Huygens' analysis of expectation

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The problem of points

A and B are playing a game. They decided that the first player who wins three rounds will win the game. Assume that A has already won two rounds and B only one. If the game is interrupted at this stage, how should A and B divide the stake m?

De rationciniis in ludo aleae

- 1655: Huygens travelled in France;

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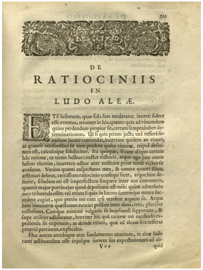

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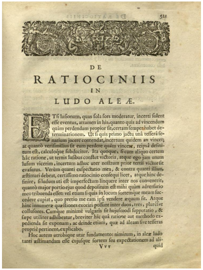
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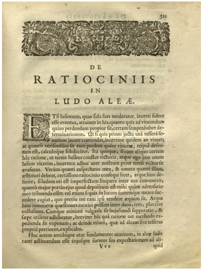

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- 1692 and 1714: Two translations in English of the treatise appeared.

The definition of expectation

I take as fundamental for such games that the chance or expectation[expectatio] to gain something is worth so much that, if one had it, one could again get the same chance in a fair game [aequa conditione certans]. Suppose, for exemple, that somebody has 3 shillings in his one hand and 7 in the other and that I'm asked to choose between them; this is so much worth for me as if I had 5 shillings for certain. Because if I have 5 shillings I can establish a fair game in which I have an even chance of getting 3 or 7 shillings.

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- The money another person has to pay to A if he wants to take her place in the game;

The first two propositions

Proposition I.

If I expect a or b, and I have an equal *chance* of gaining either of them, my expectation is worth $\frac{a+b}{2}$.

Proposition II.

If I expect a, b or c, and each of them is equally likely to happen, my expectation is worth $\frac{a+b+c}{3}$.

An exemple of Huygens' approach: Proposition III

If the number of cases of getting a is p, and the number of cases of getting b is q, assuming that all the cases are equally possible: my expectation will then be worth $\frac{pa+qb}{p+q}$.

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- With $p - 1$ of them, A *individually* makes the agreement that, if A wins, he will pay a, if he looses, he will receive a.
- With the last q players, he *individually* makes the same agreement, but substituing a with b.

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If x is the money paid by each player to take part in G^*, there are three possible outcomes:

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Putting a equal to A's gain in the third case, Huygens obtains the result stated above.

Huygens' solution of the problem of points

There are two possible outcomes for A in the next round: either A wins (so he wins the stake m) or B wins, and so both A and B have the same expectation to win m, which is, for proposition I, $\frac{1}{2}m$. Applying proposition I to this two cases, A's expectation is $\frac{3}{4}m$. By the definition of exepectation, A must receive $\frac{3}{4}$ of the stake.

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3. Game with "cyclic expectations".

Cyclic expectations: Proposition XIV

If I and another person play alternately with two dice on this condition, that I win if I throw 7, but he wins if he throws 6 first, and if I concede to him the first throw, what is the ratio of my chance to his?

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Solving the linear system in two equations, Huygens finds that x is worth $\frac{30}{61}a$, so that "the proportion of my *chance* to his *chance* is as 31 to 30"

The five problems of Huygens'treatise

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Huygens added two other problems, but he left them without solution.

Problem II

Three players, A, B and C, having 12 chips of which four are white and eight black, play on the condition that the first blindfolded player to draw a white chip wins, and that A draws first, B next and then C, then A again, and so on. The question is: What are the ratios of the chances to each other?

Part I of J. Bernoulli's *Ars Conjectandi*


JACOBI BERNOULLI,
ARS CONJECTANDI,
OPUS POSTHUMUM.
TRACTATUS
DE SERIEBUS INFINITIS,
DE LUDO PILÆ
RETICULARIS.
BASILEÆ,
Impensis THURNISIORUM, Fratrum.

In part I of his *Ars Conjectandi*, called *Pars prima, complectens Tractatum Hugenii de Ratiociniis in Ludo Aleae, cum Annotationibus Jacobi Bernoulli*, Bernoulli wrote a detailed commentary to Huygens'treatise.

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At the end of this part, he also discusses the solution of the five problems, throught two different methods, an *analytic* one, based on Huygens' work, and a *syntetic* one.

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1. A, B and C draw from the same box;
2. A, B and C draw from three boxes;
3. The players draw a chip and put it back in the box;
4. The players draw a chip and don't put it back in the box.

Bernoulli on proposition XIV

Instead of two players, Bernoulli imagines an infinite list of players who can draw only once. The players in an even place win if they draw 7, the players in a odd place win if they draw 6.

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Similarly, the probability of winning for a player in the place $2n$ is:

$$\frac{c^n e f^{n-1}}{a^{2n}} = \frac{e}{f}\left(\frac{cf}{a^2}\right)^n$$

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and:

$$\sum_{n=1}^{\infty} \frac{e}{f} \left(\frac{cf}{a^2} \right)^n = \frac{ce}{a^2 - cf}$$