D. Bernoulli and D'Alembert on smallpox inoculation

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- In France, inoculation didn't take root and it was considered with suspect and skepticism, as a desperate expedient
- Between 1750 and 1770, the debate over inoculation spread. For the French *philosophes*, it turned into a crusade against obscurantism

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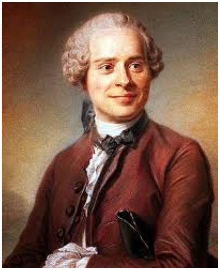
Conclusion:

Very dramatic, urgent and up-to-date problem

The decision problem



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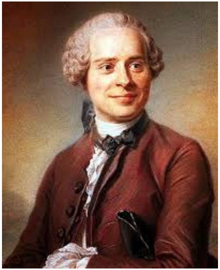

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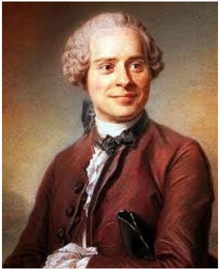

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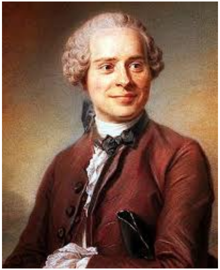

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Social perspective

Is it rational for the State to promote inoculation for all individuals at birth?

Individual perspective

Is it rational for an individual to inoculate?

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- people surviving from smallpox are protected against new infections for the rest of their life (they have been immunized);
- let $m(x)$ the mortality at age x due to causes other than smallpox. The probability for one individual to die in an infinitesimal time period dx between age x and age $x + dx$ is $m(x)dx$.

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1. Building of a populational model, a curve which describes the mortality rate $\rightarrow$ isolation of the mortality due to smallpox
2. Calculation of the variation of the life expectancy if smallpox were eradicated $\rightarrow$ comparison of two integrals
3. Evaluation of the risk of dying after inoculation

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Birth corresponds to age $x = 0$, so $S(0) = P(0) = P_0$ and $R(0) = 0$.

Between age x and age $x + dx$, each susceptible individual has a probability $q dx$ of being infected with smallpox and probability $m(x)dx$ of dying from other causes. So the variation of the number of susceptible people is $dS = -Sqdx - Sm(x)dx$, leading to the differential equation

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where $\frac{dS}{dx}$ is the derivative function $S(x)$.

The populational model (2)

During the same time interval, the number of people dying from smallpox is $pSqdx$ and the number of people who survive from smallpox is $(1-p)Sqdx$. Moreover, there are also $Rm(x)dx$ people who die from causes other than smallpox. This leads to a second differential equation:

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$$(4)\frac{S(x)}{P(x)} = \frac{1}{(1-p)e^{qx}+p}$$

Isolating mortality due to smallpox

With formula (4) and the values of $P(x)$, Bernoulli can compute the number $S(x)$ of susceptible people aged x and the number $R(x) = P(x) - S(x)$ of people aged x having survived from smallpox. The number of deaths due to smallpox between age x and age $x+1$ should be the integral $pq\int_x^{x+1} S(t)\,dt$, but the formula $pq[S(x)+S(x+1)]/2$ used by Bernoulli gives a good approximation (as the area of the trapezoid is close to the area under the curve).

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Reminder

Data collecting methods are imprecise!

Comparing two "states of humankind"

Bernoulli considers a situation where smallpox is inoculated to everybody at birth, with causing no deaths. The problem is then to estimate the increase in life expectancy if smallpox would be eradicated. Starting from the same initial population P_0, let's call $P^*(x)$ the number of people aged x when smallpox has disappeared.

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$$\frac{dP^*}{dx} = -m(x)P^*$$

So

$$P^*(x) = \frac{P(x)}{1 - p + pe^{-qx}}$$

where $P(x)$ is the population when smallpox is present.

Comparing $P(x)$ and $P^*(x)$ means to estimate the variation of life expectancy, the integrals $\frac{1}{P_0}\int_0^\infty P(x)\,dx$ and $\frac{1}{P_0}\int_0^\infty P^*(x)\,dx$.
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- Life expectancy with smallpox $= E = [\frac{1}{2}1300 + 1000 + ... + 20]/1300 \simeq 26.57$ years
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Inoculation at birth would increase life expectancy by more than three years.

Weighting risks and benefits

But inoculation is not completely safe. Let p' the probability of dying from smallpox just after inoculation, ($p' < p$); then life expectancy is $(1 - p')E^*$ if everybody went through inoculation at birth. Life expectancy remains higher if the following inequality holds:

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> *I simply wish that, in a matter which so closely concerns the wellbeing of the human race, no decision shall be made without all the knowledge which a little analysis and calculation can provide.*

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Conclusion:

Different approach to uncertainty and to the available methods to deal with it in moral decision making

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> *How are we to compare the present risk to an unknown, remote advantage? About this the analyisis of hazards can teach us nothing.*

"The last difficulty concerns the question whether (...) the mathematical expectation of different courses of action accurately measures what our preferences ought to be.
In the history of the subject, nevertheless, the theory of mathematical expectation has been very seldom disputed. D'Alembert has been almost alone in casting serious doubts upon it (...). In extreme case it seems difficult to deny some force to D'Alembert's objection; and it was with reference to extreme cases that he himself has raised it.
(...) But if doubts as to the sufficiency of the conception of mathematical expectation be sustained, it is not likely that the solution will lie, as D'Alembert suggests, and as has been exemplified above, in the discovery of some more complicated function of the probability wherewith to compound the proposed good."
J.M. Keynes, *A Treatise on Probability*, p. 315

First proposal: three fields of application of probability

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3. Speculative (physics, history) and practical science (medicine, law) → highly complex matters, it is impossible to achieve demonstrations. Probability is nevertheless necessary.

A pessimistic conclusion

> *I did not believe it necessary, as he [Bernoulli] did, to build grand calculations upon vague hypotheses, in a matter concernig human life.*

D'Alembert, *Onzième mémoire, Sur l'application du calcul des probabilités à l'inoculation de la petite vérole*

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- Gap between "moral" experience and probabilistic analysis
- Gap between "sure methods" of computing lifetime the effects and operating principles of smallpox and inoculation, which are not fully explained
- Incomplete and inaccurate data

"Thus even if we know the degree of advantage which might be obtained from each of a series of alternative courses of actions and know also the probability in each case of obtaining the advantage in question, it is not always possible by a mere process of arithmetic to determine which of the alternatives ought to be chosen. If, therefore, the question of right action is under all circumstances a determinate problem, it must be in virtue of an intuitive judgment directed to the situation as a whole, and not in virtue of an arithmetical deduction derived from a series of separate judgments directed to the individual alternatives each treated in isolation."
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It is worse to cause harm by action than by inaction.

Do not sacrifice a certain present good in the hope of a larger uncertain future good.

D'Alembert

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F. Knight, *Risk, Uncertainty and Profit*, III.VIII.9

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Models and uncertainty

making assumptions, stipulating uniformities and invariances, abstracting from the features which make a problem an isolated event $\rightarrow$ reducing *uncertainty* to *risk*.

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Bernoulli's maxim

In reckoning a probability, we must take into account all the information which we have.