Evolution and Market Behavior with Endogenous Investment Rules

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In this paper...

We investigate wealth-driven selection in incomplete asset markets populated by heterogeneous investors without rational expectations

- Which investment rules (beliefs & preferences, ...) does the market select?
- Does it exist a dominant rule?
- Can investment behaviors be “ordered”?
- Is agents’ heterogeneity a persistent property?
- What are the consequences for asset prices?
- Do asset prices reflect the most accurate beliefs?

We provide answers a simple, but rich enough (stochastic, behaviors/rules, equilibrium prices), analytically tractable model by studying the local stability of market selection equilibria.

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The Model: Assets

Discrete time. At each $t \in \mathbb{N}^+$, S states of the world and $\omega_t \in \{1, \ldots, S\}$. Bernoulli scheme $\omega = (\omega_1, \ldots, \omega_t, \ldots)$, $\omega_t = s$ with probability π_s. Ω space of sequences. (stationary, ergodic)

Repeated exchange of K short-lived assets in exogenous unitary supply ($\star$). Asset k pays $D_{k,s}$ units of the numeráire good at time t if $\omega_t = s$. D is the dividend matrix, full rank (in particular no zero rows, no zero columns). $P_{k,t}$ is price of asset k at time t.

$$D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, D = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, D = \begin{pmatrix} 1 & 1 \\ 2 & \frac{1}{2} \end{pmatrix}, D = \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 2 \end{pmatrix}$$

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The Model: Wealth and Prices

Exogenous supply

I agents buy assets to transfer wealth inter-temporally. Given asset holdings in t, $h^i_{k,t}$, wealth in $t+1$, W^i_{t+1}, is

$$W^i_{t+1} = \sum_{k=1}^{K} h^i_{k,t} D_{k,\omega_{t+1}} \quad i = 1, \ldots, I \tag{1}$$

where Walrasian Market clearing implies

$$\sum_{i=1}^{I} h^i_{k,t} = 1, \quad k = 1, \ldots, K. \tag{2}$$

and asset holdings (plus consumption) satisfy time t budget constraint (δ is saving rate)

$$\sum_{k=1}^{K} h^i_{k,t} P_{k,t} = \delta^i_t W^i_t, \quad i = 1, \ldots, I. \tag{3}$$

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I agents issue and buy securities. Assuming $\sum_k D_{k,s} = 1$ for all s and homogeneous consumption.

$$W_{t+1}^i = \sum_{k=1}^{K} h_{k,t}^i D_{k,\omega_{t+1}} + \delta_t W_t^i = \sum_{k=1}^{K} \left(h_{k,t}^i + \delta_t W_t^i \right) D_{k,\omega_{t+1}} \quad (4)$$

where, naming $\hat{h}_{k,t}^i = h_{k,t}^i + \delta_t W_t^i$, Walrasian Market clearing implies

$$\sum_{i=1}^{I} h_{k,t}^i = 0 \Leftrightarrow \sum_{i=1}^{I} \hat{h}_{k,t}^i = \delta_t \sum_i W_t^i \quad (5)$$

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Prices and Wealth Dynamics

Agent i invests on asset k at time t a fraction $\alpha^i_{k,t}$ of her wealth.

Wealth W^i_{t+1} depends on the realization of the state of the world ω_{t+1}

$$W^i_{t+1} = \sum_k \frac{\alpha^i_{k,t} W^i_t}{P_{k,t}} D_{k,\omega_{t+1}} .$$

Prices $\mathbf{P}_t$ are fixed by Walrasian market clearing

$$1 = \sum_i h^i_t \Leftrightarrow P_{k,t} = \sum_i \alpha^i_{k,t} W^i_t ,$$

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Background Literature

Simple, “exogenous”, rules (wealth fractions depend only on dividend process)

- Blume and Easley (1992) *Evolution and Market Behavior*. Market for Arrow securities and simple rules. Selection rewards log preferences with beliefs ”closest” (relative entropy) to π. Relative entropy defines order relation. (in background Log-optimality: Kelly, 1956; Breiman, 1961). When rules are not simple-counterexamples.
- *Evolutionary Finance* (a survey Evstigneev, Hens, and Schenk-Hoppe’, 2009). Simple rule in an extended framework (long-lived assets, possibly incomplete markets, more general dividend and learning processes). G-Kelly rule, i.e. invest proportionally to expected dividends, is global maximum w.r.t. order relation.

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Non-simple, “endogenous”, rules (wealth fractions depend on dividend and price process)

- Sandroni (2000) and Blume and Easley (2006): general demands, infinite horizon, perfect foresight on prices, dynamically complete markets. Find that Pareto optimality implies that, controlling for discount rates, best beliefs (relative entropy terms) gain all wealth in long run.
- Some finance applications of Heterogeneous Agents Models (Hommes 2006) and Agent Based Computational Economics (LeBaron 2006) study wealth-driven selection of CRRA rules in deterministic/simulation framework. Partial equilibrium framework. Levy, Levy, Solomon (1994, 1995, 2000), Chiarella et al (2001, 2006), Le Baron (2012). Behavioral Finance.

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Our Framework

- Short-lived assets (K)
- Endogenous investment rules (I) (L)
- No perfect foresight (incompleteness)
- Repeated trade in discrete time, temporary equilibrium
- Random Dynamical System (I) $\times$ (K) $\times$ (L)
- Local (and global) stability analysis of long-run market selection equilibria

Today we show that:

- Local stability analysis (hint global)
- Market selection landscape depends on the ecology of traders, no ordering
- Heterogeneity may be persistent (time varying)
- Asset prices may not reflect beliefs of best informed agent
- Never vanishing rule exists

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Investment Rules

Generalized CRRA

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$$\alpha^i_{k,t} = \alpha^i_k(\mathbf{P}_t, \mathbf{P}_{t-1}, ...; D, \pi) \quad k = 1, \ldots, K\,, \tag{7}$$

where $\mathbf{P}_t$ is period t price vector (CRRA included, CARA excluded).

P1 Each agent i consumes in $[0, W^i)$, or $\sum_{k=1}^K \alpha^i_k(\mathbf{P}_t, ...) = \delta^i_t = 1 - \alpha^i_{0,t} \in (0, 1]$;

P2 Portfolios are maximally diversified, or $\sum_{k=1}^K \alpha^i_k(\mathbf{P}, ...)D_{k,s} > 0$ for every s and i.

P3 Demand is strictly positive for zero contemporaneous prices, that is, for every asset k and agent i, $\alpha^i_k(\mathbf{P}_t, ...)/P_{k,t} \to c > 0$ if $P_{k,t} \to 0$.

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Normalized Prices and Wealth Dynamics

Normalizations leads to:

$$\sum_k d_{k,s} = 1, \quad \sum_i w_t^i = 1, \quad \sum_k p_{k,t} = \sum_i (1 - \alpha_{0,t}^i) w_t^i \quad \text{for all } t$$

Normalized inter-temporal budget constraint ω_{t+1}

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$$p_{k,t} = \sum_i \alpha_k^i(\mathbf{p}_t, \ldots; D, \pi) w_t^i .$$

Normalized Prices and Wealth Dynamics

Normalizations leads to:

$$\sum_k d_{k,s} = 1, \quad \sum_i w_t^i = 1, \quad \sum_k p_{k,t} = \sum_i (1 - \alpha_{0,t}^i) w_t^i \quad \text{for all } t$$

Normalized inter-temporal budget constraint ω_{t+1}

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Market Dynamics as a Random Dynamical System

$$(w_{t+1}, p_{t+1}) = \mathcal{F}(\omega_{t+1})(w_t, p_t) = \left[\begin{array}{lcl} w^1_{t+1} & = & \mathcal{W}^1(w_t, p_t; \omega_{t+1}) \\ \vdots & \vdots & \vdots \\ w^I_{t+1} & = & \mathcal{W}^I(w_t, p_t; \omega_{t+1}) \\ p_{1,t+1} & = & f_1(w_t, p_t; \omega_{t+1}) \\ p^1_{1,t+1} & = & p_{1,t} \\ \vdots & \vdots & \vdots \\ p^L_{1,t+1} & = & p^{L-1}_{1,t} \\ \vdots & \vdots & \vdots \\ p_{K,t+1} & = & f_K(w_t, p_t; \omega_{t+1}) \\ p^1_{K,t+1} & = & p_{K,t} \\ \vdots & \vdots & \vdots \\ p^L_{K,t+1} & = & p^{L-1}_{K,t} \end{array}\right.$$

$$(w_{t+1}, p_{t+1}) = \varphi(t+1, \omega, w_0, p_0) = \mathcal{F}(\omega_{t+1}) \circ \ldots \circ \mathcal{F}(\omega_2) \circ \mathcal{F}(\omega_1)(w_0, p_0).$$

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Dominance and Survival

Given (w_0, p_0) and given sequence ω we get trajectories, sequences of wealth fractions $\{w\}$ and prices $\{p\}$, and define

Definition

An agent i is said to survive on a given trajectory generated by the market dynamics if $\limsup_{t\to\infty} w_t^i > 0$ on this trajectory. Otherwise, an agent n is said to vanish on a given trajectory. A surviving agent i is said to dominate on a given trajectory if she is the unique survivor on that trajectory, that is, $\liminf_{t\to\infty} w_t^i = 1$

Survival and dominance can be characterized for subsets of Ω, e.g. almost surely.

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Long-run Market Selection Equilibria

We identify long-run market equilibria as states where w, p, αs are constant.

Technically these are deterministic fixed point of the random dynamical system.

Definition

Consider the stochastic process with elements $\omega \in \Omega$. The state (w^*, p^*) is a deterministic fixed point of the random dynamical system φ generated by the family of maps if, for almost all $\omega \in \Omega$, it holds

$$\varphi(t, \omega, w^*, p^*) = (w^*, p^*)\,, \quad \text{for every} \quad t \tag{8}$$

Survival and dominance at a market selection equilibrium

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Survival and dominance at a market selection equilibrium

Known Results: Simple Rules and Arrow Securities

Blume and Easley (1992) - 2 agents, 2 assets, no consumption

Wealth dynamics:

$$w^i_{t+1} = \begin{cases} \frac{\alpha^i w^i_t}{p_t} & \omega_{t+1} = 1 \\ \frac{(1-\alpha^i) w^i_t}{1-p_t} & \omega_{t+1} = 2 \end{cases},$$

where price (only one asset matters due to constant sum)

$$p_t = \alpha^1 w^1_t + \alpha^2 w^2_t.$$

Price is in between the αs.

Each period the market rewards the agent with a higher stake in the dividend paying asset.

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Simple Rules

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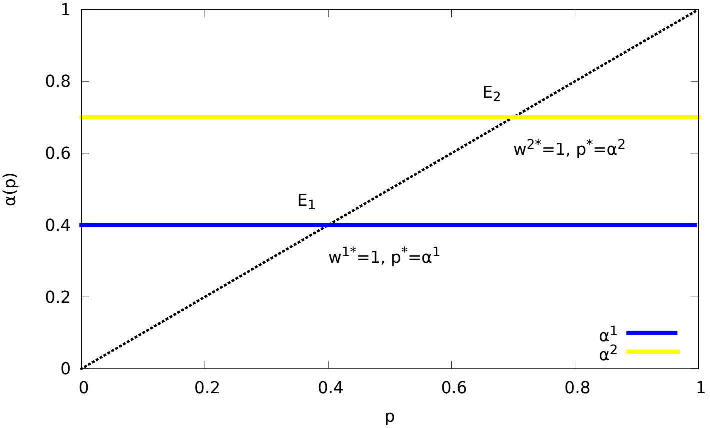

Simple Rules and Arrow Securities

Wealth ratio dynamics

When computing wealth ratios prices simplify away

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Define the Relative Entropy of α w.r.t. to π

$$I_\pi(\alpha^i) = \sum_s \pi_s \log \frac{\pi_s}{\alpha_s^i} \geq 0 \quad \text{then} \quad \frac{1}{T} \log \frac{w_T^1}{w_T^2} \to \left(I_\pi(\alpha^2) - I_\pi(\alpha^1)\right) .$$

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Simple Rules

The random walk view

Define

$$x_t = \log \frac{w_t^1}{w_t^2}$$

then the wealth dynamics is

$$x_{t+1} = x_t + \mu + \epsilon_{t+1}$$

where

$$\mu = I_\pi(\alpha^2) - I_\pi(\alpha^1)$$

and $\{\epsilon\}$ are i.i.d random variables with zero mean and finite variance.

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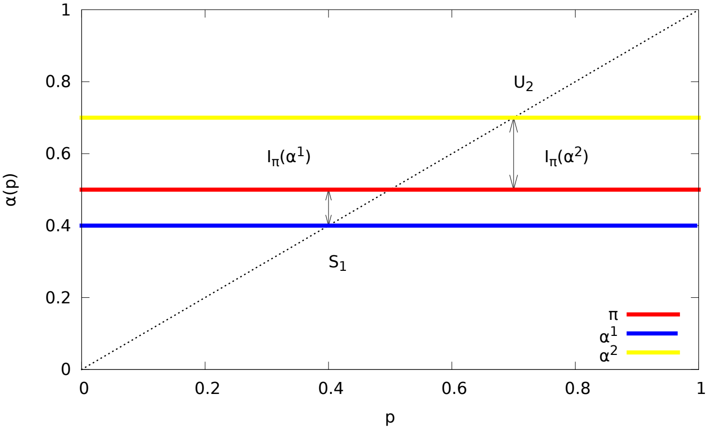
Simple Rules

Wealth selection in a plot



Two consequences:

- 1 No heterogeneity, best informed wins, rules ordered in survivability: $\alpha \succeq \beta$
- 2 The Kelly rule $\alpha_k = \pi_k$ dominates on Ω, $I(\text{Kelly}) = 0$.

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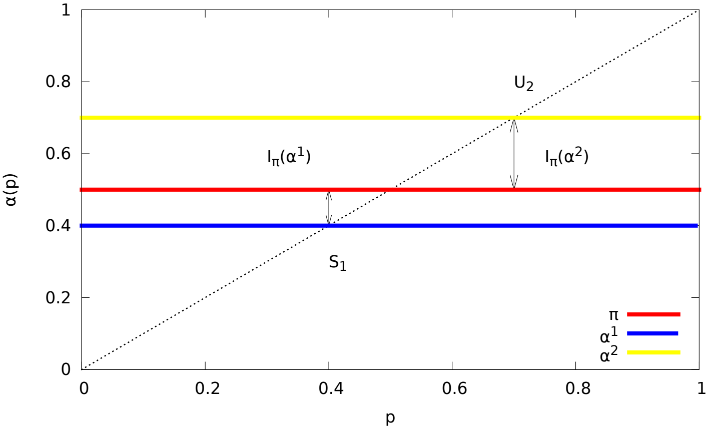

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Wealth dynamics is still:

$$w^i_{t+1} = \begin{cases} \frac{\alpha^i(p_t)w^i_t}{p_t} & \omega_{t+1} = 1 \\ \frac{(1-\alpha^i(p_t))w^i_t}{1-p_t} & \omega_{t+1} = 2 \end{cases}, \tag{9}$$

where $p_t(w_t)$ is the implicit solution of

$$p_t = \alpha^1(p_t)w^1_t + \alpha^2(p_t)w^2_t\,. \tag{10}$$

(if $w^{1*}\partial_p\alpha^1(p^*) + w^{2*}\partial_p\alpha^2(p^*) \neq 1$ OK around (w^*, p^*))

Non-simple Investment Rules

Market dynamics

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Market Selection Equilibria

2 assets, 2 agents

Theorem

Market Selection Equilibria, that is, fixed points of the random dynamical system that corresponds to the market dynamics, are given by

$$\begin{aligned} w^* &= (1,0) \\ w^* &= (0,1)\,, \end{aligned}$$

which corresponds to single survivor equilibria of $i = 1, 2$ respectively and where $p^* = \alpha^i(p^*)$, or

$$w^* = (w^{1*}, 1 - w^{1*}) \quad w^{1*} \in (0,1)$$

iff $\alpha^1(p(w^*)) = \alpha^2(p(w^*)) = p(w^*) = p^*$, which corresponds to multiple survivor equilibria

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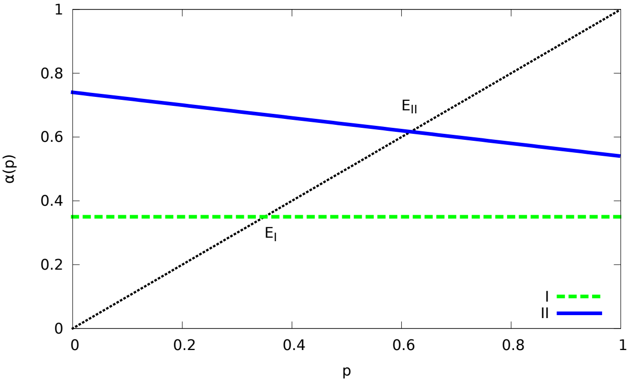
2 agents, 2 assets, non-simple investment rules

Market equilibria in a plot

$$p_t = \alpha^1(p_t)\, w_t + \alpha^2(p_t)(1 - w_t)$$



Non-simple Investment Rules

Selection

Overall we can compute

$$\frac{w^1_{t+1}}{w^2_{t+1}} = \begin{cases} \frac{\alpha^1(p_t)}{\alpha^2(p_t)} \frac{w^1_t}{w^2_t} & \omega_{t+1} = 1 \\ \\ \frac{1-\alpha^1(p_t)}{1-\alpha^2(p_t)} \frac{w^1_t}{w^2_t} & \omega_{t+1} = 2 \end{cases}$$

Now, in T periods the ratio $\frac{w^1_T}{w^2_T}$ depends on the price history.

Non-simple Investment Rules

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Non-simple Investment Rules

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Definition

A deterministic fixed point (w^*, p^*) of the random dynamical system $\varphi(t, \omega, w, p)$ is called asymptotically stable if, for almost all $\omega \in \Omega$ and there exists $U(\omega)$ of (w^*, p^*) such that for all (w, p) in $U(\omega)$ $\lim_{t\to\infty} ||\varphi(t, \omega, w, p) - (w^*, p^*)|| \to 0$.

Local Stability of Market Selection Equilibria

Definition

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A deterministic fixed point (w^*, p^*) of the random dynamical system $\varphi(t, \omega, w, p)$ is called asymptotically stable if, for almost all $\omega \in \Omega$ and there exists $U(\omega)$ of (w^*, p^*) such that for all (w, p) in $U(\omega)$ $\lim_{t\to\infty} ||\varphi(t, \omega, w, p) - (w^*, p^*)|| \to 0$.

Theorem

If the eigenvalue of the infinitely iterated map is inside the unit circle then the deterministic fixed point is asymptotically stable. For the fixed point $w^* = (1, 0)$ the eigenvalue is

$$\begin{aligned} \mu &= \left(\frac{\alpha^2(p^*)}{\alpha^1(p^*)}\right)^{\pi} \left(\frac{1-\alpha^2(p^*)}{1-\alpha^1(p^*)}\right)^{1-\pi} \\ &= \exp\Big(I_\pi(\alpha^1(p^*)) - I_\pi(\alpha^2(p^*))\Big) \end{aligned}$$

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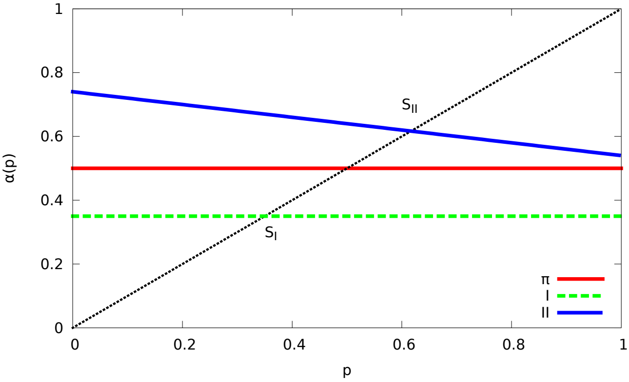
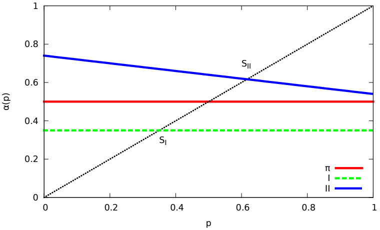
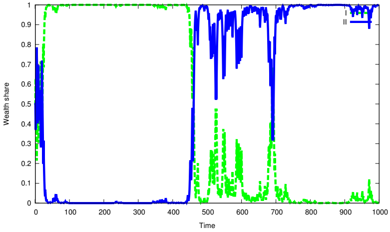
2 agents, 2 assets, non-simple rules

Stability in a plot



Coexistence of Stable (long-run) Equilibria





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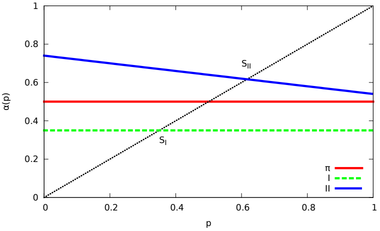

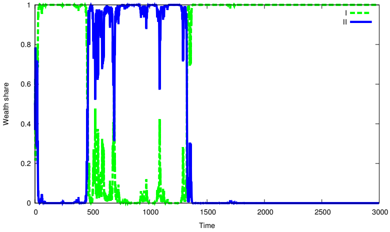

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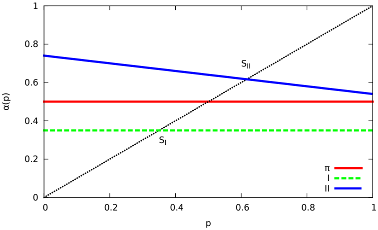

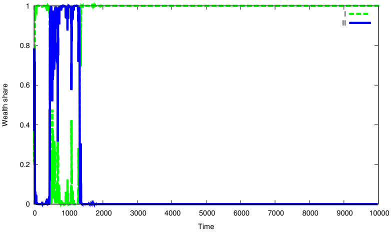

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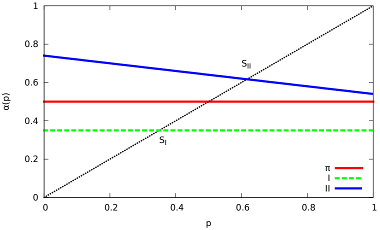

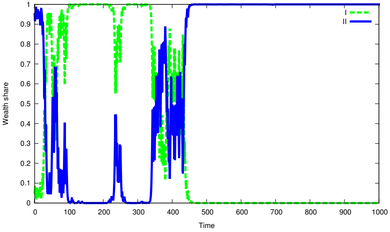

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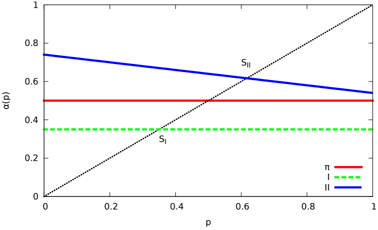

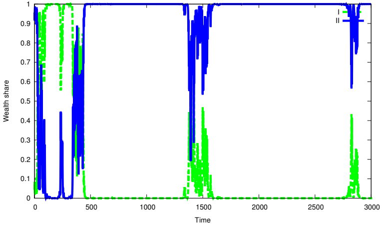

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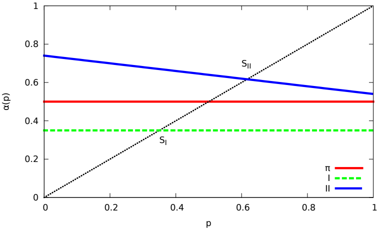
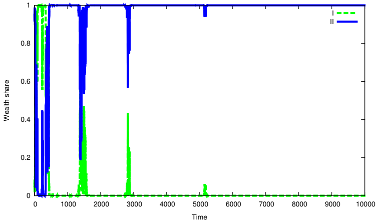
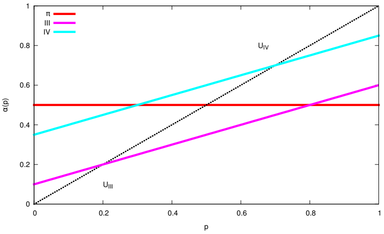
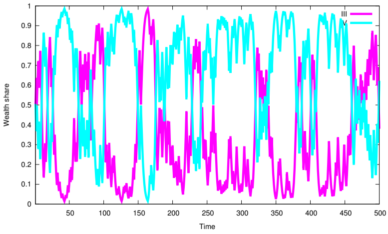




Multiple Unstable (long-run) Equilibria





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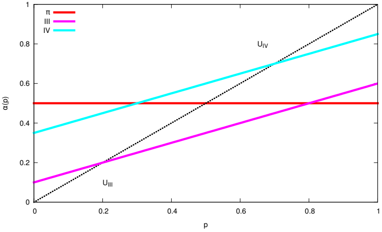
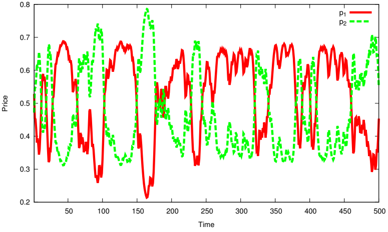
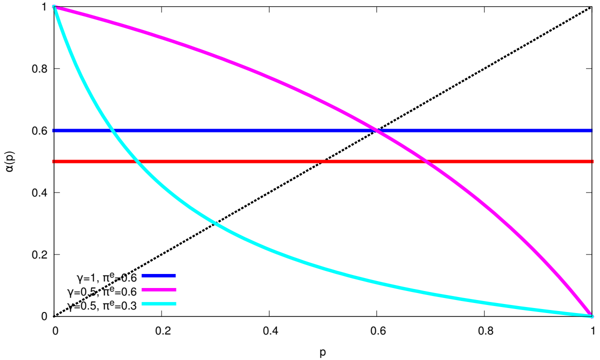




CRRA and no Aggregate Risk



$U = \sum_{\omega^T \in \Omega^T} \pi^e(\omega^T) u(w_T)$, with $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$.

Max U is solved by $\alpha(p;\pi^e,\gamma) = \dfrac{\left(\pi^e/p^{1-\gamma}\right)^{\frac{1}{\gamma}}}{\left(\pi^e/p^{1-\gamma}\right)^{\frac{1}{\gamma}} + \left((1-\pi^e)/(1-p)^{1-\gamma}\right)^{\frac{1}{\gamma}}}$

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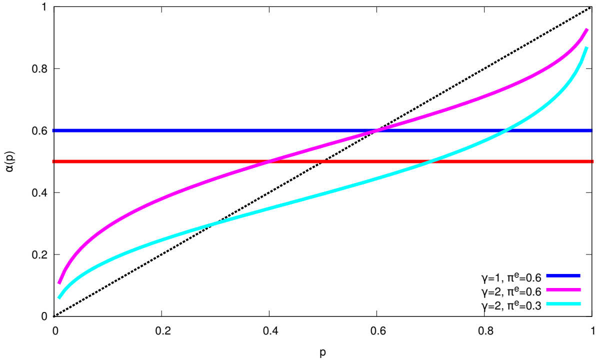

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An Order Relation on Rules

Given an asset market and two rules α and β define

$$\alpha \succeq \beta$$

iff α almost never vanishes when trading with β, and

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iff α dominates β.

Simple rules: complete, transitive
Non-simple rules: non-complete, non-transitive

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Possible Problems with Ordering

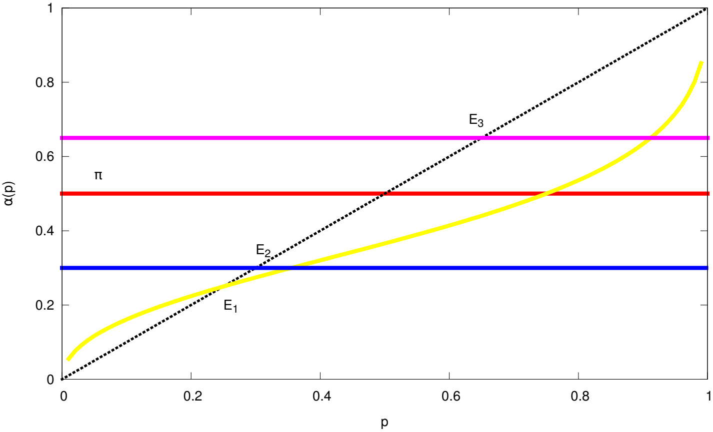
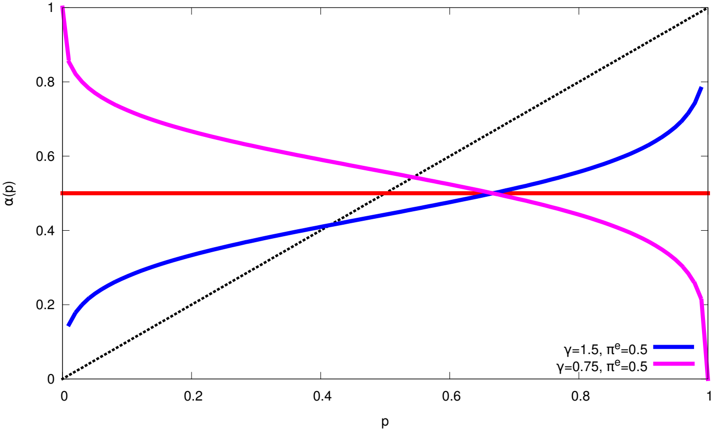


Figure: $\alpha^3 \succ \alpha^2$, $\alpha^2 \succ \alpha^1$, $\alpha^3 \sim \alpha^1$.

Aggregate Risk



$$\text{If } D = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}, \quad D_1 > D_2 \text{ then}$$

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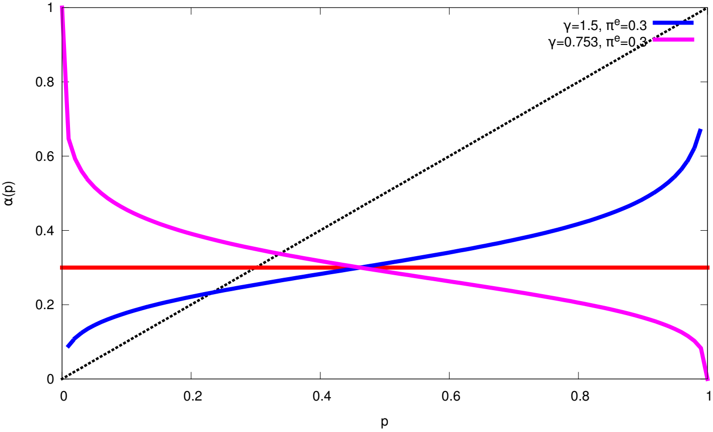

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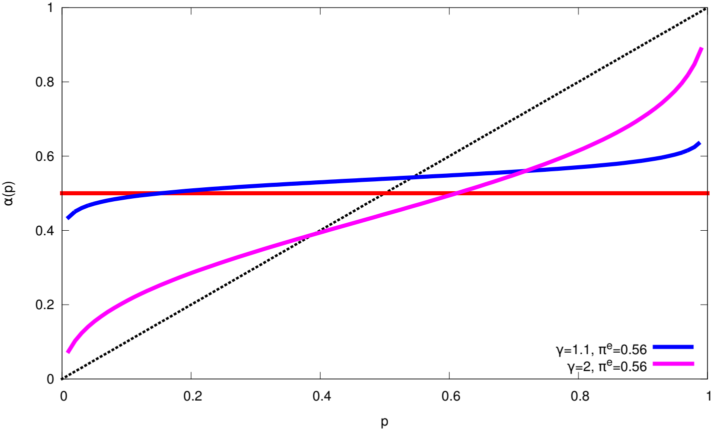

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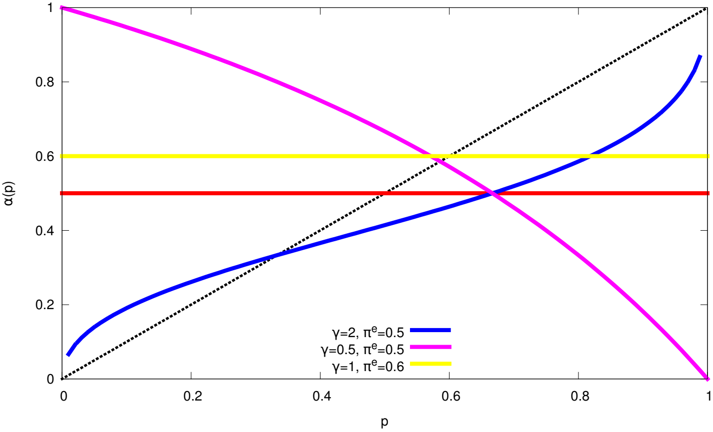

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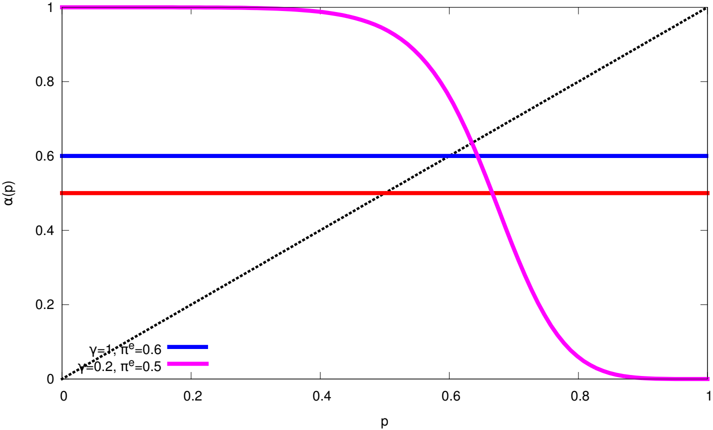
Aggregate Risk: vanishing of the informed trader

Example from Blume and Easley, JET 1992



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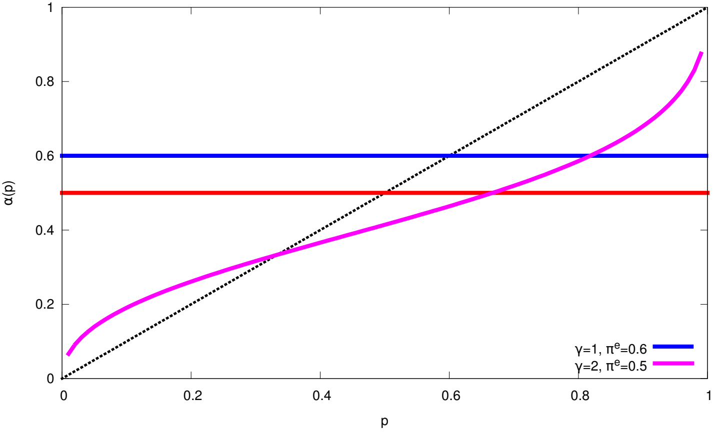

Learning from Prices

Rules αs can contain past price dependence because:

- Agents need to form price expectations when optimize over more periods
- Agents may want to use technical rules to exploit asset market imperfections
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Learning from prices means past prices feed-back into investment decisions

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Local Stability when Learning

Theorem

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If the eigenvalues of the infinitely iterated map are inside the unit circle then the deterministic fixed point is asymptotically stable. For fixed points of the type $(w^* = (1,0), \alpha^1(p^*) = p^*, p^*)$ *eigenvalues are*

$$\mu = \left(\frac{\alpha^2(p^*)}{\alpha^1(p^*)}\right)^{\pi} \left(\frac{1-\alpha^2(p^*)}{1-\alpha^1(p^*)}\right)^{1-\pi} \quad \textit{and} \quad \lambda = \left.\frac{\partial \alpha^1(p)}{\partial p}\right|_{p^*}$$

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$$\mu = 1 \quad \textit{and} \quad \lambda = w^* \left.\frac{\partial \alpha^1(p)}{\partial p}\right|_{p^*} + (1-w^*) \left.\frac{\partial \alpha^2(p)}{\partial p}\right|_{p^*}$$

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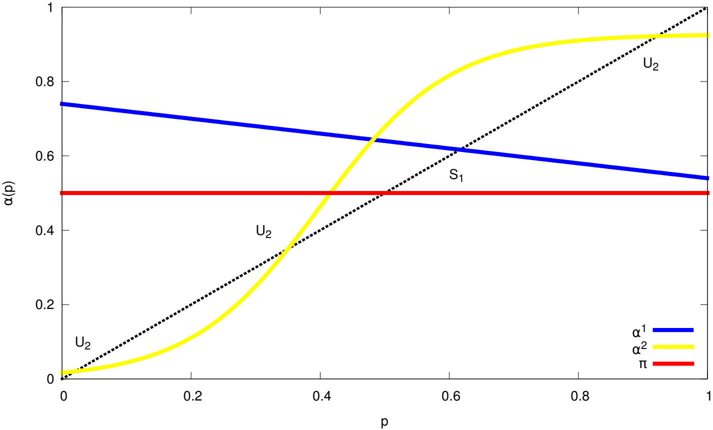
Local Stability in a Plot



Generalizations (see paper)

1. K assets, I agents, L lags
2. Ergodic and stationary process rules states of the world. Entropy w.r.t. invariant measure matters
3. Local stability for both single and multiple survival
4. Global dominating rule (generalized Kelly)

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Beyond Toy Market

Local stability single survival

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Consider the fixed point $x^* = (w^*, p^*)$ where $w^{l*} = 1$ and $p_k^* = \alpha_k^l(p^*)$ for every $k = 1, \ldots, K$. Eigenvalues are

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and solutions of the polynomial in λ of LKth degree

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Theorem

$$(\Delta_k^l)^{h,l} := -\sum_{k'=1}^{K} \{H^{-1}\}_{k,k'} (\alpha_{k'}^l)^{h,l} \;,$$

and

$$H := \begin{pmatrix} (\alpha_1^l)^{1,0} - 1 & (\alpha_1^l)^{2,0} & (\alpha_1^l)^{3,0} & \dots & (\alpha_1^l)^{K,0} \\ (\alpha_2^l)^{1,0} & (\alpha_2^l)^{2,0} - 1 & (\alpha_2^l)^{3,0} & \dots & (\alpha_3^l)^{K,0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (\alpha_K^l)^{1,0} & (\alpha_K^l)^{2,0} & (\alpha_K^l)^{3,0} & \dots & (\alpha_K^l)^{K,0} - 1 \end{pmatrix} ,$$

non-singular, with

$$(\alpha_k^i)^{h,l} := \left. \frac{\partial \alpha_k^i(p)}{\partial p_h^l} \right|_{x^*} \;, \quad i = 1, \dots, I \;, \quad l = 0, 1, \dots, L \;, \quad k, h = 1, \dots,$$

The dominant rule

A price dependent generalization of the Kelly rule

Define the function

$$I_\pi(\alpha, \mathbf{p}) = -\sum_{s=1}^{S} \pi_s \log\left(\sum_{k=1}^{K} \frac{\alpha_k}{p_k} d_{s,k}\right) ,$$

where d is the normalized dividend payoff matrix and π the invariant measure.

We define α^S as

$$\alpha^S(\mathbf{p}) = \mathrm{argmin}_{\alpha \in \Delta_c^K} \left\{\exp I_\pi(\alpha, \mathbf{p})\right\} . \tag{11}$$

If $D = I$, Arrow securities, then $\alpha^S = \pi$.

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Evolutionary stability and α^S

Theorem

Consider an ecology $\mathcal{E}$ of rules with $\alpha^S \in \mathcal{E}$. All deterministic fixed points $x^* = (w^*, p^*)$ where α^S vanishes are unstable. Moreover, there exists at least one stable deterministic fixed point in which α^S survives and long-run asset prices are equal to $p^*_k = \sum_{s=1}^{S} \pi_s d_{s,k}$, for all $k = 1, \ldots, K$.

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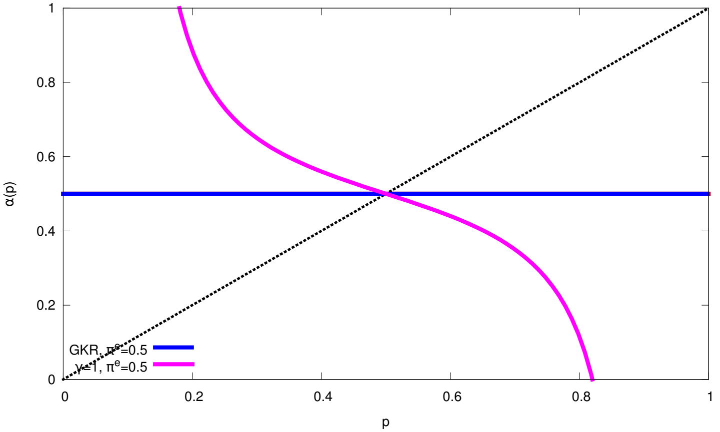
Generalized Kelly Rule (GKR) and α^S

Amir et al (2005) JME, Evstignev et al (2008) JET



$$D = \begin{pmatrix} 1-a & a \\ a & 1-a \end{pmatrix}, \quad a > 0, \quad \alpha_{GKR} = \pi_1(1-a) + (1-\pi_1)a$$

GKR and Noisy Traders

Consider a two assets market with

- two traders trading according to α^{GKR} and α^{S} respectively,
- a "noise" trader investing according to a random constant rule in $\alpha^{0} \in (0, 1)$,
- if the wealth of the noise trader is small, $w^{0} < 0.05$, she exits from the market and is replaced by a new noisy trader with random wealth in $(0.05, 01)$ and random strategy in $(0, 1)$.

w^{GKR}/w^{S} Trajectory and Approximation

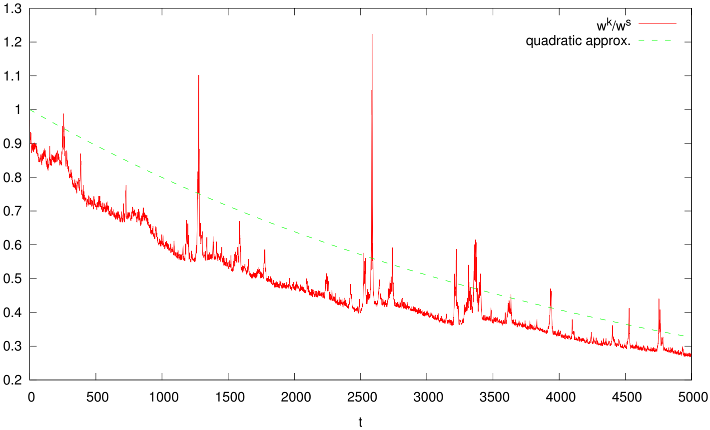


Figure: $D = \begin{pmatrix} 0.3 & 0.7 \\ 0.7 & 0.3 \end{pmatrix}$, $\pi_1 = 0.5$, $\pi_2 = 0.5$

Global Results

The non-homogeneous random walk view

Define

$$S_t = \log \frac{w_t^1}{w_t^2}$$

then wealth dynamics gives

$$S_{t+1} = S_t + \mu(S_t) + X_{t+1}(S_t)$$

with

$$\mu(s) = I_\pi(\alpha^2(s)) - I_\pi(\alpha^1(s))$$

and $\{X_t\}$ are independent but non identically distributed random variables with zero mean and finite variance.

Global Results

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A result by Lamperti

In Lamperti (1960) Theorem 3.1 states

Theorem

Take a positive Markov process $\{S_t\}$ with conditional mean $\mu(s)$ and conditional variance $v(s)$. Let $v(s)$ be bounded away from zero. Suppose that for all large enough s,

$$\mu(s) \leq \frac{\theta v(s)}{2s}$$

for some $\theta < 1$. Then $\{S_t\}$ is recurrent ($\exists r > 0 \; P(\liminf S_t \leq< r) = 1$). Conversely, if for all large s and a value of $\theta > 1$

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then $\{S_t\}$ is transient ($P(S_t \to \infty) = 1$).

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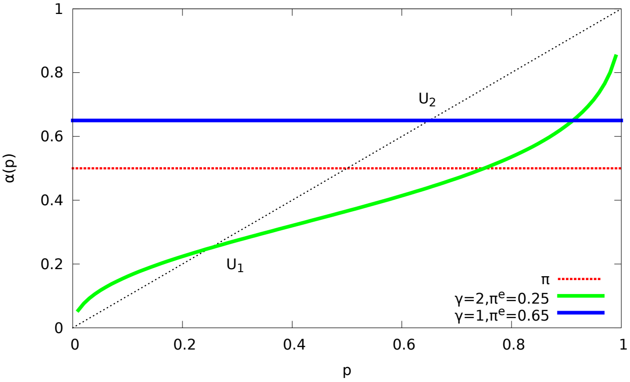
Local implies Global

S_t Recurrent



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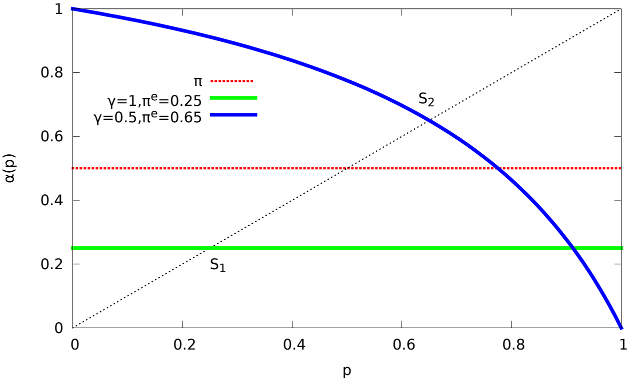

Ongoing Work and Open Issues

Open issues:

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2. More general demands (CARA, ...)
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4. Long lived assets (endowments)
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Thank You!