Delta_p u = 0$.
Viscosity solutions

Now we present a way of obtaining existence of viscosity solutions to

$$\begin{cases} \Delta_\infty u(x) = 0, & x \in \Omega, \\ u(x) = F(x), & x \in \partial\Omega, \end{cases}$$

taking the limit as $p \to \infty$ along subsequences of solutions u_p to

$$\begin{cases} \Delta_p u_p(x) = 0, & x \in \Omega, \\ u(x) = F(x), & x \in \partial\Omega. \end{cases}$$

Existence

Let us state the definitions of a weak and a viscosity solution.

Definition (Weak)

A function $u\colon \Omega \to \mathbb{R}$ is a *weak solution* in Ω if $u \in W^{1,p}(\Omega)$, verifies

$$-\int_\Omega |\nabla u|^{p-2}\nabla u \nabla \varphi = 0,$$

for every $\varphi \in C_0^\infty(\Omega)$ and

$$u = F \qquad \text{on } \partial\Omega$$

in the sense of traces.

Existence

Now, concerning viscosity solutions we have the following definition.

In our case, we have to consider the following expression

$$F(x, u, \xi, S) = |\xi|^{p-2}\mathrm{trace}(S) + (p-2)|\xi|^{p-4}\langle S\xi, \xi\rangle.$$

Existence

Definition

An upper semicontinuous function $u\colon \Omega \to \mathbb{R}$ is a *viscosity subsolution* in Ω if, whenever $\hat{x} \in \Omega$ and $\varphi \in C^2(\Omega)$ are such that $u - \varphi$ has a strict local maximum at $\hat{x}$, then

$$F(\hat{x}, \varphi(\hat{x}), \nabla\varphi(\hat{x}), D^2\varphi(\hat{x})) = \Delta_p\varphi(\hat{x}) \geq 0.$$

Existence

Definition

A lower semicontinuous function $v\colon \Omega \to \mathbb{R}$ is a *viscosity supersolution* in Ω if $-v$ is a viscosity subsolution, that is, whenever $\hat{x} \in \Omega$ and $\varphi \in C^2(\Omega)$ are such that $v - \varphi$ has a strict local minimum at $\hat{x}$, then

$$F(\hat{x}, \varphi(\hat{x}), \nabla\varphi(\hat{x}), D^2\varphi(\hat{x})) = \Delta_p\varphi(\hat{x}) \leq 0.$$

Definition

Finally, a continuous function $h\colon \Omega \to \mathbb{R}$ is a *viscosity solution* in Ω if it is both a viscosity subsolution and a viscosity supersolution.

Existence

Lemma There exists a unique weak solution and it is characterized as being a minimizer for the functional

$$F_p(u) = \int_\Omega \frac{|\nabla u|^p}{p}$$

in the set $\{u \in W^{1,p}(\Omega) \ : \ u = F \text{ on } \partial\Omega\}$.

Existence

Proof The functional F_p is coercive and weakly semicontinuous, hence the mimimum is attained. It is easy to check that this mimimum is a weak solution in the sense of the first Definition. Uniqueness comes from the strict convexity of the functional.

Existence

Proposition A continuous weak solution is a viscosity solution.

Proof Let $x_0 \in \Omega$ and a test function ϕ such that $u(\hat{x}) = \phi(\hat{x})$ and $u - \phi$ has a strict minimum at $\hat{x}$. We want to show that

$$F(\hat{x}, \phi(\hat{x}), \nabla\phi(\hat{x}), D^2\phi(\hat{x})) \leq 0,$$

that is,

$$(p-2)|D\phi|^{p-4}\Delta_\infty\phi(\hat{x}) + |D\phi|^{p-2}\Delta\phi(\hat{x}) \leq 0.$$

Existence

Assume that this is not the case, then there exists a radius $r > 0$ such that

$$(p-2)|D\phi|^{p-4}\Delta_\infty \phi(x) + |D\phi|^{p-2}\Delta\phi(x) > 0,$$

for every $x \in B(\hat{x}, r)$. Set $m = \inf_{|x-\hat{x}|=r}(u-\phi)(x)$ and let $\psi(x) = \phi(x) + m/2$. This function ψ verifies $\psi(\hat{x}) > u(\hat{x})$ and

$$\operatorname{div}(|D\psi|^{p-2}D\psi) > 0.$$

Existence

Multiplying by $(\psi - u)^+$ extended by zero outside $B(\hat{x}, r)$ we get

$$-\int_{\{\psi > u\}} |D\psi|^{p-2} D\psi D(\psi - u) > 0.$$

Taking $(\psi - u)^+$ as test function in the weak form of the problem we get

$$-\int_{\{\psi > u\}} |Du|^{p-2} Du D(\psi - u) = 0.$$

Existence

Hence,

$$
\begin{aligned}
& C(N,p) \int_{\{\psi > u\}} |D\psi - Du|^p \\
& \leq \int_{\{\psi > u\}} \langle |D\psi|^{p-2} D\psi - |Du|^{p-2} Du, D(\psi - u) \rangle \\
& < 0,
\end{aligned}
$$

a contradiction.

This proves that u is a viscosity supersolution. The proof of the fact that u is a viscosity subsolution runs as above, we omit the details.

Existence

Now we prove that there is a subsequence of u_p that converges uniformly.

Lemma There exists a subsequence of u_p and a function $u_\infty \in W^{1,\infty}(\Omega)$ such that

$$\lim_{p_j \to \infty} u_{p_j}(x) = u_\infty(x)$$

uniformly in $\overline{\Omega}$.

Existence

Proof Using that u_p is a minimizer of the associated energy functional we obtain, for any Lipschitz extension v of F,

$$\int_\Omega |Du_p|^p \leq \int_\Omega |Dv|^p \leq (\mathrm{Lip}(v,\Omega))^p|\Omega|.$$

Hence, we obtain that

$$\left(\int_\Omega |Du_p|^p\right)^{1/p} \leq \mathrm{Lip}(v,\Omega)|\Omega|^{1/p}.$$

Existence

Next, fix m, and take $p > m$. We have,

$$\begin{aligned}
&\left(\int_\Omega |Du_p|^m\right)^{1/m} \\
&\leq |\Omega|^{\frac{1}{m}-\frac{1}{p}} \left(\int_\Omega |Du_p|^p\right)^{1/p} \\
&\leq |\Omega|^{\frac{1}{m}-\frac{1}{p}} \mathrm{Lip}(v, \Omega) |\Omega|^{1/p},
\end{aligned}$$

where $|\Omega|^{\frac{1}{m}-\frac{1}{p}} \to |\Omega|^{\frac{1}{m}}$ as $p \to \infty$.

Existence

Hence, there exists a weak limit (and hence uniform since we can assume that $m > N$) in $W^{1,m}(\Omega)$ that we will denote by u_∞. This weak limit has to verify

$$\left(\int_\Omega |Du_\infty|^m\right)^{1/m} \leq |\Omega|^{\frac{1}{m}} \mathrm{Lip}(v, \Omega).$$

As the above inequality holds for every m, we get that $u_\infty \in W^{1,\infty}(\Omega)$ and moreover, $\|Du_\infty\|_{L^\infty(\Omega)} \leq \mathrm{Lip}(v, \Omega)$.

Existence

Theorem A uniform limit u_∞ of u_p as $p \to \infty$ is a viscosity solution of

$$\begin{cases} \Delta_\infty u(x) = 0, & x \in \Omega, \\ u(x) = F(x), & x \in \partial\Omega. \end{cases}$$

Existence

Proof From the uniform convergence it is clear that u_∞ is continuous and verifies $u_\infty = F$ on $\partial\Omega$.

Next, to look for the equation that u_∞ satisfies in the viscosity sense, assume that $u_\infty - \phi$ has a strict minimum at $x_0 \in \Omega$. We have to check that

$$\Delta_\infty \phi(\hat{x}) \leq 0.$$

Existence

By the uniform convergence of u_{p_i} to u_∞ there are points x_{p_i} such that $u_{p_i} - \phi$ has a minimum at x_{p_i} with $x_{p_i} \to \hat{x}$ as $p_i \to \infty$. At those points we have

$$(p_i - 2)|D\phi|^{p_i-4}\Delta_\infty\phi(x_{p_i}) + |D\phi|^{p_i-2}\Delta\phi(x_{p_i}) \leq 0.$$

If $D\varphi(\hat{x}) = 0$ then the equation is verified, hence we may assume that $D\varphi(\hat{x}) \neq 0$, and hence $D\varphi(x_{p_i}) \neq 0$ for every p_i large enough.

Existence

Therefore, we get

$$\Delta_\infty \phi(x_{p_i}) \leq \frac{1}{p_i - 2} |D\phi|^2 \Delta\phi(x_{p_i}).$$

Then passing to the limit we obtain

$$\Delta_\infty \phi(\hat{x}) \leq 0.$$

That is, u_∞ is a viscosity supersolution.

The fact that it is a viscosity subsolution is analogous, using a test function ψ such that $u_\infty - \psi$ has a strict maximum at x_0.

AMLE-Comparison with cones

L^∞ minimization problems in the calculus of variations
Let us consider the functionals

$$G_\infty(u) = \|Du\|_{L^\infty(\Omega)}$$

and

$$L(u) = \mathrm{Lip}(u, \Omega)$$

where $\mathrm{Lip}(u, \Omega)$ stands for the Lipschitz constant of u in Ω, that is,

$$\mathrm{Lip}(u, \Omega) = \sup_{x,y \in \Omega} \frac{|u(x) - u(y)|}{|x - y|}.$$

AMLE-Comparison with cones

Note that we can also write

$$\mathrm{Lip}(u,\Omega) = \inf\{L : |u(x) - u(y)| \leq L|x - y|, \forall x, y \in \Omega\}.$$

Also note that one has

$$G_\infty(u) = \mathrm{Lip}(u,\Omega)$$

if Ω is convex, but equality does not hold in general.

Our goal will be to minimize these functionals.

AMLE-Comparison with cones

First, to give an idea that this task is not easy in general, let us present an example of nonuniqueness of the minimum.
Let us consider the optimal Lpschitz extension problem, that is, given F defined on $\partial\Omega$ find a solution to

find u that minimizes $\text{Lip}(u, \overline{\Omega})$ among functions with $u = F$ on $\partial\Omega$.

Assume that F is Lipschitz (otherwise this problem does not have a minimizer). Then we have

$$\text{Lip}(u, \overline{\Omega}) \geq \text{Lip}(F, \partial\Omega)$$

for every u that extends F.

AMLE-Comparison with cones

Therefore, any Lipschitz extension u of F with $\mathrm{Lip}(u, \overline{\Omega}) = \mathrm{Lip}(F, \partial\Omega)$ is a solution to our minimization problem. Now, it is easy to construct such extensions, in fact, let

$$u^*(x) = \min_{y \in \partial\Omega} F(y) + \mathrm{Lip}(F, \partial\Omega)|x - y|$$

and

$$u_*(x) = \max_{y \in \partial\Omega} F(y) - \mathrm{Lip}(F, \partial\Omega)|x - y|.$$

AMLE-Comparison with cones

Note that we have

$$u_*(x) \leq u^*(x), \qquad \forall x \in \Omega.$$

EX 1. *Prove that u^* and u_* are solutions to our minimization problem. Moreover, show that there are the maximal and the minimal solution in the sense that any other solution u verifies*

$$u_*(x) \leq u(x) \leq u^*(x), \qquad \forall x \in \Omega.$$

AMLE-Comparison with cones

From this property we have a clear criteria for uniqueness, uniqueness for minimizers of our problem holds if and only if

$$u^*(x) = u_*(x).$$

There is no reason for these extremal solutions to coincide, and it is rare that they do. The example below shows this, no matter how nice Ω might be.

AMLE-Comparison with cones

Let $\Omega = B(0,1) \subset \mathbb{R}^2 = \{x \in \mathbb{R}^2 \ : \ |x| < 1\}$ and let $F : \partial\Omega \mapsto \mathbb{R}$ be such that $-1 \leq F \leq 1$ and the Lipschitz constant $L := \mathrm{Lip}(F, \partial\Omega)$ is large. Then, according to the definitions of u^* and u_*, we have

$$u^*(0) > u_*(0)$$

if there exists a $\delta > 0$ such that

$$F(z) - L|z| + \delta = F(z) - L + \delta < F(z) + L + \delta = F(z) + L|z| + \delta,$$

$\forall z \in \partial\Omega$.

AMLE-Comparison with cones

Since $-1 \leq F \leq 1$ this holds for $L > 1$ (taking $0 < \delta < L - 1$), in fact, we have

$$F(z) - L + \delta < 1 - L + (L - 1) = 0 < \delta \leq F(z) + L + \delta.$$

EX 2. *Let* $\Omega = (-1, 0) \cup (0, 1)$ *and let* $F(-1) = F(0) = 0$, $F(1) = 1$. *Find* u^* *and* u_*.
Modify this example to show that even if Ω *is bounded, then it is not necessarily true that*

$$\max_{\overline{\Omega}} u^* \leq \max_{\partial\Omega} F.$$

Moreover, show that $F_1 \leq F_2$ *does not necessarily imply that* $u_1^* \leq u_2^*$ *(here* u_i *is the maximal solution to the extension problem associated with* F_i*).*

AMLE-Comparison with cones

While this sort of nonuniqueness only takes place if the functional involved is not strictly convex, it is more significant here that the previously mentioned functionals are ”not local”. In fact, look at the local functional

$$G_2(u) = \int_\Omega |Du|^2\, dx.$$

AMLE-Comparison with cones

For this functional it holds that if u minimizes G_2 among functions that verify $u = F$ on $\partial\Omega$ then u restricted to a subset of Ω, D, minimizes the functional in D among functions that coincide with u on ∂D. This is what we mean by "local". This property does not hold for minimizers of G_∞ of for minimizers of $\text{Lip}(u, \Omega)$.

AMLE-Comparison with cones

This lack of locality can be corrected by a notion which is directly build from locality. Given a general nonnegative functional $G(u, D)$ which makes sense for each open subset D of the domain Ω, it is said that $u : \Omega \mapsto \mathbb{R}$ is *absolutely minimizing* for G in Ω provided that

$$G(u, D) \leq G(v, D), \qquad \text{for every } v \text{ such that } u|_{\partial D} = v|_{\partial D}.$$

AMLE-Comparison with cones

That is, u is also a minimizer for G in every subdomain D of Ω taking boundary data $u|_{\partial D}$.

When we take $G(u, \Omega)$ to be $\mathrm{Lip}(u, \Omega)$ we said that we are dealing with an *absolutely minimizing Lipschitz extension* (AMLE for short) of $F = u|_{\partial\Omega}$ in Ω.

AMLE-Comparison with cones

u is AMLE if and only if u has comparison with cones

Let us start by introducing what is a cone.

Definition

The function

$$C(x) = a|x - z| + b$$

is called a cone with slope a and vertex z.

AMLE-Comparison with cones

We also need the definition of u enjoying comparison with cones.

Definition

A continuous function u enjoys comparison with cones from above in Ω iff for every $a \in \mathbb{R}$, $V \subset\subset \Omega$ and $z \notin V$, it holds

$$u(x) - a|x - z| \leq \max_{y \in \partial V} u(y) - a|y - z|, \qquad x \in V.$$

A continuous function u has comparison with cones from below iff $-u$ has comparison with cones from above.
When both conditions hold we say that the continuous function u has comparison with cones.

AMLE-Comparison with cones

Note that the condition to have comparison with cones from above can be written as

$$u(x) - C(x) \leq \max_{y \in \partial V} u(y) - C(y), \qquad x \in V,$$

for every cone C with vertex $z \notin V$. That is, the maximum of $u - C$ is attained on ∂V.

AMLE-Comparison with cones

Assume now that u is a continuous function that has comparison with cones.
First, remark that comparison with cones from above can be rewritten as follows: for every $a, c \in \mathbb{R}$ and $z \notin V$ it holds

$$u(x) \leq c + a|x - z|, \text{ for } x \in \Omega, \text{ if it holds for } x \in \partial\Omega.$$

Similarly comparison with cones from below can be written as, for every $a, c \in \mathbb{R}$ and $z \notin V$ it holds

$$u(x) \geq c + a|x - z|, \text{ for } x \in \Omega, \text{ if it holds for } x \in \partial\Omega.$$

AMLE-Comparison with cones

Now, our aim is to show that, if u has comparison with cones then, for any $x \in V$,

$$\mathrm{Lip}(u, \partial(V \setminus \{x\})) = \mathrm{Lip}(u, \partial V \cup \{x\}) = \mathrm{Lip}(u, \partial V). \qquad (1)$$

To prove this we have to show that when $y \in \partial V$,

$$u(y) - \mathrm{Lip}(u, \partial V)|x - y| \leq u(x) \leq u(y) + \mathrm{Lip}(u, \partial V)|x - y|.$$

These inequalities hold since they hold for every $x \in \partial V$ and, from he fact that u has comparison with cones, they hold for every $x \in V$.

AMLE-Comparison with cones

Now, let $x, y \in V$, using (1) twice we obtain that

$$\mathrm{Lip}(u, \partial V) = \mathrm{Lip}(u, \partial(V \setminus \{x\})) = \mathrm{Lip}(u, \partial(V \setminus \{x, y\})).$$

Since $x, y \in \partial(V \setminus \{x, y\})$ we get that

$$|u(x) - u(y)| \leq \mathrm{Lip}(u, \partial V)|x - y|,$$

and we conclude that u is AMLE in V.

AMLE-Comparison with cones

Now, let us prove that u has comparison with cones if u is AMLE.

To this end let us observe that the Lipschitz constant of a cone $C(x) = a|x - z| + b$ is given by

$$\mathrm{Lip}(C, V) = |a|$$

and moreover, if $z \notin V$ we have

$$\mathrm{Lip}(C, \partial V) = |a|.$$

AMLE-Comparison with cones

Now, assume that $z \notin V$ and let

$$W = \left\{ x \in V \ : \ u(x) - a|x - z| > \max_{w \in \partial V} (u(w) - a|w - z|) \right\}.$$

Our goal is to show that W is empty. If it is not empty, then it is an open set, $W \subset V$, and

$$u(x) = a|x - z| + \max_{w \in \partial V} (u(w) - a|w - z|) := C(x)$$

for $x \in \partial W$.

AMLE-Comparison with cones

Therefore $u = C$ on ∂W and since u is AMLE we have $\text{Lip}(u, W) = \text{Lip}(C, \partial W) = |a|$.

Now, if $x_0 \in W$ the ray of C that contain x_0, i.e., $t \mapsto z + t(x_0 - z)$, contains a segment in W that contains x_0 and its endpoints are on ∂W.

Since $t \mapsto C(z + t(x_0 - z)) = at|x_0 - z|$ is linear on the segment with slope $a|x_0 - z|$ (hence its Lipschitz constant is $|a||x_0 - z|$) while $t \mapsto u(z + t(x_0 - z))$ also has $|a||x_0 - z|$ as Lipschitz constant on the segment and has the same boundary values at the endpoints; hence both functions are the same.

AMLE-Comparison with cones

Therefore,

$$C(z + t(x_0 - z)) = u(z + t(x_0 - z))$$

on the segment. In particular, $C(x_0) = u(x_0)$, a contradiction with the fact that $x_0 \in W$.

This proves that u has comparison with cones.

AMLE-Comparison with cones

If u has comparison with cones then $\Delta_\infty u = 0$

We know that when u has comparison with cones from above it holds

$$u(x) \leq u(y) + \max_{w \in \partial B_r(y)} \left(\frac{u(w) - u(y)}{r} \right) |x - y|$$
$$= \max_{w \in \partial B_r(y)} \frac{u(w)}{r} |x - y| + u(y) \left(1 - \frac{|x - y|}{r} \right),$$

for any $x \in B_r(y) \subset\subset \Omega$. This inequality follows since it holds trivially for $x \in \partial B_r(y)$.

AMLE-Comparison with cones

Now rewrite it as

$$u(x) - u(y) \leq \max_{w \in \partial B_r(y)} (u(w) - u(x)) \frac{|x-y|}{r - |x-y|}, \qquad (2)$$

for any $x \in B_r(y) \subset\subset \Omega$.

AMLE-Comparison with cones

Assume that u is twice differentiable at x, that is, there are a vector p and a matrix X such that

$$u(z) = u(x) + \langle p; z - x\rangle + \frac{1}{2}\langle X(z-x); z-x\rangle + o(|z-x|^2). \quad (3)$$

In fact,

$$p = Du(x), \qquad X = D^2u(x).$$

AMLE-Comparison with cones

We will prove that

$$\Delta_\infty u(x) = \langle D^2 u(x) Du(x); Du(x) \rangle = \langle Xp; p \rangle \geq 0. \qquad (4)$$

That is, comparison with cones from above implies $\Delta_\infty u \geq 0$ at points where u is twice differentiable.

We can assume that $p \neq 0$ (otherwise the inequality that we want to prove holds trivially).

AMLE-Comparison with cones

We use (3) in (2) with two choices of z. First, let us take

$$z = y = x - \lambda p$$

and expand (2) according to (3), we have,

$$\begin{aligned} & -\langle p; y - x\rangle - \langle X(y-x); y - x\rangle + o(|y-x|^2) \\ & \leq \max_{w \in \partial B_r(y)} (u(w) - u(x)) \frac{|x-y|}{r - |x-y|}. \end{aligned}$$

AMLE-Comparison with cones

Now, consider the point $w_{r,\lambda}$ at which the maximum in the right hand side is attained and use it as z in (3) to obtain, after dividing by $\lambda > 0$,

$$
\begin{aligned}
&|p|^2 + \lambda \frac{1}{2}\langle Xp; p\rangle + o(\lambda) \\
&\leq \left(\langle p; w_{r,\lambda} - x\rangle + \frac{1}{2}\langle X(w_{r,\lambda} - x); (w_{r,\lambda} - x)\rangle + o((r+\lambda)^2) \right) \\
&\qquad \times \frac{|p|}{r - \lambda |p|}.
\end{aligned}
$$

AMLE-Comparison with cones

Taking $\lambda \to 0$ we obtain

$$|p|^2 \leq \left(\langle p; \frac{w_r - x}{r} \rangle + \frac{1}{2} \langle X(\frac{w_r - x}{r}); (w_r - x) \rangle + o(r) \right) |p| \quad (5)$$

where w_r is a limit point of $w_{r,\lambda}$ and hence we have $w_r \in \partial B_r(x)$, that is, $\frac{w_r - x}{r}$ is a unit vector.

AMLE-Comparison with cones

From the previous inequality, it follows that

$$\frac{w_r - x}{r} \to \frac{p}{|p|} \tag{6}$$

as $r \to 0$. Again from the inequality (5), using that,

$$\langle p; \frac{w_r - x}{r}\rangle |p| \leq |p|^2$$

we obtain

$$0 \leq \lim_{r \to 0} \frac{1}{2} \langle X(\frac{w_r - x}{r}); (\frac{w_r - x}{r})\rangle$$

AMLE-Comparison with cones

that is,

$$0 \leq \langle X \frac{p}{|p|}; \frac{p}{|p|} \rangle,$$

which implies (4).

EX 3. *Show that this argument prove that when $\nabla u(x) = p = 0$ we obtain*

$$D^2 u(x) = X \text{ has a nonnegative eigenvalue.}$$

AMLE-Comparison with cones

Now, assume that φ is a smooth test function, that is, $u - \varphi$ has a local maximum at x, then

$$\varphi(x) - \varphi(y) \leq u(x) - u(y)$$

and

$$u(w) - u(x) \leq \varphi(w) - \varphi(x).$$

AMLE-Comparison with cones

Hence we have that (2) holds with u replaced by φ and from our previous argument, using that φ is smooth we get

$$\Delta_\infty \varphi(x) \geq 0, \qquad \text{if } D\varphi(x) \neq 0,$$

and

$$D^2\varphi(x) \text{ has a nonnegative eigenvalue} \qquad \text{if } D\varphi(x) = 0.$$

AMLE-Comparison with cones

In any case, we have

$$u - \varphi \text{ has a local maximum at } x \Rightarrow \Delta_\infty \varphi(x) \geq 0,$$

that is, if u has comparison with cones from above, then u is a viscosity subsolution to $\Delta_\infty u = 0$ in Ω.

EX 4. *Show that if u has comparison with cones from below then it is a viscosity supersolution to* $\Delta_\infty u = 0$ *in* Ω*.*

AMLE-Comparison with cones

$\Delta_\infty u = 0$ **implies comparison with cones**

First, let us compute the $\infty-$Laplacian of a radial function, we get

$$\Delta_\infty G(|x|) = G''(|x|)(G'(|x|))^2$$

when $x \neq 0$. Hence, for any small $\gamma > 0$

$$\Delta_\infty(a|x-z| - \gamma|x-z|^2) = -2\gamma(a - 2\gamma|x-z|)^2 < 0,$$

for $x \neq z$.

AMLE-Comparison with cones

Now, if u verifies $\Delta_\infty u \geq 0$ in the viscosity sense (that is, u is a viscosity subsolution to $\Delta_\infty u = 0$), then we have that $u(x) - (a|x - z| - \gamma|x - z|^2)$ cannot have a maximum in V different from z (if it has then we get a contradiction with the fact that $\Delta_\infty u \geq 0$ in the viscosity sense).

AMLE-Comparison with cones

Hence, if $z \notin V$ and $x \in V$ we must have

$$u(x) - (a|x - z| - \gamma|x - z|^2) \leq \max_{y \in \partial V} u(y) - (a|y - z| - \gamma|y - z|^2).$$

Now, just take $\gamma \to 0$ to obtain

$$u(x) - a|x - z| \leq \max_{y \in \partial V} u(y) - a|y - z|,$$

that is, we have that u has comparison with cones from above.

Analogously, one can show that if u is a viscosity supersolution to $\Delta_\infty u = 0$ then it has comparison with cones from below.