The (7,4)-conjecture for finite groups

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On many problems in Extremal Combinatorics the key step of the solution is (or would be) to understand the fine structure of the extremal system. In many cases there is a group action behind the large number of incidences. Let me give some examples.

(i) Let A be a finite subset of complex numbers and let us define the sumset of A by $A+A=\{a+b|a,b\in A\}$. Suppose that $|A+A|\le c|A|$. What can we say about the structure of A if $|A|$ is large enough, $|A|\gg 1/c$, say? The Freiman-Ruzsa theorem tells us that A is similar to an arithmetic progression (which is similar to a cyclic group). One can ask the similar question when A is a subset of a group. There are quite a few technical issues depending on the ambient group, however this research direction became quite fruitful. It includes recent work of Bourgain-Katz-Tao, Helfgott, Pyber-Szabo, Breuillard-Green-Tao, Hrushovski, Bourgain-Gamburd and Sarnak, just to mention the main directions.

(ii) Matousek conjectured that the minimum number of distinct distances between two n-element collinear point-sets is superlinear in n. (unless the two lines are orthogonal or parallel.) Elekes and Ronyai proved this conjecture which was strengthened later by Elekes and Szabo. The proof relies on the understanding of the structure of algebraic surfaces $f(x,y,z)=0$ which have large intersections with Cartesian products. Work of Szabo and - from a different direction - a result of Hrushovski show that f must be the image of the graph of the multiplication function of an appropriate algebraic group.

(iii) It is known that any line connecting two rational points on an elliptic curve intersects the curve in a third - also rational - point. Elliptic curves are the only known bases for constructing point-sets with many collinear triples without four points on a line. The group structure behind this construction is well understood, however it is not known if all such point configurations come from groups or not. Note that in this problem we have a triple-system which can be extended to a quasigroup, however it is not clear how to go further.

There are some combinatorial tools which can be used on quasigroups. The most powerful is the so called Hypergraph Removal Lemma. The simplest case states that for every dense subset of triples of the form (a,b,ab) there is a six-element subset of the quasigroup which spans at least three triples from the selected subset. This is called the (6,3)-theorem. It was proved by Ruzsa and Szemerédi. The similar statement for (7,4) is one of the most important conjectures in extremal combinatorics.

So, the conjecture is that for every $c>0$ there is a threshold n_0 so that if a quasigroup has order $n\ge n_0$ then for every c-dense subset of triples of the form (a,b,ab) there is a seven-element subset of the quasigroup which spans at least four triples from the selected subset. (c-dense means that there are at least cn^2 triples in the subset)

We will prove the conjecture for groups using the Hypergraph Regularity Lemma.