An introduction to asymmetric information and financial market microstructure (Tutorial)

Fabrizio Lillo

Scuola Normale Superiore di Pisa, University of Palermo (Italy) and Santa Fe Institute (USA)

Pisa - July 10, 2013

Outline

- Introduction to financial market microstructure
- "Classical" asymmetric information models in financial market microstructure
  - Sequential models: Glosten-Milgrom
  - Strategic models: Kyle
- How efficiency shapes market impact (Farmer, Gerig, Lillo, and Waelbroeck, 2011)

Market microstructure

Market microstructure "is devoted to theoretical, empirical, and experimental research on the economics of securities markets, including the role of information in the price discovery process, the definition, measurement, control, and determinants of liquidity and transactions costs, and their implications for the efficiency, welfare, and regulation of alternative trading mechanisms and market structures" (NBER Working Group)

"We depart from the usual approaches of the theory of exchange by (1) making the assumption of asynchronous, temporally discrete market activities on the part of market agents and (2) adopting a viewpoint which treats the temporal microstructure, i.e. moment-to-moment aggregate exchange behavior, as an important descriptive aspect of such markets." (Garman 1976)

Asymmetric information models

- Aim to model the interaction of informed traders with uninformed intermediaries (dealers, market makers, liquidity providers), often in presence of noise traders
- The security payoff is usually of a common value nature
- The primary benefit derived from ownership of the security is the resale value or terminal liquidating dividend that is the same for all holders
- But for trade to exist, we also need private value components, that is, diversification or risk exposure needs that are idiosyncratic to each agent
- Public information initially consists of common knowledge concerning the probability structure of the economy, in particular the unconditional distribution of terminal security value and the distribution of types of agents
- As trading unfolds, the most important updates to the public information set are market data, such as bids, asks, and the prices and volumes of trades
- Thus, the trading process is an adjustment from one well-defined information set to another (no stationarity, ergodicity, time-homogeneity)

Sequential and strategic models

- **Sequential trade models:** randomly selected traders arrive at the market singly, sequentially, and independently. (Copeland and Galai (1983) and Glosten and Milgrom (1985)).
  When an individual trader only participates in the market once, there is no need for her to take into account the effect her actions might have on subsequent decisions of others.
- **Strategic trader models:** A single informed agent who can trade at multiple times (Kyle 1985).
  A trader who revisits the market, however, must make such calculations, and they involve considerations of strategy In both models a trade reveals something about the agent's private information

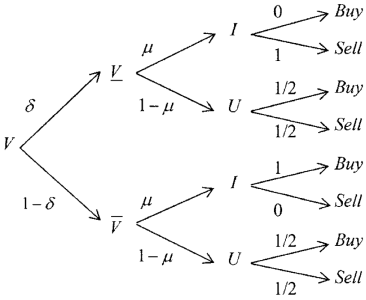
Sequential model: Glosten and Milgrom (simplified)

- One security with value (payoff) V that is either low ($\underline{V}$) or high ($\overline{V}$). The probability of the low outcome is δ.
- The value is revealed after the market closes and it is not affected by trading. It is determined by a random draw of nature before the market opens.
- Traders can be either informed (i.e. they know the value (V) outcome) or uninformed. The fraction of informed traders is μ.
- A dealer posts bid and ask quotes, B and A.
- Parameters ($\underline{V}$,$\overline{V}$, δ, μ) are known to the dealer

Glosten Milgrom model (continued)

- At each time step a trader is drawn at random from the population
  - If she is informed, she buys if $V = \overline{V}$ and sells if $V = \underline{V}$.
  - If she is uninformed, she buys and sells with equal probability
- Transaction price is set by the dealer:
  - Buys (from traders) occur at dealer's ask price A
  - Sells (from traders) occur at dealer's bid price B
- The dealer does not know whether the trader is informed.

Probability tree



Total probabilities are obtained by multiplication along the path (U=Uninformed, I=informed)

- $P(V = \underline{V}, U, Buy) = \delta(1-\mu)\frac{1}{2}$
- $P(Buy) = \frac{1+\mu(1-2\delta)}{2}$

Dealer's decision process

Let us consider the dealer

- The purchases and sales are not sensitive to quotes.
- If she is monopolist, she sets $A = \infty$ and $B = 0$
- Competition and regulation: "We assume that competition among dealers drives expected profit to zero".

Dealer's decision process

The dealer uses Bayes rule to update her beliefs on V. Assume that the first trade is a buy. The updated value of her belief on δ is

$$\delta_1(Buy) = P(\underline{V}|Buy) = \frac{P(\underline{V}, Buy)}{P(Buy)} = \frac{\delta(1-\mu)}{1+\mu(1-2\delta)} \tag{1}$$

If the first trade is instead a sell, the updated value of her belief on δ is

$$\delta_1(Sell) = P(\underline{V}|Sell) = \frac{P(\underline{V}, Sell)}{P(Sell)} = \frac{\delta(1+\mu)}{1-\mu(1-2\delta)} \tag{2}$$

Dealer's profit

Assume that the first trade is a buy.
At the end of the day, the dealer's realized profit on the first transaction is $\Pi = A - V$ (because it is a buy). Immediately after the the first trade her expectation of the profit is

$$E[\Pi|Buy] = A - E[V|Buy] = A - (\delta_1(Buy)\underline{V} + (1 - \delta_1(Buy))\overline{V}) \quad (3)$$

Because of competition $E[\Pi|Buy] = 0$, hence $A = E[V|Buy]$ and therefore

$$A = \frac{\underline{V}(1-\mu)\delta + \overline{V}(1-\delta)(1+\mu)}{1 + \mu(1-2\delta)} \quad (4)$$

Dealer's profit (continued)

Analogously the bid price is equal to

$$B = E[V|Sell] = \frac{\underline{V}(1+\mu)\delta + \overline{V}(1-\mu)(1-\delta)}{1-\mu(1-2\delta)} \tag{5}$$

Therefore the spread (before the first transaction) is set by the dealer at

$$A - B = \frac{4(1-\delta)\delta\mu(\overline{V} - \underline{V})}{1-(1-2\delta)^2\mu^2} \tag{6}$$

For $\delta = 1/2$ (equally likely terminal value V)

$$A - B = (\overline{V} - \underline{V})\mu \tag{7}$$

The (initial) spread is proportional to the fraction of informed traders. I.e. the spread is a protection of the dealer to adverse selection from trading with informed traders

Who loses money?

Assume that the first trade is a buy.
From conditional expectation

$$E[V|Buy] = E[V|U, Buy]P(U|Buy) + E[V|I, Buy]P(I|Buy) \tag{8}$$

Using $A = E[V|Buy]$ we get rearranging the terms

$$(A - E[V|U, Buy])P(U|Buy) = -(A - E[V|I, Buy])P(I|Buy) \tag{9}$$

The left hand side is the dealer's gain from trading with the uninformed, the right hand side is the dealer's gain from trading with the the informed.

Wealth is (on average) transferred from uninformed traders to informed traders. The dealer breaks even (on average).

Dynamics

What does happen after the first trade?
After the initial trade, the dealer updates her beliefs and update the quotes accordingly by using the expressions for the map from prior to posterior probabilities

$$\delta_k(Buy_k; \delta_{k-1}) = \frac{\delta_{k-1}(1-\mu)}{1+\mu(1-2\delta_{k-1})} \qquad \delta_k(Sell_k; \delta_{k-1}) = \frac{\delta_{k-1}(1+\mu)}{1-\mu(1-2\delta_{k-1})}$$

- Trade price is a martingale
- The spread declines over time and prices converges to the true value of V
- There is a price impact of prices, i.e., given a past history, a buy moves the price up and a sell moves the price down

General comments

- The Glosten Milgrom model assumes that informed traders trade only with market orders. This is unrealistic in limit order book markets.
- However this (unrealistic) assumption is used to estimate empirically the probability of informed trading (PIN)
- The Glosten Milgrom model shows that an important component of the spread is due to adverse selection
- In classical microstructure the spread is divided in three components
  - Fixed noninformational cost (like in Roll model)
  - Adverse selection cost (like in Glosten Milgrom model)
  - Inventory cost (the dealer puts a spread to control the risk of accumulating a large inventory)
- Some studies propose methods to estimate empirically the three components (Huang and Stoll, 1997)[1]

[1] They found that 90% of the spread is associated to order processing costs, and not to adverse selection (which is often found to have, within this framework, a negative contribution to the spread!)

Strategic model: Kyle

- The model describes a case of information asymmetry, the way in which information is incorporated into price, and the strategic reasoning of the dealer **AND** of the informed agent
- It is an equilibrium model
- There are several variants: single period, multiple periods, continuous time
- Three agents
  - One market maker (or dealer), MM
  - One informed trader
  - Many noise traders

Kyle model: one period

- The terminal (liquidation) value v of an asset is normally distributed with mean p_0 and variance Σ_0.
- The informed trader knows v and enters a demand x (volume).
- Noise traders submit a net order flow u, which is Gaussian distributed with mean zero and variance σ_u^2
- The MM observes the total demand $y = x + u$ and then sets a price p. All the trades are cleared at p, any imbalance is exchanged by the MM.

Kyle model: one period (continued)

- The informed trader wants to trade as much as possible to exploit her informational advantage
- However the MM knows that there is an informed trader and if the total demand is large (in absolute value) she is likely to incur in a loss. Thus the MM protects herself by setting a price that is increasing in the net order flow.
- The solution to the model is an expression of this trade-off

Informed trader

- The informed trader conjectures that the MM uses a linear price adjustment rule $p = \lambda y + \mu$, where λ is inversely related to liquidity.
- The informed trader's profit is

$$\pi = (v - p)x = x[v - \lambda(u + x) - \mu] \tag{10}$$

and the expected profit is

$$E[\pi] = x(v - \lambda x - \mu) \tag{11}$$

- The informed traders maximizes the expected profit, i.e.

$$x = \frac{v - \mu}{2\lambda} \tag{12}$$

- In Kyle's model the informed trader can loose money, but on average she makes a profit

Market maker

- The MM conjectures that the informed trader's demand is linear in v, i.e. $x = \alpha + \beta v$
- Knowing the optimization process of the informed trader, the MM solves

$$\frac{v - \mu}{2\lambda} = \alpha + \beta v \tag{13}$$

which gives

$$\alpha = -\frac{\mu}{2\lambda} \qquad \beta = \frac{1}{2\lambda} \tag{14}$$

- As liquidity drops the informed agent trades less
- The MM observes y and sets

$$p = E[v|y] \tag{15}$$

Equilibrium solution

- If X and Y are bivariate normal variables, it is

$$E[Y|X = x] = \mu_Y + \frac{\sigma_{XY}}{\sigma_X^2}(x - \mu_X) \tag{16}$$

  where μ_X and μ_Y are the mean values of X and Y, σ_{XY} their covariance, and σ_X is the variance of X.
- This can be used to find

$$E[v|y] = E[v|u + \alpha + \beta v] \tag{17}$$

- The solution is

$$\alpha = -p_0\sqrt{\frac{\sigma_u^2}{\Sigma_0}} \qquad \mu = p_0 \qquad \lambda = \frac{1}{2}\sqrt{\frac{\Sigma_0}{\sigma_u^2}} \qquad \beta = \sqrt{\frac{\sigma_u^2}{\Sigma_0}} \tag{18}$$

Equilibrium solution (continued)

- The impact is linear and the liquidity increases with the amount of noise traders

$$p = p_0 + \frac{1}{2}\sqrt{\frac{\Sigma_0}{\sigma_u^2}}y \tag{19}$$

- The informed agent trades more when she can hide her demand in the noise traders demand

$$x = (v - p_0)\sqrt{\frac{\sigma_u^2}{\Sigma_0}} \tag{20}$$

- The expected profit of the informed agent depends on the amount of noise traders

$$E[\pi] = \frac{(v - p_0)^2}{2}\sqrt{\frac{\sigma_u^2}{\Sigma_0}} \tag{21}$$

- The noise traders loose money and the MM breaks even (on average)

Kyle model in multiple periods in a nutshell

- There are N auctions, each taking place at times $0 = t_0 < t_1 < \dots < t_N = 1$
- The liquidation value of the asset is v, normally distributed with mean p_0 and variance Σ_0
- The quantity traded by noise traders in auction n is $\Delta u_n = u_n - u_{n-1}$, where u_n is a Brownian motion with zero mean and instantaneous variance σ_u^2
- x_n is the aggregate position of the informed after after the nth auction and $\Delta x_n = x_n - x_{n-1}$ is the quantity traded in this auction
- Each auction is divided in two steps:
  1. The informed and the noise traders place the aggregate demand $\Delta x_n + \Delta u_n$
  2. The market maker sets the liquidation price p_n

Kyle model in multiple periods in a nutshell

- The informed trader's trading strategy is a vector of functions $X =< X_1, ...X_N >$, where $x_n = X_n(p_1, ..., p_{n-1}, v)$
- The market maker pricing rule is a vector of function $P =< P_1, ..., P_N >$, where $p_n = P_n(x_1 + u_1, ..., x_n + u + n)$
- The profit of the informed on position acquired at auctions $n, ..., N$ is $\pi_n = \sum_{k=n}^{N}(v - p_k)x_k$
- A *sequential auction equilibrium* is a pair X, P such that
  - Profit maximization. $\forall n = 1, .., N$ and $\forall X'$ s.t. $X_1' = X_1, ...X_{n-1}' = X_{n-1}$ it is
  $$E[\pi_n(X, P)|p_1, ..., p_{n-1}, v] \geq E[\pi_n(X', P)|p_1, ..., p_{n-1}, v] \quad (22)$$
  - Market efficiency. $\forall n = 1, .., N$ it is
  $$p_n = E[v|x_1 + u_1, ..., x_n + u_n] \quad (23)$$
- A linear equilibrium is a sequential auction equilibrium in which the functions X and P are linear
- A recursive linear equilibrium is a linear equilibrium s.t. $\exists\lambda_1, ...\lambda_N$ s.t. $\forall n = 1, .., N$
$$p_n = p_{n-1} + \lambda_n(\Delta x_n + \Delta u_n) \quad (24)$$

Kyle model in multiple periods in a nutshell

Theorem. There exists a unique linear equilibrium and this equilibrium is a recursive linear equilibrium. In this equilibrium there are constants $\beta_n, \lambda_n, \alpha_n, \delta_n, \Sigma_n$ such that

$$\Delta x_n = \beta_n (v - p_{n-1}) \Delta t_n \tag{25}$$

$$\Delta p_n = \lambda_n (\Delta x_n + \Delta u_n) \tag{26}$$

$$\Sigma_n \equiv var[v | \Delta x_1 + \Delta u_1, \ldots \Delta x_n + \Delta u_n] = (1 - \beta_n \lambda_n \Delta t_n) \Sigma_{n-1} \tag{27}$$

$$E[\pi_n | p_1, \ldots, p_{n-1}, v] = \alpha_{n-1} (v - p_{n-1})^2 + \delta_{n-1} \tag{28}$$

Predictions:

- The informed agent “slices and dices” (splits) her order flow in order to hide in the noise trader order flow
- Linear price impact
- Uncorrelated total order flow
- Permanent and fixed impact
- Variance of fundamental value v declines (but does not go to zero unless $N \to \infty$)

Empirical data consistently shows a *sublinear* (concave) volume dependence of impact of individual orders

$$E[\Delta p|q] \equiv R_{so}(T=1|q) \propto q^{\psi}; \qquad \psi \in [0.1, 0.3], \tag{29}$$

or even a logarithmic dependence $R_{so}(T=1|q) \propto \ln q$.

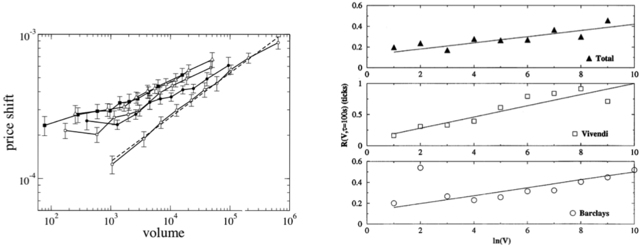


Figure: Impact of individual market orders for London Stock Exchange (left, Lillo and Farmer 2004) and Paris Bourse (right, Bouchaud and Potters 2002)

Empirical facts: impact of metaorder

- For algorithmic executions traders care most of impact of metaorder because this is a cost
- Unfortunately this is the most difficult to measure because of lack of data. Two approaches:
  - Using data of proprietary trading strategies of investors or brokers (but often not published)
  - Using data from exchanges who give exceptional access to identification codes that allow one to reconstruct the metaorders from some market participants
- Remarkably, very different studies seem to agree on the "square-root impact law", i.e.

$$R_{mo}(T|Q) = Y\sigma\sqrt{\frac{Q}{V}}, \tag{30}$$

where $Y \simeq 1$, σ is the daily volatility of the asset, and V the daily traded volume

Impact of metaorders from proprietary data

From Toth et al. 2011

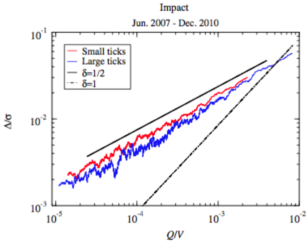
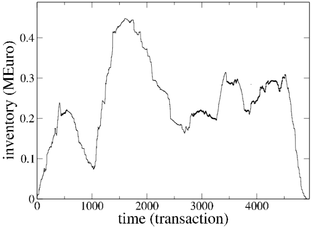


Figure: The impact of metaorders for Capital Fund Management proprietary trades on futures markets, in the period from June 2007 to December 2010. Impact is measured here as the average execution shortfall of a metaorder of size Q. The data base contains nearly 500,000 trades. We show $R_{mo}(T|Q)/\sigma$ vs Q/V on a log-log scale, where σ and V are the daily volatility and daily volume measured the day the metaorder is executed. The blue curve is for large tick sizes, and the red curve is for small tick sizes.

Impact of metaorders reconstructed from brokerage data

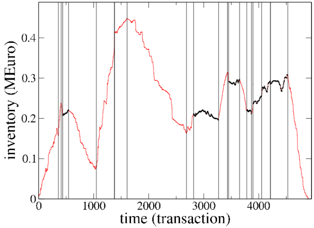
We have special access to database from exchanges containing the information of the brokers behind any order or trade.

An example: inventory time series: Credit Agricole trading Santander



- Clear trends are visible
- The identification of metaorders must be statistical: a typical regime switching problem

Credit Agricole trading Santander



Different algorithms:

- Modified t-test (G. Vaglica, F. Lillo, E. Moro, and R. N. Mantegna, Physical Review E **77**, 036110 (2008).)
- Hidden Markov Model (G. Vaglica, F. Lillo, and R. N. Mantegna, New Journal of Physics, **12** 075031 (2010)).

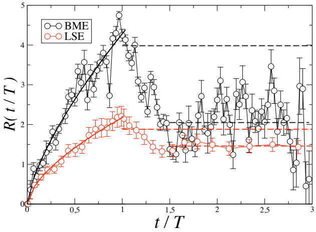


Figure: Market impact versus time. The symbols are the average value of the market impact of the metaorder as a function of the normalized time to completion t/T. The rescaled time $t/T = 0$ corresponds to the starting point of the metaorder, while $t/T = 1$ corresponds to the end of the metaorder.

We find approximately the square root law

$$E[r|N] = A\epsilon s N^{\beta} \qquad \beta \simeq 1/2 \tag{31}$$

Order flow

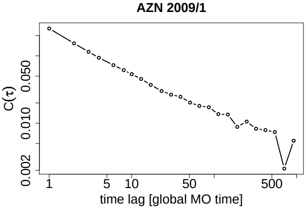
- We focus here on orders that trigger transactions, i.e. *market orders*
- A buy market order moves the price up and a sell market order moves the price down (on average)
- The flow of market orders reflects the supply and demand of shares
- A market order is characterized by a volume v and a sign $\epsilon = +1$ for buy orders and $\epsilon = -1$ for sell orders.
- We consider the time series in market order time, i.e. time advances of one unit when a new market order arrives.
- The unconditional sample autocorrelation function of signs is

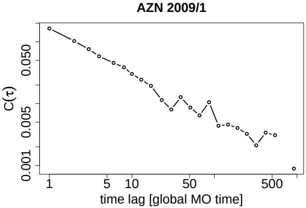
$$C(\tau) = \frac{1}{N} \sum_t \epsilon_t \epsilon_{t+\tau} - \left(\frac{1}{N} \sum_t \epsilon_t \right)^2 ,$$

where N is the length of the time series.

Market order flow is very persistent in time

It has been shown (Bouchaud *et al.*, 2004, Lillo and Farmer, 2004) that the time series of market order signs is a long memory process.





$C(\tau)$ of market order signs ϵ (left) and signed volumes ϵv (right).
The autocorrelation function decays asymptotically as

$$C(\tau) \sim \tau^{-\gamma} = \tau^{2H-2}$$

where H is the Hurst exponent. For the investigated stocks $H \simeq 0.75$ (i.e. $\gamma \simeq 0.5$).

Summary of empirical evidences

Kyle's model provides a good framework to understand asymmetric information and strategic decision between informed and market makers.
However, contrary to the model's predictions, data show that

- Impact is non linear
- Impact has a permanent and a temporary component
- Order flow is strongly autocorrelated in time

We present a modeling framework that explains these facts as result of the asymmetric information about the duration (or size) of the large trade.

What is the origin of long-memory in order flow?

Two explanations have been proposed

- Herding among market participants (LeBaron and Yamamoto 2007). Agents herd either because they follow the same signal(s) or because they copy each other trading strategies. Direct vs indirect interaction
- Order splitting (Lillo, Mike, and Farmer 2005). To avoid revealing true intentions, large investors break their trades up into small pieces and trade incrementally (Kyle, 1985). Convert heavy tail of large orders volume distributions in correlated order flow.

Is it possible to quantify **empirically** the contribution of herding and order splitting to the autocorrelation of order flow?
Note that this is part of the question on the origin of *diagonal effect* raised in Biais, Hillion and Spatt (1995).

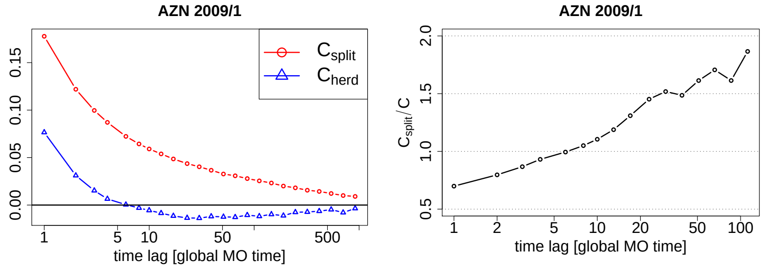


Figure: Left panel. The splitting and the herding term of the correlation of MO signs (the two terms sum up to $C(\tau)$) for the first half year of 2009 for AZN. Right panel. The splitting ratio of MO signs (defined as the ratio of the splitting term in the correlations and the entire correlation) for the first half year of 2009 for AZN.

Splitting dominates herding as an explanation of the autocorrelated order flow.

Autocorrelation of order flow reflects metaorder size distribution

Lillo, Mike, and Farmer 2005 propose a model where N agents slice and dice their large orders.
The number of trades for a large order is Pareto distributed $p(L) \sim L^{-(\alpha+1)}$
The model predicts that $\gamma = \alpha - 1$, i.e. $\alpha \simeq 1.5$.

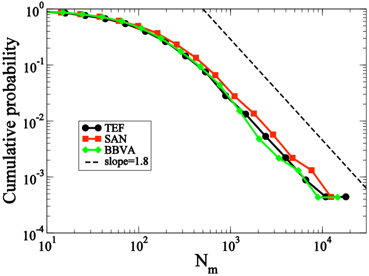


Figure: Distribution of the size of the statistically reconstructed metaorder in the Spanish stock market

- Metaorder size is asymptotically power law distributed
- The tail exponent is consistent with the splitting model
- Recently Bershova and Rakhlin (2013) found a tail exponent of 1.56 by investigating metaorders of clients of AllianceBernstein

A model for fair pricing condition of market impact

- Portfolio managers make long-term decision to either buy or sell a given asset, and then incrementally executes small trades until she has bought or sold the desired quantity. We call this agent a *directional trader*.
- Following Kyle, the counterpart of the trades is a *market maker* who provides liquidity.
- The directional trader's metaorder to either buy or sell is broken up and executed in a series of smaller trades. These trades take place over N time intervals each of length τ, labeled by the index $k = 1, \ldots, N$.
- The size is chosen from a distribution p_N which is given, and which is public information. Moreover we assume that the distribution p_N has compact support, i.e. that the maximal number of intervals is a known value M.

A model for fair pricing condition of market impact

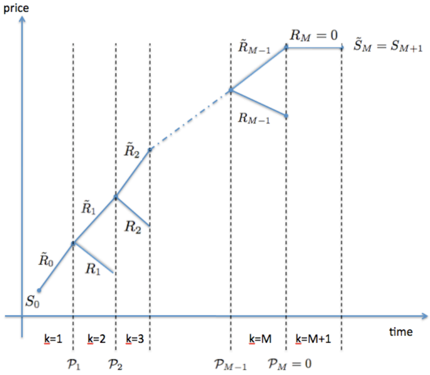


Figure: Sketch of the model

Market efficiency

The only source of uncertainty for the market maker is the size of the metaorder.
The existence of metaorders causes order flow to be correlated.
Let $\mathcal{P}_k$ be the probability that the metaorder will continue based on the knowledge that it is still active at timestep k.
If the market is efficient

$$\mathcal{P}_k \tilde{R}_k - (1 - \mathcal{P}_k) R_k = 0, \qquad k = 1, 2, ..., M - 1 \tag{32}$$

where $\tilde{R}_k = \tilde{S}_{k+1} - \tilde{S}_k$ and $R_k = (\tilde{S}_k - S_{k+1})$. $\tilde{R}_k$ is the expected return if the order continues to $k + 1$, and $-R_k$ is the expected return if it stops at k.
Moreover $\mathcal{P}_M = 0$ which implies that $R_M = 0$.

Breakeven when averaged over all sizes

Assume that the market maker sells during N intervals to the directional trader at prices $\tilde{S}_i$ and then buys them back at price S_{N+1}, so that her profit for a trade of size N is

$$\Pi_N \equiv \sum_{i=1}^{N} \tilde{S}_i - N S_{N+1}$$

The condition that the market maker will *breakeven when averaged over all sizes* can be written

$$E_1[\Pi_N] \equiv \sum_{N=1}^{M} p_N \Pi_N = 0$$

This obviously also implies that the directional trader will also break even.
It can be shown that market efficiency implies breakeven when averaged over all sizes if the support of p_N is compact.

Determining the continuation probability

$$\mathcal{P}_k = \frac{\sum_{i=k+1}^{M} p_i}{\sum_{i=k}^{M} p_i}. \qquad (33)$$

For Pareto distributed metaorder sizes

$$p_N = \frac{1}{\zeta(\alpha)} \frac{1}{N^{\alpha+1}}, \qquad N \geq 1 \qquad (34)$$

where the normalization constant $\zeta(\alpha)$ is the Riemann zeta function it is

$$\mathcal{P}_k = \frac{\zeta(1+\alpha, k+1)}{\zeta(1+\alpha, k)} \simeq \left(\frac{k}{k+1}\right)^{\alpha} \sim 1 - \frac{\alpha}{k}. \qquad (35)$$

where $\zeta(s, a)$ is the generalized Riemann zeta function.

Fair pricing condition

Efficiency at the end of the first interval implies that $\Pi_1 > 0$, i.e. the market maker makes profit for short orders (as originally suggested by Glosten 1985). Breakeven averaged over all size implies that market maker loses money for other order sizes.

Efficiency at the end of the last interval implies that $\Pi_M < 0$, i.e. the market maker loses money on orders of maximal size.

We postulate that

$$\Pi_N = \sum_{i=1}^{N} \tilde{S}_i - N S_{N+1} = 0 \qquad N = 2, 3, ..., M-1$$

This means that

$$p_1 \Pi_1 + p_M \Pi_M = 0$$

We showed that fair pricing comes from a Nash equilibrium in a competitive environment of informed traders

More recently, Donier et al. (2013) justifies the fair pricing condition in terms of competition between market makers

Bershova and Rakhlin (2013) verified empirically fair pricing on real metaorder of AllianceBernstein

Solution

The system of efficiency (martingale) conditions and fair pricing conditions has solution

$$\tilde{R}_k = \frac{1}{k} \frac{p_k}{\sum_{i=k+1}^{M} p_i} \frac{1 - p_1}{\sum_{i=k}^{M} p_i} \tilde{R}_1 \qquad k = 2, 3, ..., M - 1$$

$$R_k = \frac{\mathcal{P}_k}{1 - \mathcal{P}_k} \tilde{R}_k \qquad k = 1, 2, ..., M - 1$$

There are two undetermined parameters:

- $\tilde{R}_0$ is related to market conditions, such as volatility, and to the intensity of the order flow imbalance.
- $\tilde{R}_1$ fixes the scale of the impact.

As a realistic case let us consider the case of Pareto distributed sizes, $p_N \propto N^{-(1+\alpha)}$. The impact $\tilde{S}_k - \tilde{S}_1$ behaves asymptotically for k large but $k \ll M$ as

$$\tilde{S}_k - \tilde{S}_1 \sim \begin{cases} k^{\alpha-1} & \text{for } \alpha \neq 1 \\ \log(k+1) & \text{for } \alpha = 1 \end{cases}$$

In the important case of $\alpha = 1.5$ we get

- A square root law for temporary impact
- Permanent impact is $1/\alpha = 2/3$ of the temporary impact

Conclusions

- Order flow is a long memory process
- Efficiency is restored via an asymmetric liquidity mechanism
- The origin is order splitting; herding plays a minor role
- The exponent $\gamma \simeq 0.5$ of the autocorrelation function can be related to a tail exponent $\alpha \simeq 1.5$ of the metaorder size (via the splitting model).
- The fair pricing model predicts a square root impact and a $2/3$ reversion, similarly to what observed in real (reconstructed) metaorders.
- The model postulates the fair pricing condition, similarly to what observed in real (reconstructed) metaorders.
- The recent paper (2013) of Bershova and Rakhlin confirms most of the predictions of this model on a set of metaorders of AllianceBernstein