# Strategies and Interactive Beliefs in Dynamic Games

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# Introduction

- *Topic:* formal analysis of strategic thinking in dynamic games, backward and forward induction reasoning.
- *Tools:* epistemic (type) structures where states determine truth value of conditional statements.
- *Distinctive feature:* we use **only epistemic conditionals** of the form:
  "if **i** learned **h** he would believe **E** with probability **p**";
  state=(actual actions,epistemic conditionals).
- *Motivation:* strategies cannot be (irreversibly) chosen, they are just beliefs on own contingent choices obtained from planning (objective *vs* subjective strategies).
- *Focus:* generic perfect information games.

# Background literature

- *Hierarchies of probabilitstic beliefs*: Mertens & Zamir (IJGT 1985), Brandenburger & Dekel (JET 1993), Heifetz & Samet (JET 1998)
- *Rationalizability in games:* Bernheim (Ecma 1984), Pearce (Ecma 1984)
- *Dynamic interactive epistemology:* Ben Porath (REStud 1997), Battigalli & Siniscalchi (JET 1999,2002)

# Road map

1. Preview on conditionals and Backward Induction (BI): semi-formal analysis of an example
2. Setup: perfect information (PI) games and epistemic structures
3. Strategies as epistemic constructs: some results
4. Conclusions

# Preview: two kinds of type structures

Variants of Battigalli & Siniscalchi JET99: types are implicit representations of hierarchies of conditional beliefs. But, what are first-order beliefs about?

- $\mathbf{H}$=histories, $\mathbf{Z}$=paths (terminal nodes), $\mathbf{S_i}$,$\mathbf{S_{-i}}$ ,$\mathbf{S}$=strategies
- "Traditional" $\mathbf{S}$-*based structures*: $(\mathbf{\bar{T}_i}, \bar{\beta}_\mathbf{i})_{\mathbf{i}\in\mathbf{I}}$, states in $\mathbf{\Omega = S \times T}$, $(\mathbf{s_i}, \mathbf{t_i})$=state of $\mathbf{i}$ ($\mathbf{s_i}$ is "objective"), $\mathcal{H}_\mathbf{i}$=conditioning events,

$$\begin{aligned}\mathcal{H}_\mathbf{i} &= \{\mathbf{S_{-i}(h)} \times \mathbf{\bar{T}_{-i}} : \mathbf{h} \in \mathbf{H}\} \\ \bar{\beta}_\mathbf{i} &: \mathbf{\bar{T}_i} \to \mathbf{\Delta}^{\mathcal{H}_\mathbf{i}}(\mathbf{S_{-i}} \times \mathbf{\bar{T}_{-i}}) \subset [\mathbf{\Delta}(\mathbf{S_{-i}} \times \mathbf{\bar{T}_{-i}})]^{\mathcal{H}_\mathbf{i}}\end{aligned}$$

- Our $\mathbf{Z}$-based structures: $(\mathbf{T_i}, \beta_\mathbf{i})_{\mathbf{i}\in\mathbf{I}}$, states in $\mathbf{\Omega = Z \times T}$, $\mathbf{t_i}$=epist.state of $\mathbf{i}$ (beliefs about others+plan),

$$\begin{aligned}\mathbf{H} &\cong \{\mathbf{Z(h)} \times \mathbf{T_{-i}} : \mathbf{h} \in \mathbf{H}\} \\ \beta_\mathbf{i} &: \mathbf{T_i} \to \mathbf{\Delta}^\mathbf{H}(\mathbf{Z} \times \mathbf{T_{-i}}) \subset [\mathbf{\Delta}(\mathbf{Z} \times \mathbf{T_{-i}})]^\mathbf{H}\end{aligned}$$

# Preview on conditionals and BI: an example

## Stackelberg Minigame

| outputs: | low (left) | high (right) |
|---|---|---|
| high (up) | **3**,*1* | **0**,*0* |
| low (down) | **2**,*2* | **1**,*3* |

tree:

(**3**,*1*) (**0**,*0*)
**L** ↖ ↗ **R**
*Bob*
↑ **U**
**Ann**
↓ **D**
*Bob*
**l** ↙ ↘ **r**
(**2**,*2*) (**1**,*3*)

# Preview on conditionals and BI: an example

## Stackelberg Minigame

- *Ann believes "***Bob would go Right given Up***"*
  =belief about a behavioral conditional
- **Given Up**, *Ann would believe "***Bob goes Right***"*
  =conditional belief about behavior
- In **traditional analysis:** behavioral conditionals and beliefs about conditionals and opponents' beliefs
- In **our analysis:** actual actions (paths) and *conditional* beliefs about actions and beliefs

# Preview on conditionals and BI: traditional analysis

States $\omega$ specify strategies $\mathbf{s_i} = \sigma_\mathbf{i}(\omega)$ **objectively**
*'Bob would go Right given Up'* **false** at $\omega$ iff $\sigma_\mathbf{Bob}(\omega) \in \{\mathbf{L.r}, \mathbf{L.l}\}$

$\mathbf{Rat_{Bob}} \subset [\mathbf{L.r}] = \{\omega : \sigma_\mathbf{i}(\omega) = \mathbf{L.r}\} \Rightarrow$
$\mathrm{B}_\mathbf{Ann}(\mathbf{Rat_{Bob}}) \subset \mathrm{B}_\mathbf{Ann}(\mathbf{L.r})$ (uncond. belief)

States also specify conditional beliefs

By built in independence:

$\mathrm{B}_\mathbf{Ann}(\mathbf{L.r}) \subset \mathrm{B}_\mathbf{Ann}(\mathbf{L}|\mathbf{U}) \cap \mathrm{B}_\mathbf{Ann}(\mathbf{r}|\mathbf{D}) =$
$= \mathrm{B}_\mathbf{Ann}(\mathbf{util} = \mathbf{3}|\mathbf{U}) \cap \mathrm{B}_\mathbf{Ann}(\mathbf{util} = \mathbf{1}|\mathbf{D})$

$\Rightarrow \quad \mathbf{Rat_{Ann}} \cap \mathbf{Rat_{Bob}} \cap \mathrm{B}_\mathbf{Ann}(\mathbf{Rat_{Bob}}) \subset$
$\subset [\mathbf{U}, \mathbf{L.r}] \subset [\mathbf{U}, \mathbf{L}]$

$\Rightarrow$ **BI strategies and path obtain**

$(\mathbf{3}, \mathit{1})$ $\quad$ $(\mathbf{0}, \mathit{0})$
**L** $\nwarrow$ $\quad$ $\nearrow$ **R**
*Bob*
$\uparrow$ **U**
**Ann**
$\downarrow$ **D**
*Bob*
**l** $\swarrow$ $\quad$ $\searrow$ **r**
$(\mathbf{2}, \mathit{2})$ $\quad$ $(\mathbf{1}, \mathit{3})$

# Preview on conditionals and BI: our analysis

"strategy" is an epistemic concept:
**i**'s cond. beliefs $\Rightarrow$ (beliefs about others and) contingent plan of **i**

Rat. Planning: $\mathbf{RP_{Bob}}=\mathrm{B}_{\mathbf{Bob}}(\mathbf{L}|\mathbf{U})\cap\mathrm{B}_{\mathbf{Bob}}(\mathbf{r}|\mathbf{D})$

Material Consistency=path cons. with plan

Rationality=Material cons.+Rat.plan.
$\mathbf{R_{Bob}} \subset [\mathbf{U},\mathbf{L}] \cup [\mathbf{D},\mathbf{r}]$

But no event $[\mathbf{L.r}]$. $\mathrm{B}_{\mathbf{Ann}}(\mathbf{L.r})$ not expressible!

$\mathrm{B}_{\mathbf{Ann}}(\mathbf{R_{Bob}}) \subset \mathrm{B}_{\mathbf{Ann}}([\mathbf{U},\mathbf{L}] \cup [\mathbf{D},\mathbf{r}]) \not\subset$
$\not\subset \mathrm{B}_{\mathbf{Ann}}(\mathbf{L}|\mathbf{U})\cap\mathrm{B}_{\mathbf{Ann}}(\mathbf{r}|\mathbf{D})$ e.g. if $\mathrm{B}_{\mathbf{Ann}}(\mathbf{D})$

$(\mathbf{3},\mathit{1})$ $\qquad$ $(\mathbf{0},\mathit{0})$
$\mathbf{L} \nwarrow$ $\qquad$ $\nearrow \mathbf{R}$
*Bob*
$\uparrow \mathbf{U}$
**Ann**
$\downarrow \mathbf{D}$
*Bob*
$\mathbf{l} \swarrow$ $\qquad$ $\searrow \mathbf{r}$
$(\mathbf{2},\mathit{2})$ $\qquad$ $(\mathbf{1},\mathit{3})$

Recall: $\mathbf{R}_{\mathbf{Bob}} \subset [\mathbf{U}, \mathbf{L}] \cup [\mathbf{D}, \mathbf{r}]$, but
$\mathrm{B}_{\mathbf{Ann}}(\mathbf{R}_{\mathbf{Bob}}) \not\subset \mathrm{B}_{\mathbf{Ann}}(\mathbf{L}|\mathbf{U}) \cap \mathrm{B}_{\mathbf{Ann}}(\mathbf{r}|\mathbf{D})$ e.g. if $\mathrm{B}_{\mathbf{Ann}}(\mathbf{D})$ (Ann plans **D**)

Since there is no event $[\mathbf{L.r}]$,
$\mathrm{B}_{\mathbf{Ann}}(\mathbf{L.r})$ not expressible

If $\mathrm{B}_{\mathbf{Ann}}(\mathbf{D})$ (Ann plans **D**) the following is possible

$$\mathrm{B}_{\mathbf{Ann}}([\mathbf{U}, \mathbf{L}] \cup [\mathbf{D}, \mathbf{r}]) \cap \neg \mathrm{B}_{\mathbf{Ann}}(\mathbf{L}|\mathbf{U})$$

$$\exists \omega \in [\mathbf{D}, \mathbf{r}] \cap \mathbf{R}_{\mathbf{Ann}} \cap \mathbf{R}_{\mathbf{Bob}} \cap \mathrm{B}_{\mathbf{Ann}}(\mathbf{R}_{\mathbf{Bob}})$$

$(\mathbf{3}, \mathit{1})$ $(\mathbf{0}, \mathit{0})$
$\mathbf{L} \nwarrow$ $\nearrow \mathbf{R}$
*Bob*
$\uparrow \mathbf{U}$
**Ann**
$\downarrow \mathbf{D}$
*Bob*
$\mathbf{l} \swarrow$ $\searrow \mathbf{r}$
$(\mathbf{2}, \mathit{2})$ $(\mathbf{1}, \mathit{3})$

$\Rightarrow$ **Imperfect Nash** equilibrium **may obtain**

- Ann *strongly believes* **E** if she believes **E** *given* each **C** with $\mathbf{C} \cap \mathbf{E} \neq \emptyset$
- $[\mathbf{U}] \cap \mathbf{R}_{\mathbf{Bob}} \neq \emptyset$, $[\mathbf{D}] \cap \mathbf{R}_{\mathbf{Bob}} \neq \emptyset$
- hence $\mathrm{SB}_{\mathbf{Ann}}(\mathbf{R}_{\mathbf{Bob}}) \subset \mathrm{B}_{\mathbf{Ann}}(\mathbf{R}_{\mathbf{Bob}}|\mathbf{U}) \cap \mathrm{B}_{\mathbf{Ann}}(\mathbf{R}_{\mathbf{Bob}}|\mathbf{D})$
- but $\mathbf{R}_{\mathbf{Bob}} \subset [\mathbf{U}, \mathbf{L}] \cup [\mathbf{D}, \mathbf{r}]$
- hence $\mathrm{SB}_{\mathbf{Ann}}(\mathbf{R}_{\mathbf{Bob}}) \subset \mathrm{B}_{\mathbf{Ann}}(\mathbf{L}|\mathbf{U}) \cap \mathrm{B}_{\mathbf{Ann}}(\mathbf{r}|\mathbf{D})$ (BI cond. beliefs)
- thus $\mathbf{R}_{\mathbf{Ann}} \cap \mathbf{R}_{\mathbf{Bob}} \cap \mathrm{SB}_{\mathbf{Ann}}(\mathbf{R}_{\mathbf{Bob}}) \subset [\mathbf{U}, \mathbf{L}]$ (BI path)

- Suppose
  (i) Ann's conditional beliefs about Bob's beliefs (hence his plan) are *independent* of her actions (**Ind**),
  (ii) Ann strongly believes that Bob is materially consistent:
- $\mathbf{Ind} \cap \mathrm{B}_{\mathbf{Ann}}(\mathbf{RP}_{\mathbf{Bob}}) \cap \mathrm{SB}_{\mathbf{Ann}}(\mathbf{MC}_{\mathbf{Bob}}) \subset \mathrm{B}_{\mathbf{Ann}}(\mathbf{L}|\mathbf{U}) \cap \mathrm{B}_{\mathbf{Ann}}(\mathbf{r}|\mathbf{D})$
- Thus
  $\mathbf{R}_{\mathbf{Ann}} \cap \mathbf{R}_{\mathbf{Bob}} \cap \mathbf{Ind} \cap \mathrm{B}_{\mathbf{Ann}}(\mathbf{RP}_{\mathbf{Bob}}) \cap \mathrm{SB}_{\mathbf{Ann}}(\mathbf{MC}_{\mathbf{Bob}}) \subset [\mathbf{U}, \mathbf{L}]$
- (Note: such independence and belief in consistency are implicit in traditional analysis)

- Route 1 (SB, with higher-level epistemic assumptions) leads to the BI-path in generic PI games
- Route 2 (OAI) leads to the BI-path in generic two-stage PI games, not in longer games such as the Centipede (even with higher-level epistemic assumptions)

# "Centipede" example: strong belief analysis

## Forward induction reasoning yields the BI outcome

$$\begin{array}{ccccccc}
\textbf{Ann} & \overset{\textbf{C}}{\longrightarrow} & \mathit{Bob} & \overset{c}{\longrightarrow} & \textbf{Ann} & \overset{\textbf{C}'}{\longrightarrow} & (\mathbf{1}... \\
\end{array}$$

- $\mathbf{R_{Ann}} \subset \neg[\mathbf{C}, \mathbf{c}, \mathbf{C}']$
- $\mathbf{R_{Bob}} \cap \mathrm{SB}_{\mathbf{Bob}}(\mathbf{R_{Ann}}) \subset [\mathbf{D}] \cup [\mathbf{C}, \mathbf{d}]$
- $\mathbf{R_{Ann}} \cap \mathrm{SB}_{\mathbf{Ann}}(\mathbf{R_{Bob}} \cap \mathrm{SB}_{\mathbf{Bob}}(\mathbf{R_{Ann}})) \subset [\mathbf{D}]$

## "Centipede" example: initial common belief analysis

Not enough to get the BI outcome

$$
\begin{array}{ccccccc}
 & \mathbf{C} & & c & & \mathbf{C}' & \\
\mathbf{Ann} & \longrightarrow & Bob & \longrightarrow & \mathbf{Ann} & \longrightarrow & (\mathbf{0},3) \\
\downarrow \mathbf{D} & & \downarrow d & & \downarrow \mathbf{D}' & & \\
(\mathbf{1},0) & & (\mathbf{0},2) & & (\mathbf{3},0) & &
\end{array}
$$

- $\mathbf{R_{Ann}} \subset \neg[\mathbf{C}, \mathbf{c}, \mathbf{C}']$
- But, **without** $\mathrm{SB}_{\mathbf{Bob}}(\mathbf{R_{Ann}})$, $\mathbf{R_{Bob}}$ has no behavioral implication! Thus
- $\mathbf{R_{Ann}} \cap \mathbf{R_{Bob}} \cap \mathbf{Ind} \cap \mathrm{B}_{\mathbf{Ann}}(\mathbf{RP_{Bob}}) \cap \mathrm{SB}_{\mathbf{Ann}}(\mathbf{MC_{Bob}}) \subset \neg[\mathbf{C}, \mathbf{c}, \mathbf{C}']$
- $\Rightarrow$ initial common belief in $\mathbf{R} \cap \mathbf{Ind} \cap \mathrm{SB}(\mathbf{MC})$ only buys $\neg[\mathbf{C}, \mathbf{c}, \mathbf{C}']$

(We can show this by example and as corollary of a general theorem)

# Setup

## Perfect Information (PI) games

- $\mathbf{i} \in \mathbf{I}$, *players*
- $\mathbf{h} \in \mathbf{H}$, *histories/nodes* ($\mathbf{H_i}$, owned by $\mathbf{i}$), **H finite**
- $\mathbf{z} \in \mathbf{Z} \subset \mathbf{H}$, *terminal* histories/*paths*; $\mathbf{Z(h) = \{z : h \preceq z\}}$
- $\mathbf{a} \in \mathbf{A(h)}$, *actions* at $\mathbf{h} \in \mathbf{H \backslash Z}$
- $\mathbf{u_i : Z} \to \mathbb{R}$, *utility/payoff* s.t. **no relevant ties**

- $\omega \in \mathbf{\Omega} = \mathbf{X} \times \mathbf{T} = \mathbf{X} \times \prod_{\mathbf{i} \in \mathbf{I}} \mathbf{T_i}$, *states of the world*
- $\mathbf{x} \in \mathbf{X}$, *eXternal* (non-epistemic) *states*, **finite** (e.g. $\mathbf{X} = \mathbf{S}$ or $\mathbf{X} = \mathbf{Z}$)
- (implicitly understood *path function* $\pi : \mathbf{X} \to \mathbf{Z}$)
- for $\mathbf{Y} \subset \mathbf{X}$, $[\mathbf{Y}] = \mathbf{Y} \times \mathbf{T}$ eXternal events
- $\beta_\mathbf{i} : \mathbf{T_i} \to \mathbf{B_i}$, $\beta_\mathbf{i}(\mathbf{t_i})$ epistemic state of **i** at $\omega$ ($\mathbf{B_i} = \mathbf{Beliefs_i}$ to be specified)

- Conditioning events (hypotheses) correspond to histories:

$$[h] = \{(x,t) \in X \times T_{-i} : \pi(x) \in Z(h)\}$$

- Probability measures concentrated on conditioning events $C = [h]$, $D = [h']$ ... related by *chain rule*:

$$E \subset C \subset D \Rightarrow \mu_i(E|D) = \mu_i(E|C)\mu_i(C|D) \qquad \text{(ch.r)}$$

- Write

$$\mu_{i,h}(\cdot) = \mu_i(\cdot|[h]), \mu_i = (\mu_{i,h}(\cdot))_{h\in H} \in [\Delta(X \times T)]^H$$

Conditional probability systems (*CPS*'s) on $(X \times T, H)$:

$$\Delta^H(X \times T_{-i}) = \left\{\mu_i : \mu_{i,h}([h]) = 1 \text{ and (ch.r) holds}\right\}$$

# Setup

We focus on epistemic **type structures** where eXternal states are complete sequences of actions: $\mathbf{X = Z}$

$$\mathbf{\Omega = Z \times \prod_{i \in I} T_i}, \ \beta_{\mathbf{i}} \mathbf{: T_i \to \Delta^H(Z \times T_{-i})}, \ (\mathbf{i \in I})$$

meaning:

- **no knowledge** about **z at the outset**
- **information** about moves **cannot** directly **disclose** anything about types/**beliefs**

# Type structures

$$\mathbf{\Omega = Z \times \prod_i T_i,\ \beta_i : T_i \to \Delta^H(Z \times T_{-i}),}$$

Technical assumptions (in a sense, w.l.o.g.): for each $\mathbf{i \in I}$

- $\mathbf{T_i}$ **compact metrizable**, hence $\mathbf{\Delta^H(Z \times T_{-i})}$ compact metrizable (B&S JET99)
- $\boldsymbol{\beta_i}$ **continuous**

**Definition**

A type structure is *complete* if $\boldsymbol{\beta_i}$ is **onto** for every $\mathbf{i \in I}$.

Main example: the canonical structure of hierarchies of beliefs.

# Canonical structure: types as hierarchies of beliefs

- First-order beliefs: $\mathbf{T_i^1 = \Delta^H(Z)}$
- Second-order beliefs:

$$\mathbf{T_i^2} = \left\{ (\mu_i^1, \mu_i^2) \in \mathbf{T_i^1} \times \mathbf{\Delta^H} \left( \mathbf{Z} \times \mathbf{T_{-i}^1} \right) : \forall \mathbf{h} \in \mathbf{H}, \mathrm{marg}_{\mathbf{Z}} \mu_{i,h}^2(\cdot) =$$

- ... recursive def. of $\mathbf{T_i^n} \subset \mathbf{T_i^{n-1}} \times \mathbf{\Delta^H(Z} \times \mathbf{T_{-i}^{n-1})}$ ...
- $\mathbf{T_i}$ = projective limit of $\mathbf{(T_i^n)_{n\in\mathbb{N}}}$, set of *infinite hierachies of beliefs* satisfying common full belief in coherence
- homeomorphism $\beta_i : \mathbf{T_i} \to \mathbf{\Delta^H(Z \times T_{-i})}$ by a generalization of Kolmogorov's extension theorem

Standard (monotonic) belief operators:
[for $\mathbf{E} \subset \mathbf{Z} \times \mathbf{T}$, let $\mathbf{E}_{\mathbf{t}_i} = \{(\mathbf{z}, \mathbf{t}_{-i}) : (\mathbf{z}, \mathbf{t}_i, \mathbf{t}_{-i}) \in \mathbf{E}\}$ section at $\mathbf{t}_i$]

- conditional belief $\mathrm{B}_i(\mathbf{E}|\mathbf{h}) = \{(\mathbf{z}, \mathbf{t}_i, \mathbf{t}_{-i}) : \beta_i(\mathbf{t}_i)(\mathbf{E}_{\mathbf{t}_i}|[\mathbf{h}]) = \mathbf{1}\}$,
- initial belief $\mathrm{B}_i(\mathbf{E}) = \mathrm{B}_i(\mathbf{E}|\mathbf{h}^0)$ ($\mathbf{h}^0$=empty hist., root)

Nonmonotonic belief operator

- strong belief $\mathrm{SB}_i(\mathbf{E}) = \bigcap_{\mathbf{h}:\mathbf{E}\cap[\mathbf{h}]\neq\emptyset} \mathrm{B}_i(\mathbf{E}|\mathbf{h})$

Given $\mu_i \in \Delta^H(Z \times T_{-i})$ derive

$$\mu_i(a|h) = \mu_i([h,a]|[h])$$

for all $h \in H \backslash Z$, $a \in A(h)$.
Consider only the conditional prob. of $i$'s opponents' actions:

$$\mu_i(a|h),\ h \in H_{-i},\ a \in A(h)$$

$\Rightarrow$ subjective decision tree $\Gamma_i(\mu_i)$ for $i$
$\mu_i$ is *consistent with dynamic programming* on $\Gamma_i(\mu_i)$ iff (OSD)

$$\forall h \in H_i,\ \mu_i\left(\arg\max_{a \in A(h)} V_i((h,a),\mu_i)|h\right) = 1,$$

$$\text{with } V_i((h,a),\mu_i) = \sum_{z \in Z(h,a)} u_i(z)\mu_i(z|h,a)$$

The plan of $\mathbf{i}$ at $(\mathbf{z}, \mathbf{t_i}, \mathbf{t_{-i}})$ (plan of $\mathbf{t_i}$) is derived from $\beta_\mathbf{i}(\mathbf{t_i})$

### Definition

Pl. $\mathbf{i}$ *plans rationally* at $(\mathbf{z}, \mathbf{t_i}, \mathbf{t_{-i}})$ if

$$\forall \mathbf{h} \in \mathbf{H_i},\ \beta_\mathbf{i}(\mathbf{t_i}) \left( \arg \max_{\mathbf{a} \in \mathbf{A(h)}} \mathbf{V_i}((\mathbf{h}, \mathbf{a}), \mu_\mathbf{i}) | \mathbf{h} \right) = \mathbf{1}.$$

Event: $\mathbf{RP_i}$

- *$RP_\mathbf{i}$ is just a property of* $\mathbf{i}$*'s beliefs/types:* $\mathbf{i}$ expects to take locally maximizing actions conditional on each $\mathbf{h} \in \mathbf{H_i}$.
- **Interpretation:** $\mathbf{i}$ has beliefs about others and computes his plan (beliefs about himself) by dynamic programming ("folding back") on the corresponding subjective decision tree.

Connect beliefs to behavior:

**Definition**

Pl. $\mathbf{i}$ is *materially consistent at* $(\mathbf{z}, \mathbf{t_i}, \mathbf{t_{-i}})$ if he does not violate his plan on path $\mathbf{z}$

$$\begin{aligned} \forall \mathbf{h} &\in \mathbf{H_i}, \forall \mathbf{a} \in \mathbf{A(h)}, \\ \mathbf{(h, a)} &\preceq \mathbf{z} \Rightarrow \beta_{\mathbf{i}}(\mathbf{t_i})(\mathbf{a}|\mathbf{h}) > \mathbf{0}. \end{aligned}$$

Event: $\mathbf{MC_i}$

**Definition**

Pl. $\mathbf{i}$ is *materially rational at* $(\mathbf{z}, \mathbf{t_i}, \mathbf{t_{-i}})$ if he plans rationally and does not violate his plan at $(\mathbf{z}, \mathbf{t_i}, \mathbf{t_{-i}})$.Event: $\mathbf{R_i} = \mathbf{MC_i} \cap \mathbf{RP_i}$.

- Recall: results apply to **finite PI games with NRT**, in such games the **BI strategy** is **unique** and (mixed) **Nash** equilibria **yield a unique path** with prob. 1.
- Similar to traditional analysis: initial common belief in material rationality does not yield BI or Nash paths in games of depth $\mathbf{d} > \mathbf{2}$ (e.g. Centipede).

- *Unlike traditional analysis:* in games of **depth 2**, correct belief in rationality does **not** yield **BI**, only a **Nash path** (cf. initial example).

## Proposition

*In a game* $\mathbf{\Gamma}$ *of depth 2,* $\forall \mathbf{z} \in \mathbf{Z}$*,* $\mathbf{z}$ *is a Nash path if and only if*

$$\mathbf{(z, t)} \in \bigcap_{\mathbf{i}} \mathbf{R_i} \cap \mathrm{B}_{\mathbf{i}}(\mathbf{R_{-i}})$$

*for some* $\mathbf{t}$*, in some type structure* $(\mathbf{T_i}, \beta_{\mathbf{i}})_{\mathbf{i} \in \mathbf{I}}$ *for* $\mathbf{\Gamma}$*.*

# Strategies as beliefs

## Common Strong Belief in Material Rationality

- One way to obtain elementary BI (games of depth 2) is to assume that the first mover strongly believes in the material rationality of the co-player.
- We can go further and replicate "traditional" results on common strong belief in rationality and Nash eq. (Battigalli-Friedenberg, 2010), or EFR and BI in complete structures (Battigalli-Siniscalchi, 2002).

- $\mathbf{R}_\mathbf{i}^0 = \mathbf{R}_\mathbf{i}$, $\mathbf{R}_\mathbf{i}^{\mathbf{k}+1} = \mathbf{R}_\mathbf{i}^\mathbf{k} \cap \mathrm{SB}_\mathbf{i}(\mathbf{R}_{-\mathbf{i}}^\mathbf{k})$
- $\mathbf{CSBR} = \bigcap_{\mathbf{k},\mathbf{i}} \mathbf{R}_\mathbf{i}^\mathbf{k}$, correct Common Strong Belief in **R**
- $\pi : \mathbf{S} \to \mathbf{Z}$ strategy-path function

## Proposition

*(i) In any type structure*

$$proj_{\mathbf{Z}}\mathbf{CSBR} \subset \{\mathbf{Nash}\text{-}paths\}$$

*(ii) In a* **complete** *(or otherwise "sufficiently rich") type structure*

$$proj_{\mathbf{Z}}\mathbf{CSBR} = \pi(\mathbf{EFR}) = \{\mathbf{BI}\text{-}path\}.$$

- Reason for non-BI result: Ann may initially believe in $\mathbf{R_{Bob}}$ but she need not believe in $\mathbf{R_{Bob}}$ conditional on taking an unplanned action.
- This cannot happen if Ann's beliefs about Bob's type *conditional on her own actions* do not depend on the conditioning action, and she strongly (hence always) believes in $\mathbf{MC_{Bob}}$.
- *Own-action* **independence**=**i**'s conditional beliefs about $\mathbf{t_{-i}}$ do not depend on **i**'s actions (event $\mathbf{Ind_i}$)

## Proposition

*In every "rich" (e.g., complete) type structure for a game of depth 2*

$$proj_{\mathbf{Z}}\left(\bigcap_{\mathbf{i}} \mathbf{R_i} \cap \mathbf{Ind_i} \cap \mathrm{B}_{\mathbf{i}}\left(\mathbf{R_{-i}}\right) \cap \mathrm{SB}_{\mathbf{i}}(\mathbf{MC_{-i}})\right) = \{\mathbf{BI}\text{-}path\}.$$

What about longer games (e.g. Centipede)?

- $\mathbf{RInd}_i^0 = \mathbf{R}_i \cap \mathbf{Ind}_i \cap \mathrm{SB}_i(\mathbf{MC}_{-i})$,
- $\mathbf{RInd}_i^{k+1} = \mathbf{RInd}_i^k \cap \mathrm{B}_i(\mathbf{RInd}_{-i}^k)$
- $\mathbf{CBRInd} = \bigcap_{i,k} \mathbf{RInd}_i^k$

All the paths consistent with the "Dekel-Fudenberg procedure", $\pi(\mathbf{S}^\infty \mathbf{W})$, are also consistent with **CBRInd**:

**Proposition**

*There are type structures (including the canonical one) such that* $\forall \mathbf{s} \in \mathbf{S}^\infty \mathbf{W}$, $\exists (\mathbf{z}, \mathbf{t}) \in \mathbf{CBRInd}$ *such that* $\mathbf{z} = \pi(\mathbf{s})$.

**Corollary**

*There are non-BI paths consistent with* **CBRInd** *in Centipede (of depth* $\mathbf{d} > 2$*).*

# Conclusions

- Many results of the "traditional analysis" with behavioral conditionals in the state of the world make a lot of sense. They are built on often implicit assumptions of consistency of plans with actual behavior, strong belief in consistency (perceived intentionality) and self/opponents independence.
- We rule out behavioral conditionals, but allow for epistemic ones. This forces an interpretation of strategies as epistemic constructs and imposes discipline: in this setup strategies cannot be chosen, they can only be planned.
- Consistency and own-action independence have to be assumed explicitly. This seems fitting for a formal analysis of strategic reasoning.

# Conclusions

## Imperfect information (with perfect recall)

- $\mathbf{h_i} \in \mathbf{H_i}$ information sets (personal histories)
- Conditions/hypotheses for $\mathbf{i}$: $\mathbf{[h_i]}$ and $\mathbf{[h_i, a_i]}$ ($\mathbf{a_i} \in \mathbf{A_i(h_i)}$)
- Value of action $\mathbf{a_i}$ at $\mathbf{h_i}$: $\mathbb{E}_{\beta_{\mathbf{i}}(\mathbf{t_i})}[\mathbf{u_i}|\mathbf{h_i}, \mathbf{a_i}]$
- Potential conflict between our analysis and standard decision theory.
- *Own-action independence* and *strong belief in material consistency* allow to reconcile our analysis with traditional theory.

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