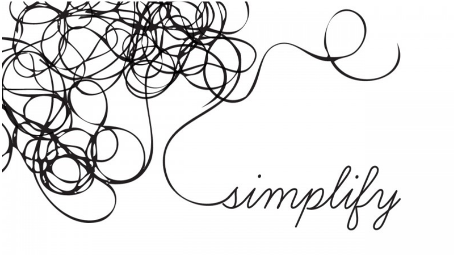
Mathematical Abstraction: Turing's Analysis of Computation

Rossella Marrano

Scuola Normale Superiore

The problem: mathematical modelling



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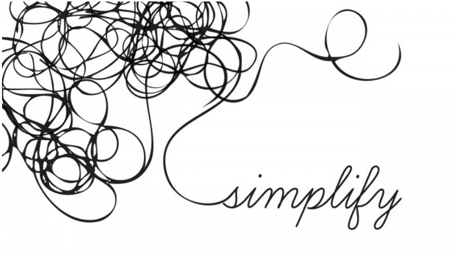

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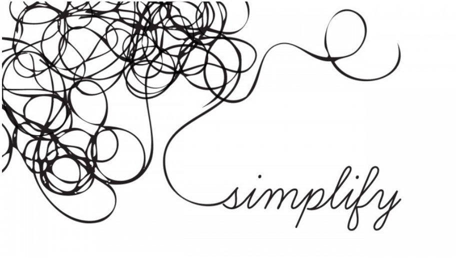

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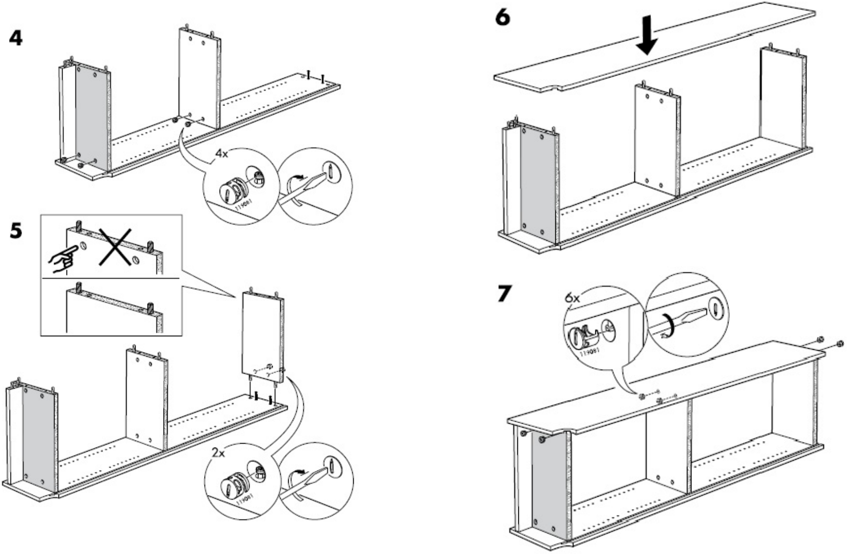
Case-study

Turing's analysis of computation

- mathematical abstraction
- role of the agent

Algorithm (we all do it)



Algorithm (mathematicians have always been doing)

input: two positive integers a and b with $a \geq b$
output: greatest common divisor $GCD(a, b)$

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input: two positive integers a and b with $a \geq b$
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method:

Divide a by b:

$$a = q_1 b + r_1$$

if $r_1 > 0$, divide it by b:

$$b = q_2 r_1 + r_2$$

if $r_2 > 0$, keep dividing:

$$r_1 = q_3 r_2 + r_3$$

$$(\dots)$$

$$r_n = q_{n+2} r_{n+1} + r_{n+2}$$

Continue until $r_k = 0$

The last non-zero remainder r_{k-1} is $GCD(a, b)$.

2000 years later: the confluence of ideas

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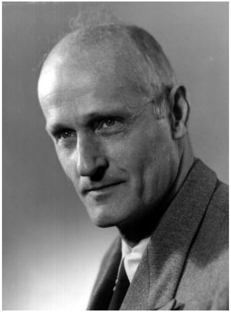
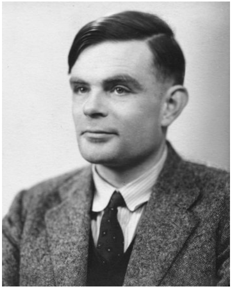
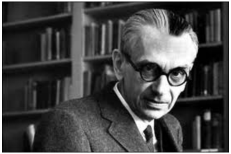
Alonzo Church
(1903-1995)



Stephen Kleene
(1909-1994)



Alan M. Turing
(1912-1954)



Kurt Gödel (1906-1978)

Turing's analysis

> If one is given a puzzle to solve one will usually, if it proves to be difficult, ask the owner whether it can be done. Such a question should have a quite definite answer, yes or no, at any rate provided the rules describing what you are allowed to do are perfectly clear. Of course the owner of the puzzle may not know the answer. One might equally ask, 'How can one tell whether a puzzle is solvable?', but this cannot be answered so straightforwardly. (Turing, 1954)

Turing's problem

Investigate the class of problems that can be effectively solved.

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Central idea

> We may compare a man in the process of computing a real number to a machine [...].

Computing is normally done by writing certain symbols on paper. We may suppose this paper is divided into squares like a child's arithmetic book. In elementary arithmetic the two-dimensional character of the paper is sometimes used. But such a use is always avoidable, and I think that it will be agreed that the two-dimensional character of paper is no essential of computation. I assume then that the computation is carried out on one-dimensional paper, i.e. on a tape divided into squares.

[...]

I shall also suppose that the number of symbols which may be printed is finite.

[...] The behaviour of the computer at any moment is determined by the symbols which he is observing, and his 'state of mind' at that moment. We may suppose that there is a bound B to the number of symbols or squares which the computer can observe at one moment. If he wishes to observe more, he must use successive observations. We will also suppose that the number of states of mind which need to be taken into account is finite.

[...] Let us imagine the operations to be performed by the computer to be split up into 'simple operations' which are so elementary it is not easy to imagine them further divided. Every such operation consists of some change of the physical system consisting of the computer and his tape. We know the state of the system if we know the sequence of symbols on the tape, which of these are observed by the computer (possibly with a special order), and the state of mind of the computer. We may suppose that in a simple operation not more than one symbol is altered. Any other changes can be split up into changes of this kind.

Abstraction hypotheses

I one-dimensional character of the support

II finite number of printed symbols

III finite number of observed symbols

IV finite number of states of mind

V all the performed operations are elementary ones:
- not more than one symbol is altered
- the squares whose symbols are changed are always the 'observed' squares
- changing observed square is one-step process

VI any calculation step is determined only by the computer's state of mind and the observed symbols

VII (implicit) no practical limitations

We may now construct a machine to do the work of this computer. To each state of mind of the computer corresponds an 'm-configuration' of the machine. The machine scans B squares corresponding to the B squares observed by the computer.

[...]

The move which is done, and the succeeding configuration, are determined by the scanned symbol and the m-configuration.

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- paper

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- tape

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Human computer

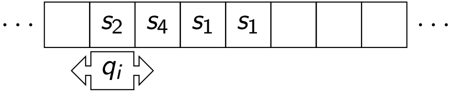
- paper
- mental states
- observation

Machine

- tape
- m-configurations
- scanning

Turing Machine

- $q_0, \ldots, q_n$
- $s_1, \ldots, s_m$
- indefinitely extensible tape
- read-write head



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  - move to the right (R)
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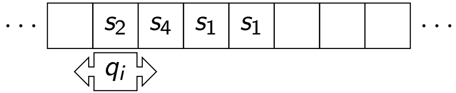

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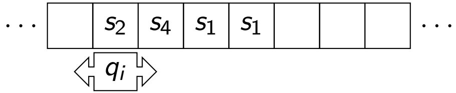

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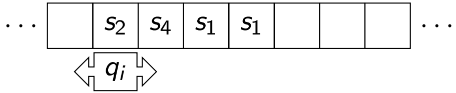

List of instructions

⟨ state, symbol, symbol to write or head action, next state ⟩

$\langle q_i, s_k, s_m, q_j \rangle$

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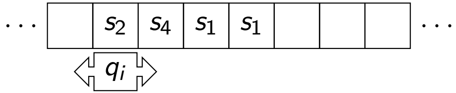

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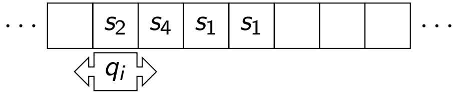

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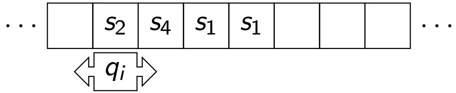

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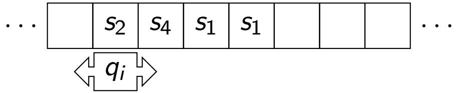

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There are T-unsolvable problems

There are T-solvable but unfeasible in practice problems

Church-Turing Thesis

Fact

All T-solvable problems are effectively solvable.

Church-Turing Thesis

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All T-solvable problems are effectively solvable.

> `Logical computing machines can do anything that could be described as "rule of thumb" or "purely mechanical".`
>
> `(Turing, 1948)`

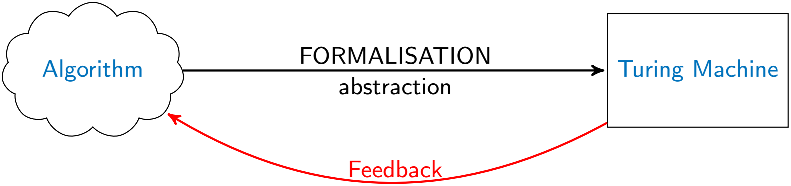
Thesis

All effectively solvable problems are T-solvable.

- strength and solid consensus
- the thesis cannot be proved


Algorithm
Turing Machine


Algorithm
FORMALISATION
abstraction
Turing Machine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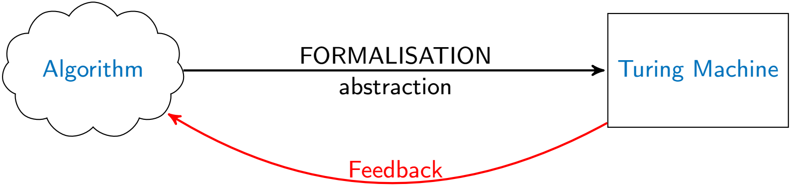
Feedback

Feedback I



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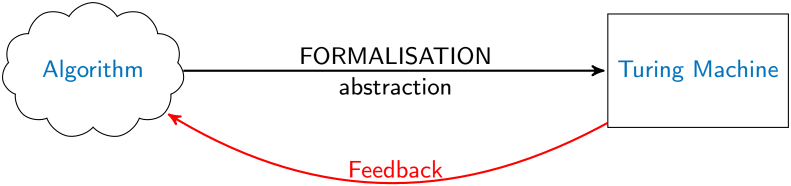

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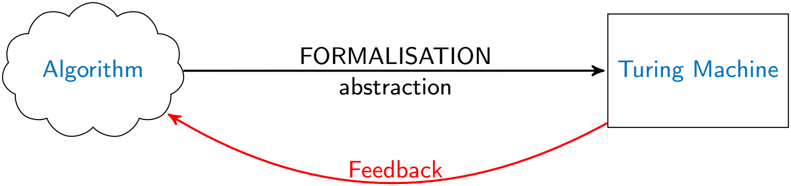

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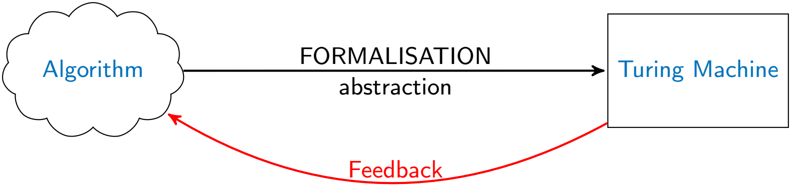

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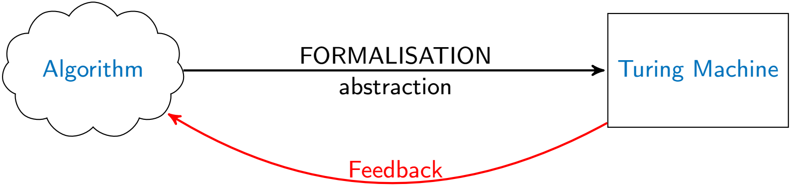
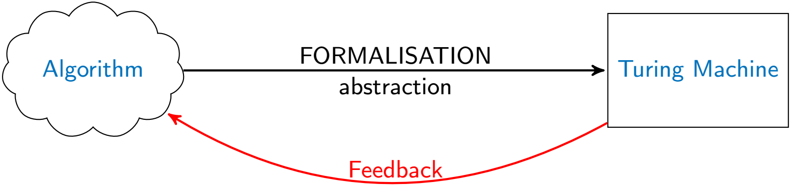
- The 'human nature' of the computer is irrelevant.



Feedback II (CTT)



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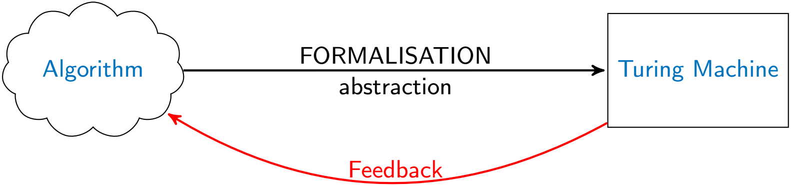

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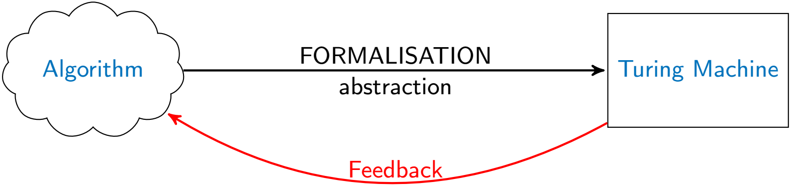

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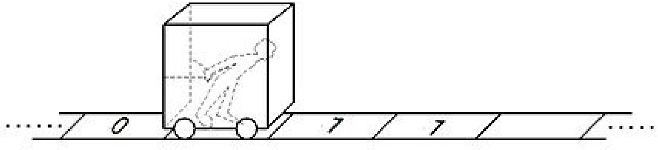
- abstraction hypotheses do not change the nature of the intuitive concept we start with
- mathematical abstraction in this case is costless
- the intuitive notion (sharpened by the act of modelling) is itself a mathematical concept

Analogy VS. Modelling

- Turing doesn't model the human being, he models the intuitive notion of algorithmic procedure



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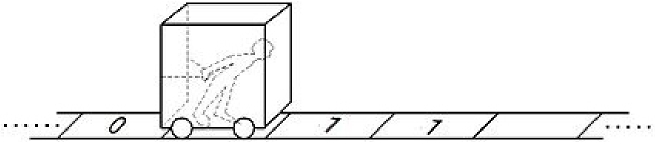

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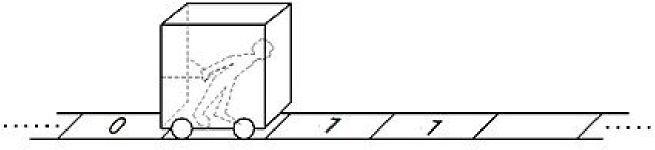

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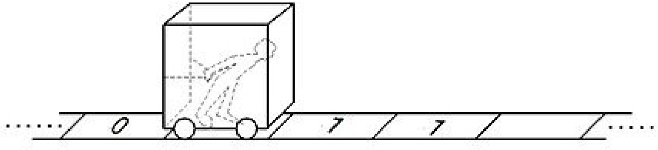

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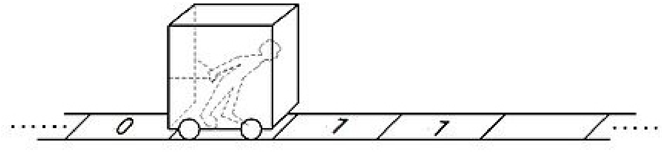
But we can take Turing Machines to be agents again!



TMs as idealised agents

- no resource limitations:
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- no cognitive limitations
- no mistakes
- no interactions with the environment

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Most demanding normative standard for rationality

In principle VS. feasible