Convexity and Risk

(based on Cerreia–Vioglio (2009) and Cerreia–Vioglio, Dillenberger, and Ortoleva (2013))

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Games and Decisions Conference 2013

July 2013

Preferences under Risk

- We consider a Decision Maker, DM, who has preferences over probability distributions
- Why probability distributions?
- In Economics and Finance, probability distributions are used to model: value of an asset, value of an investment, behavior of an opponent, mixed strategies

Formalization of the Decision Framework

- Probability distributions?
- C is a set of consequences which is a subset of a compact metric space
- Δ is the set of all Borel probability measures over C
- Elements of Δ are denoted by $p, q, r \in \Delta$ and they are called lotteries

Examples

- C is a finite set. Δ can be identified with the simplex of $\mathbb{R}^{|C|}$
- C is a bounded and closed interval, $[w, b]$, of $\mathbb{R}$
- In this tutorial, we will focus on this latter, very *special* but also very *important*, case

The Mathematical Framework

- $C([w,b])$: space of continuous functions on $[w,b]$ endowed with the supnorm
- Define $\mathcal{V}$ as

$$\{v \in C([w,b]) : v(b) = 1 \text{ and } v \text{ is increasing}\}$$

- Define $\mathcal{U}_{\text{nor}}$ as

$$\{v \in C([w,b]) : v(w) = 0 = v(b) - 1 \text{ and s. increasing}\}$$

- $ca([w,b])$: space of all finite Borel signed measures on $[w,b]$
- $ca([w,b])$ is the norm dual of $C([w,b])$. $ca([w,b])$ is endowed with the weak*-topology
- Δ is endowed with the relative topology
- Fact: Δ is convex and compact

Preferences

- How do we model the DM's preferences over lotteries?
- A binary relation $\succsim$ on Δ. More formally, a binary relation is a subset of $\Delta \times \Delta$
- $p \succsim q$ is interpreted as "p is weakly better than q"
- $\succ$ denotes the asymmetric part (Strict Preference)
- $\sim$ denotes the symmetric part (Indifference)

Utility Function

Definition

Let $\succsim$ be a binary relation on Δ. $V : \Delta \to \mathbb{R}$ is a utility function of $\succsim$ if and only if

$$p \succsim q \iff V(p) \geq V(q)$$

- We also say that V represents $\succsim$
- Not each binary relation admits a utility function
- Notice that if $f : V(\Delta) \to \mathbb{R}$ is strictly increasing than $f \circ V$ also represents $\succsim$

Key Tenets of Rationality

- **Preorder** $\succsim$ is reflexive and transitive
- **Weak Monotonicity** For each $x, y \in [w, b]$

$$x \geq y \iff \delta_x \succsim \delta_y$$

- Those are key tenets of rationality for a DM with preferences over **monetary** lotteries

A Technical Property

- **Continuity** If $\{p_n\}_{n\in\mathbb{N}}, \{q_n\}_{n\in\mathbb{N}} \subseteq \Delta$, $p_n \succsim q_n$ for all $n \in \mathbb{N}$, $p_n \to p$, and $q_n \to q$ then $p \succsim q$
- Intuitively, the DM will not experience a big change in his preferences given some small change in the prospects he is facing

Expected Utility: references

- von Neumann–Morgenstern (1944)
- Nagumo–Kolmogorov–De Finetti (1929, 1930, 1931). See also Hardy–Littlewood–Polya (1934)
- Herstein–Milnor (1953)

Expected Utility: the model

- Consider a strictly increasing $v \in C([w, b])$
- Define $U : \Delta \to \mathbb{R}$ by

$$U(p) = \int_{[w,b]} v(x)\, dp(x) = \mathbb{E}_p(v) \qquad \forall p \in \Delta$$

- A DM could rank lotteries according to the criterion U, that is,

$$p \succsim q \overset{\text{def}}{\iff} U(p) \geq U(q)$$

Expected Utility: certainty equivalents

- Define $V : \Delta \to [w, b]$ by

$$V(p) = v^{-1}(U(p)) = v^{-1}(\mathbb{E}_p(v)) \qquad \forall p \in \Delta \qquad (1)$$

  Notice that $V(\delta_x) = x$ for all $x \in [w, b]$
- Thus, $V(p)$ is the amount of money that, if received with certainty, is equivalent to $V(p)$ under (1)
- It is immediate to see that

$$p \succsim q \overset{\text{def}}{\iff} U(p) \geq U(q) \iff V(p) \geq V(q)$$

von Neumann–Morgenstern

- **Completeness** If $p, q \in \Delta$ then either $p \succsim q$ or $q \succsim p$
- **Independence** For each $p, q, r \in \Delta$ and $\lambda \in (0, 1]$,

$$p \succsim q \Longrightarrow \lambda p + (1 - \lambda) r \succsim \lambda q + (1 - \lambda) r$$

von Neumann–Morgenstern Representation

Theorem

Let $\succsim$ be a binary relation on Δ. The following statements are equivalent:

(i) $\succsim$ *satisfies Preorder, Completeness, Continuity, Weak Monotonicity, and Independence;*

(ii) *there exists a strictly increasing $v \in C([w, b])$ such that*

$$p \succsim q \Longleftrightarrow \mathbb{E}_p(v) \geq \mathbb{E}_q(v)$$

Moreover, v is unique up to an affine transformation

Expected Utility: a discussion

- If we restrict ourselves to elements of $\mathcal{U}_{\mathrm{nor}}$ then we have full fledged uniqueness
- Notice that v is also such that

$$p \succsim q \Longleftrightarrow v^{-1}\left(\mathbb{E}_p\left(v\right)\right) \geq v^{-1}\left(\mathbb{E}_q\left(v\right)\right)$$

Definition

Let $\succsim$ be a binary relation on Δ. $\succsim$ is an Expected Utility (EU) binary relation if and only if it satisfies Preorder, Completeness, Continuity, Weak Monotonicity, and Independence

Risk Attitudes: references

- Pratt (1964)
- Yaari (1969)
- Rothschild–Stiglitz (1970)

Absolute Risk Attitudes: definitions

- For each $p \in \Delta$, $e(p)$ denotes the expected value of p

Definition

Let $\succsim$ be a binary relation on Δ. $\succsim$ is risk averse if and only if $\delta_{e(p)} \succsim p$ for all $p \in \Delta$

Definition

Let $\succsim$ be a binary relation on Δ. $\succsim$ is a mean preserving spread averter if and only if

$$p \succsim_{MPS} q \Longrightarrow p \succsim q$$

Absolute Risk Attitudes: a characterization

Theorem

Let $\succsim$ *be an EU binary relation on* Δ. *The following statements are equivalent:*

(i) $\succsim$ *is risk averse;*

(ii) $\succsim$ *is a mean preserving spread averter;*

(iii) v *is concave.*

Comparative Risk Attitudes

Definition

Let $\succsim_1$ and $\succsim_2$ be two binary relations on Δ. $\succsim_1$ is more risk averse than $\succsim_2$ if and only if for each $p \in \Delta$ and $x \in [w, b]$

$$p \succsim_1 \delta_x \Longrightarrow p \succsim_2 \delta_x$$

Theorem

Let $\succsim_1$ and $\succsim_2$ be EU binary relations. The following statements are equivalent:

(i) $\succsim_1$ *is more risk averse than* $\succsim_2$;

(ii) *there exists a strictly increasing, continuous, and concave function f such that $v_1 = f \circ v_2$.*

Removing Completeness: references

- Aumann (1962)
- Dubra, Maccheroni, Ok (2004)
- Baucells and Shapley (2006)

Multi–Expected Utility

Theorem

Let $\succsim$ be a binary relation on Δ. The following statements are equivalent:

(i) *$\succsim$ satisfies Preorder, Continuity, Weak Monotonicity, and Independence;*

(ii) *there exists a set, $\mathcal{W}$, of strictly increasing and continuous functions such that*

$$p \succsim q \Longleftrightarrow \mathbb{E}_p\left(v\right) \geq \mathbb{E}_q\left(v\right) \qquad \forall v \in \mathcal{W}$$

Moreover, $\mathcal{W}$ can be chosen to be a subset of $\mathcal{U}_{\mathrm{nor}}$. In this case, it is unique up to the closed and convex hull.

Multi–Expected Utility: a discussion

- Notice that we could rewrite the aforementioned representation result in the following way

$$p \succsim q \iff v^{-1}\left(\mathbb{E}_p\left(v\right)\right) \geq v^{-1}\left(\mathbb{E}_q\left(v\right)\right) \qquad \forall v \in \mathcal{W}$$

Definition

Let $\succsim$ be a binary relation on Δ. $\succsim$ is a Multi–Expected Utility (EU) binary relation if and only if it satisfies Preorder, Continuity, Weak Monotonicity, and Independence

Experimental Evidence

- One of the most prominently observed behavior pattern: **certainty effect**; overvalue risk-free prospects, even in violation of EU
- Allais paradox (common-ratio effect)
  - Choose between:
    - (A) 3000 for sure
    - (B) 4000 with probability 0.8 and 0 with probability 0.2
  - Choose between:
    - (C) 3000 with probability 0.25 and 0 with probability 0.75
    - (D) 4000 with probability 0.2 and 0 with probability 0.8
- Majority violate Expected Utility by choosing (A,D) or (B,C)
- Violations are *systematic*; most of them are of the (A,D) type
- **Many** more related experiments

Getting Rid of Independence: references

- Kahneman and Tversky (1979)
- Quiggin (1982)
- Machina (1982)
- Fishburn (1983)
- Dekel (1986)
- Becker and Sarin (1987)
- Yaari (1987)
- Chew (1989)
- Chew, Epstein, and Segal (1991)
- Gul (1991)

A Different Approach: references

- Maccheroni (2002)
- Cerreia–Vioglio (2009)
- Cerreia–Vioglio, Dillenberger, and Ortoleva (2013)
- Ghirardato, Maccheroni, and Marinacci (2004) (**under Knightian Uncertainty**)
- Gilboa, Maccheroni, Marinacci, and Schmeidler (2010) (**under Knightian Uncertainty**)
- Cerreia–Vioglio, Ghirardato, Maccheroni, Marinacci, and Siniscalchi (2011) (**under Knightian Uncertainty**)

Getting Rid of Independence

- For the moment, let us just consider a binary relation that satisfies *Preorder*, *Completeness*, *Continuity*, and *Weak Monotonicity*
- Can we retrieve a part of this binary relation which is standard, that is, it still satisfies Independence?
- In other words, is the DM an "EU guy" for some of his choices?
- Define a derived binary relation $\succsim'$

$$p \succsim' q \Longleftrightarrow \lambda p + (1-\lambda) r \succsim \lambda q + (1-\lambda) r \quad \forall \lambda \in (0,1], \forall r \in \Delta$$

- $\succsim'$ captures the DM ranking for which the DM is sure
- $\succsim'$ is a Multi-EU binary relation

Getting Rid of Independence: the story

- The DM is EU... when he is sure of his rankings. Failures of Independence arise in his behavior when forced to choose and he is not definitely sure about p being better than q
- Failures of Independence happen in completing preferences

Getting Rid of Independence: a result (Cer '09)

Theorem

Let $\succsim$ be a binary relation on Δ that satisfies Preorder, Completeness, Continuity, and Weak Monotonicity. The following statements are true:

(a) *There exists a convex set $\mathcal{W}' \subseteq \mathcal{U}_{\mathrm{nor}}$ such that*

$$p \succsim' q \iff \mathbb{E}_p(v) \geq \mathbb{E}_q(v) \qquad \forall v \in \mathcal{W}';$$

(b) *For each $p, q \in \Delta$, if $p \succsim' q$ then $p \succsim q$;*

(c) *If $\succsim''$ is another binary relation that satisfies (a) and (b) then $p \succsim'' q$ then $p \succsim' q$;*

(d) *If $\succsim''$ is another binary relation that satisfies (a) and (b) then $\bar{\mathrm{co}}(\mathcal{W}') \subseteq \bar{\mathrm{co}}(\mathcal{W}'')$.*

Getting Back Some Independence: NCI

- **Negative Certainty Independence** For each $p, q \in \Delta$, $x \in [w, b]$, $\lambda \in (0,1)$

$$p \succsim \delta_x \Longrightarrow \lambda p + (1-\lambda)\, q \succsim \lambda \delta_x + (1-\lambda)\, q$$

- It was first proposed by Dillenberger (2010)
- Let us reread NCI in terms of $\succsim'$ and its interpretation

$$p \succsim \delta_x \Longrightarrow p \succsim' \delta_x$$

- In other words,

$$p \not\succsim' \delta_x \Longrightarrow \delta_x \succ p$$

- When in doubt, go with certainty!
- **Certainty Effect!**

Cautious Expected Utility (CDO '13)

Theorem

Let $\succsim$ be a binary relation on Δ. The following statements are equivalent:

(i) *$\succsim$ satisfies Preorder, Completeness, Continuity, Weak Monotonicity, and NCI;*

(ii) *There exists a set $\mathcal{W}$ of strictly increasing and continuous functions such that $V : \Delta \to \mathbb{R}$, defined by*

$$V(p) = \inf_{v \in \mathcal{W}} v^{-1}\left(\mathbb{E}_p(v)\right) \qquad \forall p \in \Delta,$$

is a continuous utility function for $\succsim$.

Moreover, $\mathcal{W}$ can be chosen to also represent $\succsim'$ (that is, $= \mathcal{W}'$). In this case, it is unique up to the closed convex hull.

Absolute Risk Attitudes: a characterization

Definition

Let $\succsim$ be a binary relation on Δ. $\succsim$ is a Cautious Expected Utility binary relation if and only if it satisfies Preorder, Continuity, Weak Monotonicity, and NCI

Theorem

Let $\succsim$ be a Cautious EU binary relation on Δ. The following statements are equivalent:

(i) *$\succsim$ is a mean preserving spread averter;*

(ii) *Each v in $\mathcal{W}$ is concave.*

We can also carry a comparative static exercise!

Preference for Randomization

- **Mixing** For each $p, q \in \Delta$ and $\lambda \in (0,1)$

$$p \sim q \Longrightarrow \lambda p + (1-\lambda)\, q \succsim p$$

- It is not hard to show that Cautious EU binary relations satisfy Mixing
- If $\succsim$ satisfies Preorder, Completeness, Continuity, and Mixing then for each $q \in \Delta$ the set

$$\{p \in \Delta : p \succsim q\}$$

is convex

A Duality Class

Call $\mathcal{U}(\mathbb{R} \times \Delta)$ the class of functions $U : \mathbb{R} \times \mathcal{V} \to [-\infty, \infty]$ such that

- P.1 For each $v \in \mathcal{V}$ the function $U(\cdot, v) : \mathbb{R} \to [-\infty, \infty]$ is increasing
- P.2 $\lim_{t \to \infty} U(t, v) = \lim_{t \to \infty} U(t, v')$ for all $v, v' \in \mathcal{V}$
- P.3 U is $\diamond$-even quasiconvex
- P.4 U is linearly continuous, that is, the function $V_U : \Delta \to [-\infty, \infty]$, defined by

$$V_U(p) = \inf_{v \in \mathcal{V}} U(\mathbb{E}_p(v), v) \qquad \forall p \in \Delta,$$

  is real valued and continuous

A Representation (Cer '09)

Theorem

Let $\succsim$ be a binary relation on Δ. The following statements are equivalent:

(i) *$\succsim$ satisfies Preorder, Completeness, Continuity, Weak Monotonicity, and Mixing;*

(ii) *There exists a function U in $\mathcal{U}(\mathbb{R} \times \mathcal{V})$ such that*

$$p \succsim q \Longleftrightarrow \inf_{v \in \mathcal{V}} U\left(\mathbb{E}_p(v), v\right) \geq \inf_{v \in \mathcal{V}} U\left(\mathbb{E}_q(v), v\right)$$

- U can be proven to be "essentially" unique and also an index of risk aversion (comparative statics is possible)

Particular Cases

- Cautious EU binary relations

$$U(t,v) = \begin{cases} v^{-1}(t) & v \in \mathcal{W} \\ \infty & otherwise \end{cases} \qquad \forall (t,v) \in \mathbb{R} \times \Delta$$

- Maccheroni's binary relations

$$U(t,v) = \begin{cases} t & v \in \mathcal{W} \\ \infty & otherwise \end{cases} \qquad \forall (t,v) \in \mathbb{R} \times \Delta$$

Conclusions

- There are several experimental studies providing evidence for the violation of the axiom of Independence
- There are also several models out there that address this issue
- DMO '13 offer as motivation the Certainty Effect (Negative Certainty Independence)
- One read of this model: a procedure of cautious extension of incomplete preferences
- Certainty Effect implies a preference for hedging (Mixing/Preference for Randomization)
- Mixing yields a generalized version of the pessimistic criterion characterized in DMO '13 under the same interpretation (see Cer '09)

A Message

- Decision Theory is fun! Often, in answering relevant Economics questions, it forces you to solve nontrivial mathematical problems!

Preorder: a discussion

- **Reflexive** $p \succsim p$ for all $p \in \Delta$
- **Transitive** If $p \succsim q$ and $q \succsim r$ then $p \succsim r$

Preorder

Uniqueness

- Assume v_1 and v_2 "both represent" $\succsim$
- There exists $\alpha > 0$ and $\beta \in \mathbb{R}$ such that

$$v_1 = \alpha v_2 + \beta$$

A Problem of Choice under Risk

Consider two lotteries

$$p \begin{array}{l} \overset{.5}{\nearrow} \; 1\ Mil\ \$ \\ \\ \underset{.5}{\searrow} \; 0\ \$ \end{array} \qquad\qquad q \xrightarrow{1} 480k\ \$$$

Notice that $e(p) = 500k\ \$$. Personally, $q \succ p$. Thus, $\delta_{e(p)} \succ q \succ p$.

Risk Aversion

A Problem of Choice under Risk

- Assume $\mathcal{W}_1$ and $\mathcal{W}_2$ "both represent" $\succsim$
- $\bar{\text{co}}\left(\mathcal{W}_1\right) = \bar{\text{co}}\left(\mathcal{W}_2\right)$

Sketch of the Proof

1. Define the Aumann cone

$$C\left(\succsim\right)=\left\{\lambda\left(p-q\right):\lambda>0 \text{ and } p\succsim q\right\}$$

2. Prove it is indeed a weak* closed and convex cone (non trivial use of Banach–Steinhaus and Krein–Smulian theorems)
3. Compute the polar cone

$$\mathcal{W}=\left\{v\in C\left(\left[w,b\right]\right):\int_{\left[w,b\right]}v\left(x\right)dr\leq 0 \text{ for all } r\in C\left(\succsim\right)\right\}$$

4. By an application of the Hahn–Banach Theorem and point 2, it follows that $\mathcal{W}$ is the set that does the job