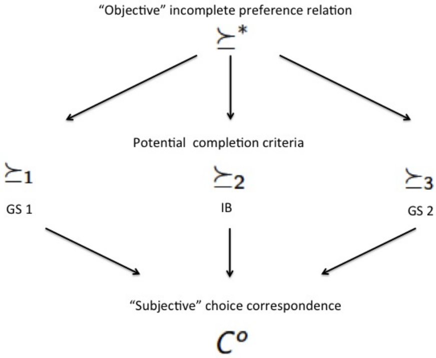
From Preferences to Choice: a Completion Approach

Francesco Girlanda

Pisa, July 2013

- In a world characterized by an "objective" incomplete preference relation, we want to study the mental process that lead to the choices that will be eventually made
- The Decision Maker considers several "potential" completion criteria and he aggregates them in a "subjective" choice correspondence
- The attitude of the Decision Maker toward the criteria will influence the aggregation process



Related Literature

- **Gilboa, Maccheroni, Marinacci and Schmeidler (2010)**: model with 2 preference relations that can be considered a bridge between a representation à la **Bewley (2002)** and à la **Gilboa and Schmeidler (1989)**
- **Crès, Gilboa and Vieille (2011)**: model in which there are several experts that adopt the Decision Maker's utility function in providing their advice in a situation of uncertainty. The Decision Maker aggregates experts' opinions in such a way that the decision maker's valuation of an act is the minimal weighted valuation over all weights vectors in a set of probability vectors over the experts
- **Maccheroni, Marinacci and Rustichini (2006)**: characterization of variational representation of preferences

Framework (1/2)

- Anscombe and Aumann (1963) model
- L is the set of finite support probability distributions over the set of outcomes X
- S is the set of states of the world and it is endowed with an algebra of events Σ
- $\triangle(\Sigma)$ is the set of finitely additive probabilities on Σ
- $\mathscr{F}$ is the set of acts and it consists of all simple measurable functions $f : S \to L$

Framework (2/2)

- $\mathscr{F}_c$ the set of constant acts
- $\mathfrak{I}$ the set of all non empty finite subsets of $\mathscr{F}$
- $\succeq^*$ is a preference relation representing "objective rationality"
- $\{\succeq_i\}_{i=1}^N$ are "subjective" preference relations representing potential completion criteria
- $C^o : \mathfrak{I} \to \mathfrak{I}$ represents the choices effectively implemented and it is the result of the aggregation of the criteria $\{\succeq_i\}_{i=1}^N$

A preference relation $\succeq$ is **Bewley** if it satisfies the following axioms:

- **reflexivity** For any $f \in \mathscr{F}$ we have that $f \succeq f$
- **transitivity** If $f, g, h \in \mathscr{F}$ $f \succeq g$ and $g \succeq h$ then $f \succeq h$
- **non degeneracy** there are $f, g \in \mathscr{F}$ such that $f \succ g$
- **monotonicity** For every $f, g \in \mathscr{F}$, if $f(s) \succeq g(s)$ for all $s \in S$ implies $f \succeq g$
- **continuity** For all $f, g, h, \in \mathscr{F}$ the sets $\{\lambda \in [0,1] : \lambda \cdot f + (1-\lambda) \cdot g \succeq h\}$ and $\{\lambda \in [0,1] : h \succeq \lambda \cdot f + (1-\lambda) \cdot g\}$ are closed in $[0,1]$
- **c-completeness**: If $\forall f, g, \in \mathscr{F}_c$ either $f \succeq g$ or $g \succeq f$
- **independence** For every $f, g, h \in \mathscr{F}$ and $\alpha \in (0,1)$ $f \succeq g$ if and only if $\alpha \cdot f + (1-\alpha) \cdot h \succeq \alpha \cdot g + (1-\alpha) \cdot h$

Axioms (2/5)

A preference relation $\succeq$ is **Invariant Biseparable** if it satisfies the following axioms:

- **reflexivity** For any $f \in \mathscr{F}$ we have that $f \succeq f$
- **transitivity** If $f, g, h \in \mathscr{F}$ $f \succeq g$ and $g \succeq h$ then $f \succeq h$
- **non degeneracy** there are $f, g \in \mathscr{F}$ such that $f \succ g$
- **monotonicity** For every $f, g \in \mathscr{F}$, if $f(s) \succeq g(s)$ for all $s \in S$ implies $f \succeq g$
- **continuity** For all $f, g, h, \in \mathscr{F}$ the sets $\{\lambda \in [0,1] : \lambda \cdot f + (1-\lambda) \cdot g \succeq h\}$ and $\{\lambda \in [0,1] : h \succeq \lambda \cdot f + (1-\lambda) \cdot g\}$ are closed in $[0,1]$
- **completeness:** For all $f, g \in \mathscr{F}$ either $f \succeq g$ or $g \succeq f$
- **c-independence** For every $f, g \in \mathscr{F}$, $h \in \mathscr{F}_c$ and $\alpha \in (0,1)$ $f \succeq g$ if and only if $\alpha \cdot f + (1-\alpha) \cdot h \succeq \alpha \cdot g + (1-\alpha) \cdot h$

A choice correspondence C^o is **Invariant Biseparable** if it satisfies the following axioms:

- **WARP** If $A, B \in \mathfrak{I}$ are such that $B \subseteq A$ and $C^o(A) \cap B \neq \emptyset$ then $C^o(B) = C^o(A) \cap B$;
- **non degeneracy** there are $f, g \in \mathscr{F}$ such that $f = C^o(\{f, g\})$;
- **monotonicity** For every $f, g \in \mathscr{F}$, if $f(s) \in C^o(\{f(s), g(s)\})$ for all $s \in S$ implies $f \in C^o(\{f, g\})$;
- **continuity** For any $f, g, h \in \mathscr{F}$ the sets $\{\lambda \in [0,1] : \lambda \cdot f + (1-\lambda) \cdot g \in C^o(\{\lambda \cdot f + (1-\lambda) \cdot g, h\})\}$ and $\{\lambda \in [0,1] : h \in C^o(\{\lambda \cdot f + (1-\lambda) \cdot g, h\})\}$ are closed in $[0,1]$;
- **c-independence** For every $A \in \mathfrak{I}$, $h \in \mathscr{F}_c$ and $\alpha \in (0,1)$ $C^o(\alpha \cdot A + (1-\alpha) \cdot h) = \alpha \cdot C^o(A) + (1-\alpha) \cdot h$

Axioms (4/5)

Two preference relations $\succeq^*$ and $\succeq$ can satisfy the following axioms:

- **Consistency** $f \succeq^* g$ implies $f \succeq g$;
- **Caution** For $f \in \mathscr{F}$ and $g \in \mathscr{F}_c$, $f \not\succeq^* g$ implies $g \succeq f$

The "subjective" choice correspondence $C^o : \mathfrak{I} \rightarrow \mathfrak{I}$ and the potential completion criteria $\{\succeq_i\}_{i=1}^N$ can be related by the following axioms:

- **Consistency Toward Criteria** If $f \succeq_i g$ for $i = 1, \ldots, N$ implies that $f \in C^o(\{f, g\})$
- **Caution Toward Criteria** For $f \in \mathscr{F}$ and $g \in \mathscr{F}_c$ if $\exists i \in \{1, 2, \ldots, N\}$ such that, $f \not\succeq_i g$ implies $g \in C^o(\{f, g\})$

- For each act $f \in \mathscr{F}$ we denote with $c_i^f \in \mathscr{F}_c$ the certainty equivalent of the act f with respect to the preference relation $\succeq_i$
- The certainty equivalent of act $f \in \mathscr{F}$ with respect to the choice correspondence $C^o : \mathfrak{I} \to \mathfrak{I}$ is defined as the constant act $c_o^f \in \mathscr{F}_c$ such that we have both $c_o^f \in C^o\left(\left\{f, c_o^f\right\}\right)$ and $f \in C^o\left(\left\{f, c_o^f\right\}\right)$
- **Criteria Uncertainty Aversion (CUA)** For every act $f \in \mathscr{F}$, $f_j \in \mathscr{F}$ $j = 1, \dots J$, and every number $\alpha_j \geq 0$ such that $\sum_{j=1}^{J} \alpha_j = 1$, if $f \succeq_i \sum_{j=1}^{J} \alpha_j \cdot c_i^{f_j}$ for $i = 1, \dots, N$ then $f \in C^o\left(\left\{f, \sum_{j=1}^{J} \alpha_j \cdot c_o^{f_j}\right\}\right)$

Lemma 1. $\succeq^*$ is **Bewley**; $\{\succeq_i\}_{i=1}^N$ and C^o are **invariant biseparable**; $\{\succeq_i\}_{i=1}^N$ are **consistent** with respect to $\succeq^*$; C^o satisfies **criteria uncertainty aversion**. Under these assumptions there exists a nonempty closed and convex set $\mathscr{C}$ of probabilities on Σ, a nonconstant function $u : X \to \mathbb{R}$, several monotonic, constant additive and positively homogenous linear functionals $\{I_i : B_0(\Sigma) \to \mathbb{R}\}_{i=1}^N$ and $I_o : B_0(\Sigma) \to \mathbb{R}$ and a closed and convex set $\Gamma \subseteq \triangle(\{1, 2, \ldots, N\})$ such that for every $f, g \in \mathscr{F}$ and $A \in \mathfrak{I}$ we have that:

$$f \succeq^* g \;\Leftrightarrow \int_S E_{f(s)} u \cdot dp(s) \geq \int_S E_{g(s)} u \cdot dp(s) \;\forall p \in \mathscr{C}$$

$$f \succeq_i g \;\Leftrightarrow I_i(E_f u) \geq I_i(E_g u) \; for \; i = 1, \ldots, N$$

$$C^o(A) = \underset{f \in A}{argmax} \{I_o(f)\} = \underset{f \in A}{argmax} \left\{ \underset{\gamma \in \Gamma}{min} \sum_{j=1}^N \gamma_j \cdot I_j(f) \right\}$$

Moreover, in this case, $\mathscr{C}$ is unique and u is unique up to positive affine transformations.

Proof of Lemma 1. (Sketch)

- Step1: find the functional representations of all the preference relations and of the choice correspondence
- Step 2: notice that all the subjective preference relations are consistent with $\succeq^*$ and we have that $u^* = u^o = u^1 = \cdots = u^N := u$
- Step 3: By consistency we have that $\succeq^* \subseteq \succeq_i$ and Proposition A.1 of GMM(2004) delivers $\mathscr{C}^i \subseteq \mathscr{C}^*$. Hence we have that for any $f \in \mathscr{F}$:

$$\min_{p \in \mathscr{C}^*} \int_S E_{f(s)} u \cdot dp(s) \leq \min_{p \in \mathscr{C}^i} \int_S E_{f(s)} u \cdot dp(s) \leq I_i(E_f u)$$

- Step 4: denote with $R = R(I)$ the range of the vector $I = (I_1(\cdot), \ldots, I_N(\cdot))$and show that that there exists a function $\phi : R \to \mathbb{R}$ such that for each $f \in \mathscr{F}$ we have that $I_o(f) = \phi(I(f))$
- Step 5: extend ϕ by successive steps to $\mathbb{R}^{\mathbb{N}}$ retaining monotonicity, concavity, positive homogeneity and constant additivity
- Step 6: by an application of the supporting hyperplane theorem we have that $\phi(x) = \min_{\gamma \in \Gamma} \sum_{j=1}^{N} \gamma_j \cdot x_j$ for all $x \in R$

Corollary. Under the assumptions of Lemma 1, if the there exists $\bar{i} \in \{1, \ldots, N\}$ such that $\succeq_{\bar{i}}$ satisfies cautiousness with respect to $\succeq^*$ and the standard vector $e_{\bar{i}} \in \Gamma \subseteq \triangle(\{1, 2, \ldots, N\})$, where $e_{\bar{i}}$ is the standard vector of $\mathbb{R}^{\mathbb{N}}$ that assigns probability 1 to the element $\bar{i}$, then we have that the following holds:

$$f \succeq^* g \;\Leftrightarrow \int_S E_{f(s)} u \cdot dp(s) \geq \int_S E_{g(s)} u \cdot dp(s) \;\forall p \in \mathscr{C}$$

$$C^o(A) = \underset{f \in A}{argmax} \left\{ \min_{p \in \mathscr{C}} \int_S E_{f(s)} u \cdot dp(s) \right\}$$

Moreover, in this case, $\mathscr{C}$ is unique and u is unique up to positive affine transformations.

- **CUA** alone is not enough for having a GMMS(2010) representation result even if at least one of the potential criteria satisfies **caution**, in fact it is necessary that our decision maker consider possible to use only the cautious completion criteria

Lemma 2. $\succeq^*$ is **Bewley**; $\{\succeq_i\}_{i=1}^N$ and C^o are **invariant biseparable**; $\{\succeq_i\}_{i=1}^N$ are **consistent** with respect to $\succeq^*$; C^o satisfy **consistency toward criteria** and **caution toward criteria**. If there exists $\bar{i} \in \{1,\ldots,N\}$ such that $\succeq_{\bar{i}}$ satisfies **caution** with respect to $\succeq^*$ then we have that there exists a nonempty closed and convex set $\mathscr{C}$ of probabilities on Σ and a nonconstant function $u : X \to \mathbb{R}$ such that:

$$f \succeq^* g \;\Leftrightarrow \int_S E_{f(s)} u \cdot dp(s) \geq \int_S E_{g(s)} u \cdot dp(s) \;\forall p \in \mathscr{C}$$

$$C^o(A) = \underset{f \in A}{argmax} \left\{ \underset{p \in \mathscr{C}}{min} \int_S E_{f(s)} u \cdot dp(s) \right\}$$

Moreover, in this case, $\mathscr{C}$ is unique and u is unique up to positive affine transformations.

Lemma 2 (2/2)

Proof of Lemma 2. (Sketch) It is possible to show that there exists $\mathscr{C}^*$ and a unique u such that for any act $f \in \mathscr{F}$:

$$I_o(E_f u) \geq \min_{p \in \mathscr{C}^*} \int_S E_{f(s)} u \cdot dp(s)$$

If there exists $\bar{i}$ such that $\succeq_{\bar{i}}$ satisfies caution then by Theorem 3 of GMMS(2010) we have that:

$$I_o(E_f u) \geq \min_{p \in \mathscr{C}^*} \int_S E_{f(s)} u \cdot dp(s) = \min_{p \in \mathscr{C}^{\bar{i}}} \int_S E_{f(s)} u \cdot dp(s) = I_{\bar{i}}(E_f u)$$

Suppose by contra that $I_o(E_f u) > \min_{p \in \mathscr{C}^*} \int_S E_{f(s)} u \cdot dp(s)$, then it is possible to find a constant act $g \in \mathscr{F}_c$ such that the following holds:

$$I_o(E_f u) > u(g) > \min_{p \in \mathscr{C}^*} \int_S E_{f(s)} u \cdot dp(s) = I_{\bar{i}}(E_f u)$$

But this latter inequality contradicts **Caution Toward Criteria** because for $\bar{i}$ we have that $f \not\succeq_{\bar{i}} g$ but $f \in C^o(\{f, g\})$.
Q.E.D.

Conclusions

- We constructed a framework in which it is possible to model and study the mental aggregation process that lead to the choice of a particular completion criteria
- We showed that only a really cautious decision maker will satisfy **GMMS (2010)** representation theorem

Work in Progress:

- Find a set of axioms that lead to a representation of the type $C^{o}(A) = \underset{f \in A}{argmax} \left\{ \underset{\gamma \in \Gamma}{min} \left[\sum_{j=1}^{N} \gamma_j \cdot I_j(f) + c(\gamma) \right] \right\}$ in order to model the idea that a decision maker could be biased toward some potential completion criteria

Thank you.

Bibliography

Simone Cerreia-Vioglio, Fabio Maccheroni, Massimo Marinacci, and Aldo Rustichini.
Niveloids and their extensions: Risk measures on small domains.
Technical report, Igier Working Paper n.458, 2012.

Hervé Crès, Itzhak Gilboa, and Nicolas Vieille.
Aggregation of multiple prior opinions.
Journal of Economic Theory, 146(6):2563–2582, 2011.

Paolo Ghirardato, Fabio Maccheroni, and Massimo Marinacci.
Differentiating ambiguity and ambiguity attitude.
Journal of Economic Theory, 118(2):133–173, 2004.

Itzhak Gilboa, Fabio Maccheroni, Massimo Marinacci, and David Schmeidler.
Objective and subjective rationality in a multiple prior model.
Econometrica, 78(2):755–770, 2010.

Itzhak Gilboa and David Schmeidler.
Maxmin expected utility with non-unique prior.
Journal of Mathematical Economics, 18(2):141–153, 1989.