Mixed Extensions of Decision Problems under Uncertainty

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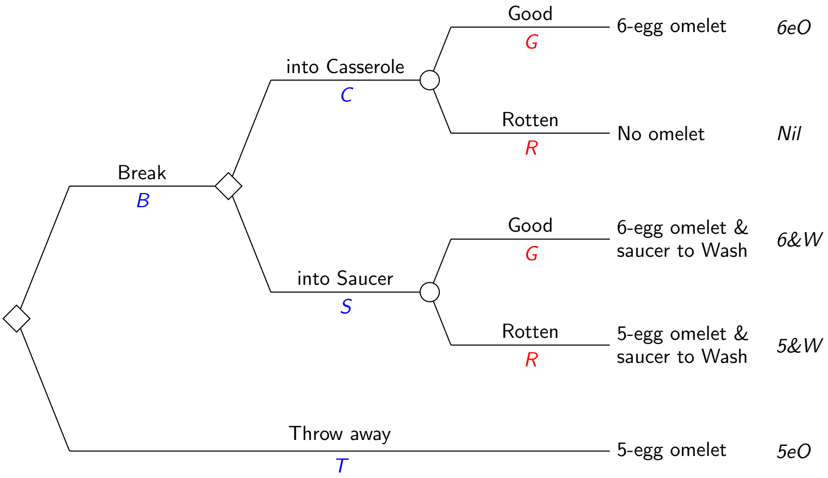
Minimal references

- Herstein and Milnor (1953)
- Kuhn (1953)
- Savage (1954)
- Milnor (1954)
- Luce and Raiffa (1957)
- Raiffa *vs* Ellsberg (1961)
- Anscombe and Aumann (1963)
- Marschak and Radner (1972)

The omelet tree



Normal omelets

Decision Strategy	State	
	G	*R*
BC	*6eO*	*Nil*
BS	*6&W*	*5&W*
TC	*5eO*	*5eO*
TS	*5eO*	*5eO*

Decision Plan	State	
	G	*R*
BC	*6eO*	*Nil*
BS	*6&W*	*5&W*
T	*5eO*	*5eO*

Table 1 of Savage (1954)

... If two different acts had the same consequences in every state of the world, there would from the present point of view be no point in considering them two different acts at all... Or, more formally, an act is a function attaching a consequence to each state *of the world...* (ibidem)

Decision frameworks under uncertainty

A **decision framework** is a quartet (A, S, C, ρ) where

- A is a set of **(conceivable pure) actions**
- S is a finite set of **states**
- C is a set of **consequences**
- and a **consequence function**

$$\begin{array}{rccc} \rho: & A \times S & \to & C \\ & (a,s) & \mapsto & \rho(a,s) \end{array}$$

associates consequences with actions and states

Savage acts and consequentialism

Savage's identification of an action $a \in A$ with the section $\rho_a \in C^S$

$$\begin{array}{rccc} \rho_a : & S & \rightarrow & C \\ & s & \mapsto & \rho(a, s) \end{array}$$

... an act is a function attaching a consequence to each state ...

amounts to identify two actions a and b if

$$\rho(a, s) = \rho(b, s) \quad \forall s \in S$$

two such actions are called **realization equivalent**, denoted $a \approx b$

Consequentialism

The DM is indifferent between realization equivalent actions

Reduced decision frameworks

Definition

A decision framework (A, S, C, ρ) is **purely reduced** iff $a \approx b \implies a = b$

In this case $a \hookrightarrow \rho_a$ is an embedding of A into the set C^S of **Savage acts**, $\{\rho_a\}_{a \in A}$ is the set of acts that are **conceivable** in (A, S, C, ρ)

Definition

A **Marschak-Radner framework** (A, S, C, ρ) is a purely reduced decision framework in which all binary acts are conceivable

Savage's is the special case $\left(C^S, S, C, \langle \cdot, \cdot \rangle\right)$ in which $\langle f, s \rangle := f(s)$

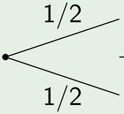
Mixed actions

A **mixed action** $\alpha \in \Delta(A)$ is a chance distribution of pure actions

Example (Luce and Raiffa, 1957)

... Consider, for example...

$$\frac{1}{2}\delta_b + \frac{1}{2}\delta_w \quad =$$



	B	W
b	1	0
w	0	1

... the security level of each act is 0, but if we permit randomization between b and w the security level can be raised to $1/2$...

Anscombe-Aumann acts I

Mixed actions in $\Delta(A)$ (chance distributions of pure actions) induce **lotteries** in $\Delta(C)$ (chance distributions of consequences)

If the DM *commits to* α, the chance of obtaining c in state s is

$$\rho_\alpha(c \mid s) := \alpha(\{a \in A : \rho(a,s) = c\})$$

That is, each mixed action $\alpha \in \Delta(A)$ induces an **Anscombe-Aumann act**

$$\begin{array}{rccc} \rho_\alpha : & S & \to & \Delta(C) \\ & s & \mapsto & \rho_\alpha(\cdot \mid s) \end{array}$$

that associates with each $s \in S$ the distribution of consequences resulting from the choice of α in state s

Anscombe-Aumann acts II

Two mixed actions [α and β] are now **realization equivalent** [$\alpha \approx \beta$] iff they induce the same distribution of consequences in every state [$\rho_\alpha = \rho_\beta$]

Theorem

Let (A, S, C, ρ) be a purely reduced decision framework

1. $\rho_{\delta_a}(s) = \delta_{\rho(a,s)}$ *for all* $a \in A$ *and* $s \in S$
2. $\rho_{q\alpha+(1-q)\beta} = q\rho_\alpha + (1-q)\rho_\beta$ *for all* $\alpha, \beta \in \Delta(A)$ *and* $q \in [0,1]$
3. $\{\rho_\alpha\}_{\alpha\in\Delta(A)} = \Delta(C)^S$ *if and only if* $\{\rho_a\}_{a\in A} = C^S$
4. *if* $a \in A$ *and* $\beta \in \Delta(A)$ *are realization equivalent, then* $\beta = \delta_a$

Anscombe-Aumann acts III

Typically $f \in \Delta(C)^S$ is interpreted as describing **ex-post** randomization

1. the DM commits to f
2. "observes" the realized state s
3. "observes" the consequence c randomly generated by $f(s)$
4. and receives c

Here $\alpha \in \Delta(A)$ may be interpreted as describing **ex-ante** randomization

1. the DM commits to α
2. "observes" the action a randomly generated by α
3. "observes" the realized state s
4. and receives $c = \rho(a, s)$

$$\forall s \in S \quad \rho_\alpha(s) = \sum_{a \in A} \alpha(a)\, \rho_{\delta_a}(s) = \sum_{a \in A} \alpha(a)\, \delta_{\rho(a,s)} = f_\alpha(s)$$

Lotteries

In a Marschak-Radner framework, denote by $cS \in A$ an action such that

$$\rho_{cS} \equiv c$$

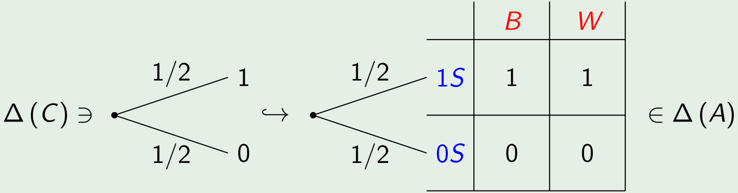
Lotteries can then be embedded into mixed actions

$$\begin{array}{rccc} \epsilon : & \Delta(C) & \to & \Delta(A) \\ & \sum_{c \in C} \gamma(c)\,\delta_c & \hookrightarrow & \sum_{c \in C} \gamma(c)\,\delta_{cS} \end{array}$$

The mixed action $\epsilon(\gamma)$ delivers c with probability $\gamma(c)$ in every state

$$\rho_{\epsilon(\gamma)} \equiv \gamma$$

Example



Why lotteries and acts?

The celebrated *von Neumann–Morgenstern Expected Utility Theorem* gives necessary and sufficient conditions on a preorder $\succsim$ on $\Delta(C)$ that guarantee the existence of a *payoff function* $u : C \to \mathbb{R}$ representing it

$$\gamma \succsim \zeta \Longleftrightarrow \sum_{c \in C} \gamma(c)\, u(c) \geq \sum_{c \in C} \zeta(c)\, u(c)$$

Starting with Savage (1954) and Anscombe and Aumann (1963) the vast majority of decision theory considered *acts* rather than *actions* for mathematical convenience and this generated a hiatus with game theory and statistics

The paper on which this tutorial is based aims to eliminate it

Preferences and risk attitudes

Let (A, S, C, ρ) be a Marschak-Radner framework

DM's **preferences** are represented by a binary relation $\succsim$ on $\Delta(A)$

DM's **risk attitudes** are derived by

$$\gamma \succsim_{\epsilon} \zeta \iff \sum_{c \in C} \gamma(c)\,\delta_{cS} \succsim \sum_{c \in C} \zeta(c)\,\delta_{cS} \quad [\epsilon(\gamma) \succsim \epsilon(\zeta)]$$

through the embedding ϵ of lotteries into mixed actions

Rational preferences

Rational preferences

A binary relation $\succsim$ *on* $\Delta(A)$ *is a* **rational preference** *iff*

1. $\succsim$ *is complete and transitive*
2. $\rho_\alpha(s) \succsim \rho_\beta(s)$ *for all* $s \in S$ *implies* $\alpha \succsim \beta$
3. $\succsim$ *satisfies the von Neumann-Morgenstern axioms on* $\Delta(C)$

See Gilboa, Maccheroni, Marinacci, and Schmeidler (2010) and Cerreia-Vioglio, Ghirardato, Maccheroni, Marinacci, and Siniscalchi (2011)

Proposition

Rational preferences are consequentialist, that is, $\alpha \approx \beta \implies \alpha \sim \beta$

Very weak dominance

Let $u : C \to \mathbb{R}$ be a von Neumann-Morgenstern **payoff function** and

- $\geqslant_u$ the (**very weak**) **dominance** preorder, that is,

$$\alpha \geqslant_u \beta \iff \sum_{a \in A} \alpha(a) u(\rho(a,s)) \geq \sum_{a \in A} \beta(a) u(\rho(a,s)) \;\; \forall s \in S$$

- $\Upsilon_u(\mathcal{B})$ the set of **dominant actions** in $\mathcal{B} \subseteq \Delta(A)$, that is,

$$\Upsilon_u(\mathcal{B}) := \{\alpha \in \mathcal{B} : \alpha \geqslant_u \beta \; \forall \beta \in \mathcal{B}\}$$

Rationality and dominance

Theorem

Let $\beth$ be the class of all non-empty finite parts of $\Delta(A)$

TFAE for a binary relation $\succsim$ on $\Delta(A)$

- $\succsim$ *is a rational preference*
- $\succsim$ *is generated by a choice correspondence $\Gamma : \beth \to \beth$ satisfying WARP and for which there exists $u : C \to \mathbb{R}$ such that for all $\mathcal{B} \in \beth$, $\Upsilon_u(\mathcal{B}) \subseteq \Gamma(\mathcal{B})$ and equality holds if $\mathcal{B} \subseteq \Delta(C)$*

Consistency checks:

1. The payoff function u represents $\succsim$ on $\Delta(C)$

$$\gamma \succsim \zeta \iff \sum_{c \in C} \gamma(c)\, u(c) \geq \sum_{c \in C} \zeta(c)\, u(c)$$

2. $\alpha \geqslant_u \beta \iff \rho_\alpha(s) \succsim \rho_\beta(s)$ for all $s \in S$, denoted $\alpha \succcurlyeq_S \beta$

Subjective probabilities

Continuity

If $\alpha, \beta, \eta \in \Delta(A)$*, then* $\{q \in [0,1] : q\alpha + (1-q)\beta \succsim \eta\}$ *and* $\{q \in [0,1] : q\alpha + (1-q)\beta \precsim \eta\}$ *are closed in* $[0,1]$

Theorem (Cerreia-Vioglio et alii, 2011)

If $\succsim$ *is a continuous and non-trivial rational preference, then there exist a non-constant* $u : C \to \mathbb{R}$ *and a closed and convex* $\mathcal{M} \subseteq \Delta(S)$ *such that given* $\alpha, \beta \in \Delta(A)$

$$q\alpha + (1-q)\eta \succsim q\beta + (1-q)\eta \quad \forall q \in [0,1] \text{ and } \eta \in \Delta(A)$$

$$\Longleftrightarrow$$

$$\mathrm{E}_{\alpha\times\mu}[u \circ \rho] \geq \mathrm{E}_{\beta\times\mu}[u \circ \rho] \quad \forall \mu \in \mathcal{M}$$

in this case u *is cardinally unique and* $\mathcal{M}$ *is unique*

Classical maxminimization

Extreme caution

If $\alpha \in \Delta(A)$, $\gamma \in \Delta(C)$, *and* $\alpha \not\succsim_S \gamma$, *then* $\gamma \succ \alpha$

Theorem (Milnor, 1954)

Let $\succsim$ *be a binary relation on* $\Delta(A)$ *TFAE*

- $\succsim$ *is a continuous and extremely cautious rational preference*
- *there exists* $u : C \to \mathbb{R}$ *such that*

$$\alpha \succsim \beta \iff \min_{s\in S} \sum_{a\in A} \alpha(a)\, u(\rho(a,s)) \geq \min_{s\in S} \sum_{a\in A} \beta(a)\, u(\rho(a,s))$$

Subjective expected utility

Independence

If $\alpha, \beta, \eta \in \Delta(A)$, *then* $\alpha \sim \beta$ *implies* $\frac{1}{2}\alpha + \frac{1}{2}\eta \sim \frac{1}{2}\beta + \frac{1}{2}\eta$

Theorem (Anscombe and Aumann, 1963)

Let $\succsim$ *be a binary relation on* $\Delta(A)$ *TFAE*

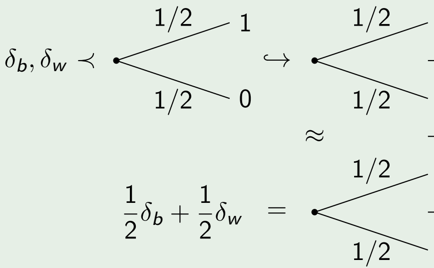
- $\succsim$ *is a continuous and independent rational preference*
- *there exist* $u : C \to \mathbb{R}$ *and* $\mu \in \Delta(S)$ *such that*

$$\alpha \succsim \beta \iff \mathrm{E}_{\alpha\times\mu}[u \circ \rho] \geq \mathrm{E}_{\beta\times\mu}[u \circ \rho]$$

$$\mathrm{E}_{\alpha\times\mu}[u \circ \rho] = \sum_{s\in S}\mu(s)\sum_{a\in A}\alpha(a)\,u(\rho(a,s)) = \int_S u(\rho_\alpha(s))\,d\mu(s)$$

Preference for randomization

Example (Ellsberg paradox – Raiffa version – Klibanoff discussion)



	B	W
$1S$	1	1
$0S$	0	0
b	1	0
w	0	1

Uncertainty aversion

If $\alpha, \beta \in \Delta(A)$, *then* $\alpha \sim \beta$ *implies* $\frac{1}{2}\alpha + \frac{1}{2}\beta \succsim \alpha$

Games against nature

A continuous rational preference which is uncertainty averse has the form

$$\alpha \succsim \beta \Longleftrightarrow \min_{\sigma \in \mathcal{S}} U(\alpha, \sigma) \geq \min_{\sigma \in \mathcal{S}} U(\beta, \sigma)$$

where, for each $\sigma \in \mathcal{S} \subseteq \Delta(S)$, $U(\eta, \sigma)$ is an increasing transformation of $\mathrm{E}_{\eta \times \sigma}[u \circ \rho]$; thus the choice correspondence $\Gamma : \beth \to \beth$ has the form

$$\Gamma(\mathcal{B}) = \arg\max_{\beta \in \mathcal{B}} \min_{\sigma \in \mathcal{S}} U(\beta, \sigma) \quad \forall \mathcal{B} \in \beth$$

See Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio (2011)

Maxmin expected utility

C-Independence

If $\alpha, \beta \in \Delta(A)$, $\gamma \in \Delta(C)$, *and* $q \in (0,1)$, *then*

$$\alpha \succ \beta \iff q\alpha + (1-q)\gamma \succ q\beta + (1-q)\gamma$$

Theorem (Gilboa and Schmeidler, 1989)

Let $\succsim$ *be a binary relation on* $\Delta(A)$ *TFAE*

- $\succsim$ *is a continuous and C-independent rational preference*
- *there exist* $u : C \to \mathbb{R}$ *and* $\mathcal{S} \subseteq \Delta(S)$ *such that*

$$\alpha \succsim \beta \iff \min_{\sigma \in \mathcal{S}} \mathrm{E}_{\alpha \times \sigma}[u \circ \rho] \geq \min_{\sigma \in \mathcal{S}} \mathrm{E}_{\beta \times \sigma}[u \circ \rho]$$

Variational preferences

Weak C-Independence

If $\alpha, \beta \in \Delta(A)$, $\gamma, \zeta \in \Delta(C)$, *and* $q \in [0,1]$, *then*

$$q\alpha + (1-q)\gamma \succsim q\beta + (1-q)\gamma \implies q\alpha + (1-q)\zeta \succsim q\beta + (1-q)\zeta$$

Theorem (Maccheroni, Marinacci, and Rustichini, 2006)

Let $\succsim$ *be a binary relation on* $\Delta(A)$ *TFAE*

- $\succsim$ *is a continuous and weakly C-independent rational preference*
- *there exist* $u : C \to \mathbb{R}$ *and a proper* $c : \Delta(S) \to [0, \infty]$ *such that*

$$\alpha \succsim \beta \iff \min_{\sigma} \{\mathrm{E}_{\alpha \times \sigma}[u \circ \rho] - c(\sigma)\} \geq \min_{\sigma} \{\mathrm{E}_{\beta \times \sigma}[u \circ \rho] - c(\sigma)\}$$