Distributed quantum dense coding

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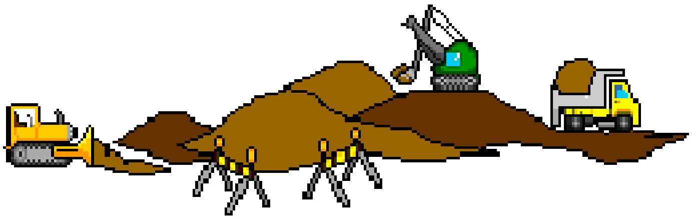
I. Dense coding with bipartite systems

II. Distributed dense coding: N senders

III. Distributed dense coding: 2 receivers

Pisa, 15.12. 2004



Quantum information in Düsseldorf

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Motivation and Outline

D. Bruß, G. M. D'Ariano, M. Lewenstein, C. Macchiavello, A. Sen(De), and U. Sen;

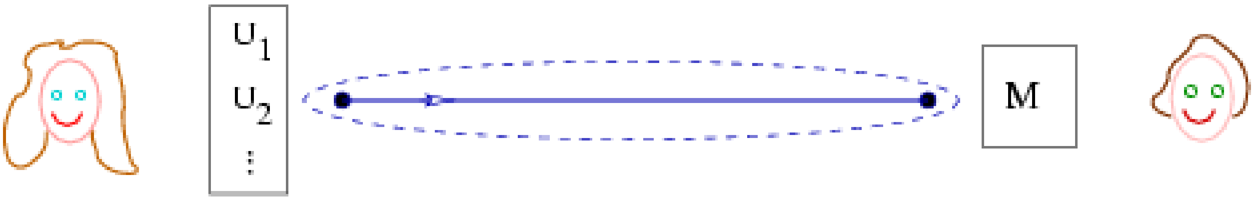
Phys. Rev. Lett. **93***, 210501 (2004).*

* Which states are useful for dense coding?

* Equivalence of dense coding and quantum teleportation?

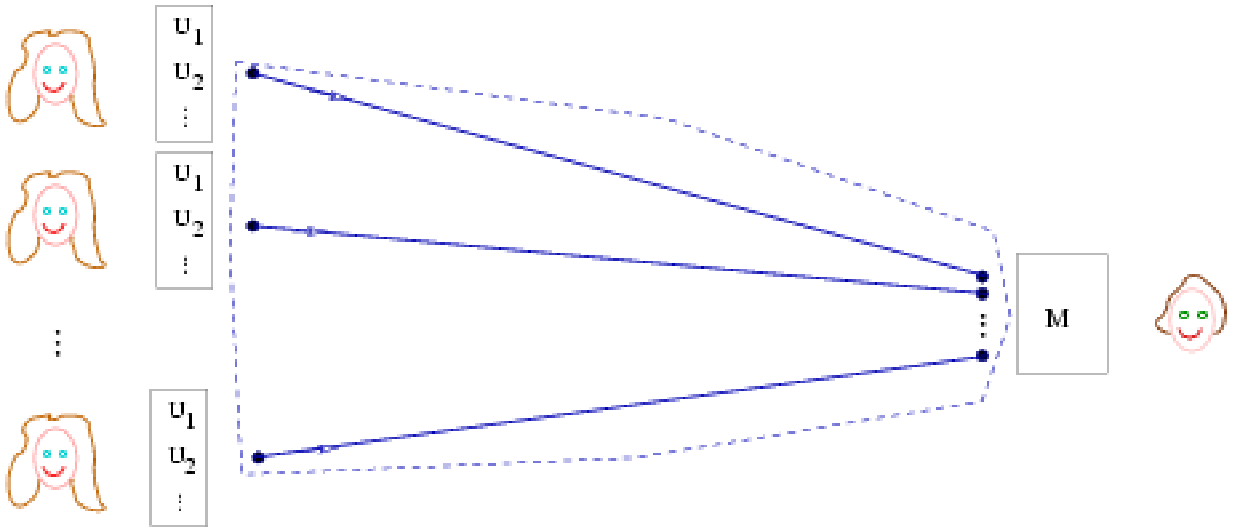
* Generalization to multipartite scenario

Dense coding with bipartite systems:

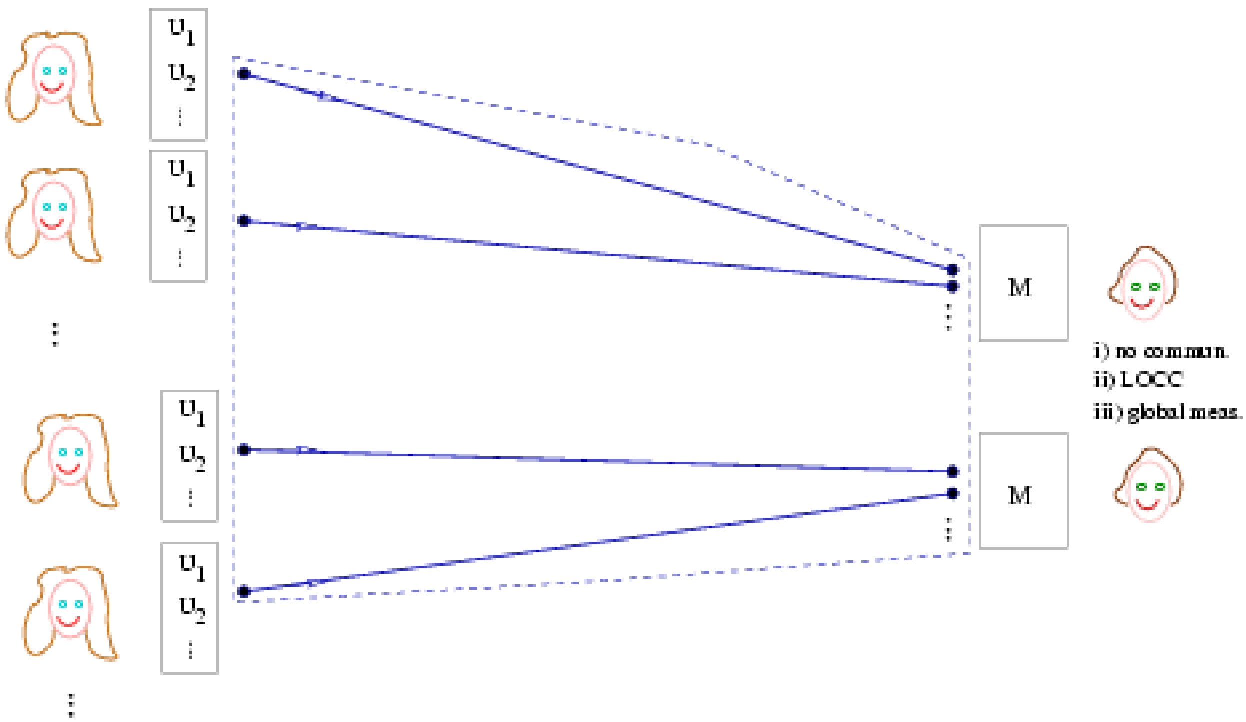


Multipartite scenario

N senders:



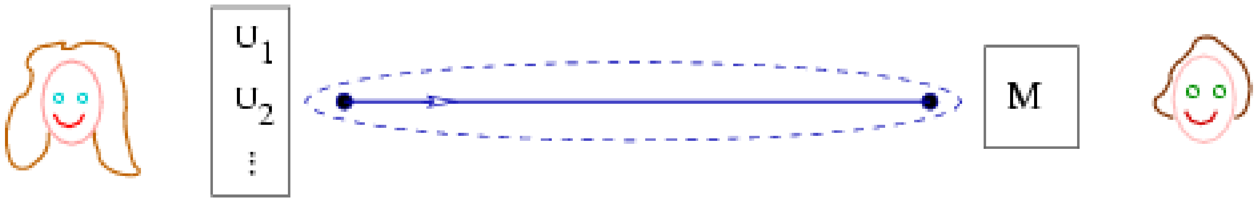
2 receivers:



I. Dense coding with bipartite systems

Superdense coding with 2 qubits:

C.H. Bennett and S.J. Wiesner, Phys. Rev. Lett. ***69****, 2881 (1992).*



"encode 2 classical bits into 1 qubit + entanglement"

The protocol:

0. Alice + Bob share $|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$

1. Alice: $U_i \in \{\mathbb{1}, \sigma_x, \sigma_y, \sigma_z\}$ with $p_i = 1/4$

2. Bob: Bell measurement $\leadsto$ 2 bits of info

Dense coding with higher dim. bipartite states

The protocol:

0. Alice + Bob share ρ^{AB}

1. Alice:

$$\rho^{AB} \quad \xrightarrow{\{U_i\}} \quad \{p_i, \rho_i^{AB}\}$$

$$\text{with} \qquad \rho_i^{AB} = U_i \otimes I_{d_B} \rho^{AB} U_i^\dagger \otimes I_{d_B}$$

2. Bob: Holevo bound for Bob's info

$$S(\overline{\rho}) - \sum_i p_i S(\rho_i^{AB}) = \sum_i p_i S(\rho_i^{AB} \parallel \overline{\rho})$$

with

$S(\xi) = -\mathrm{tr}(\xi \log \xi)$ — von Neumann entropy

$S(\varrho \parallel \xi) = \mathrm{tr}(\varrho \log \varrho - \varrho \log \xi)$ — relative entropy

$\overline{\rho} = \sum_i p_i \rho_i^{AB}$ — average state

Capacity of dense coding:

$$\chi = \max_{\{U_i\},\{p_i\}} \sum_i p_i S(\rho_i^{AB} \parallel \overline{\rho})$$

Capacity of bipartite dense coding

Lemma: Maximal information transfer is reached for complete set of orthogonal unitary operators $\{W_i\}$ with $\frac{1}{d_A^2}\sum_j W_j^\dagger \Xi W_j = Tr[\Xi]\mathbb{1}$ for all Ξ, and $p_i = 1/d_A^2$.

Proof: *[see T. Hiroshima, J. Phys. A **34**, 6907 (2001)]*

Step 1.

$$\rho^{AB} \quad \xrightarrow{\{W_i\}} \quad \{\frac{1}{d_A^2}, \rho_i'\}$$

with $$\rho_i' = W_i \otimes I_{d_B} \rho^{AB} W_i^\dagger \otimes I_{d_B}$$

Average state: $\overline{\rho}' = \frac{1}{d_A} I_{d_A} \otimes \rho^B$

Capacity: $\chi' = \frac{1}{d_A^2}\sum_i S(\rho_i' \parallel \overline{\rho}')$

Step 2.

Consider arbitrary U: $\sigma_{AB} = U \otimes I_{d_B} \rho^{AB} U^\dagger \otimes I_{d_B}$

$\rightsquigarrow$ Capacity: $\chi' = S(\sigma_{AB} \parallel \overline{\rho}')$

Step 3.

Consider arbitrary ensemble

$$\mathcal{E} = \{p_i, \rho_i = U_i \otimes I_{d_B} \rho^{AB} U_i^\dagger \otimes I_{d_B}\}$$

with capacity $\chi_{\mathcal{E}} = \sum_i p_i S(\rho_i \parallel \overline{\rho})$

Donald's identity:

$$\chi' = \sum_i p_i S(\rho_i \parallel \overline{\rho}) + S(\overline{\rho} \parallel \overline{\rho}') = \chi_{\mathcal{E}} + S(\overline{\rho} \parallel \overline{\rho}') \geq \chi_{\mathcal{E}}$$

$\rightsquigarrow$ Capacity of bipartite dense coding:

$$\chi = \log_2 d_A + S(\rho^B) - S(\rho^{AB})$$

Dense codeability

A state is useful for dense coding (dense codeable) if

$$\chi > \log_2 d_A \quad \rightsquigarrow \quad S(\rho^B) > S(\rho^{AB}) \quad [*]$$

Observations and conclusions:

- Separable states: $[*]$ is never fulfilled
- $[*] \rightsquigarrow \rho^{AB}$ violates reduction criterion

  K. Vollbrecht and M. Wolf, quant-ph/0202058

  (remember reduction criterion: $I_{d_A} \otimes \rho^B \geq \rho^{AB}$)

  $\rightsquigarrow \rho^{AB}$ is distillable

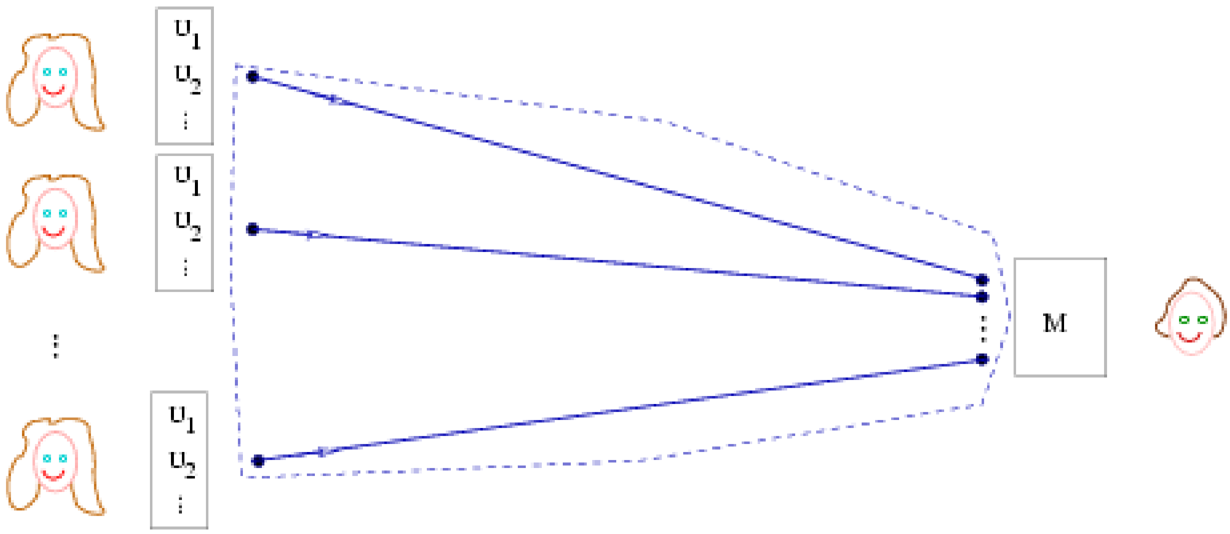
  M. Horodecki and P. Horodecki, Phys. Rev. A ***59****, 4206 (1999)*

  $\rightsquigarrow$ bound entanglement not useful for dense coding

  (for $d \times d$: *see M. Horodecki et al, Quant. Inf. Comp.* **1***, 70 (2001)*)
- Any pure bipartite entangled state useful for DC
- $\exists$ mixed entangled states in 2×2, which are not useful for DC (e.g. Werner states with $F < 0.748$)

  $\rightsquigarrow$ dense coding and teleportation are inequivalent!

II. Distributed dense coding: N senders



The protocol:

0. N Alices + Bob share $\rho^{A_1 \dots A_N B}$
1. jth Alice: apply $U_{i_j}^{A_j}$ with probability $p_{i_j}^{A_j}$, joint action?
2. Bob: global measurement, Holevo bound can be achieved

   for product encodings

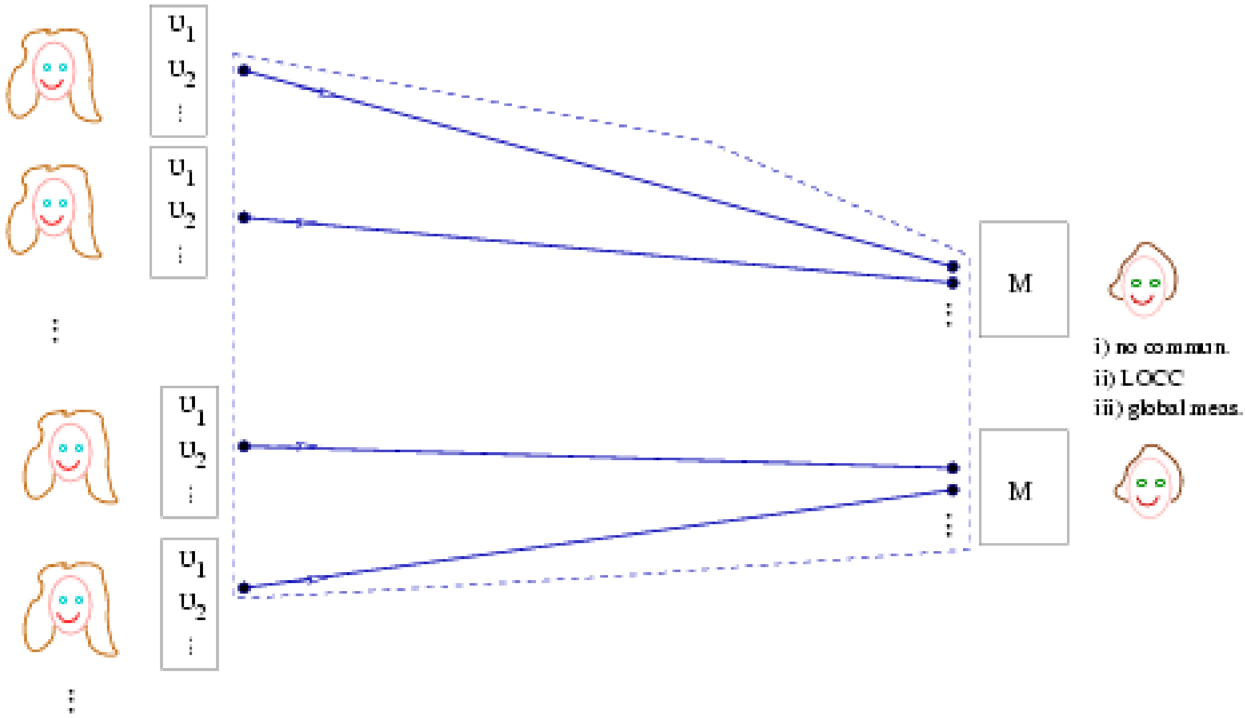
   $\rightsquigarrow$ complete orthogonal *local* set $\{W_i = \otimes_j W_{i_j}^{A_j}\}$

Capacity of dense coding with single receiver:

$$\begin{aligned}\chi^{A_1 \dots A_N B} &= \log_2 d_{A_1} + \dots + \log_2 d_{A_N} \\ &\quad + S(\rho^B) - S(\rho^{A_1 \dots A_N B})\end{aligned}$$

Surprisingly: Alices do not need to perform global unitaries to achieve capacity!

III. Distributed dense coding: 2 receivers



0. N Alices + 2 Bobs share $\rho^{A_1 \ldots A_N B_1 B_2}$

1. jth Alice: apply $U_{i_j}^{A_j}$ with probability $p_{i_j}^{A_j}$

   $A_1, \ldots A_k$ sends to B_1; $A_{k+1}, \ldots A_N$ sends to B_2

2. Bobs:

   i) no communication: $\rightsquigarrow$ DC capacities are additive

   ii) LOCC: $\rightsquigarrow$ Holevo-like upper bound

   P. Badziąg, M. Horodecki, A. Sen(De), and U. Sen, Phys. Rev. Lett. **91**, *117901 (2003)*

   iii) global measurement: $\rightsquigarrow$ one receiver

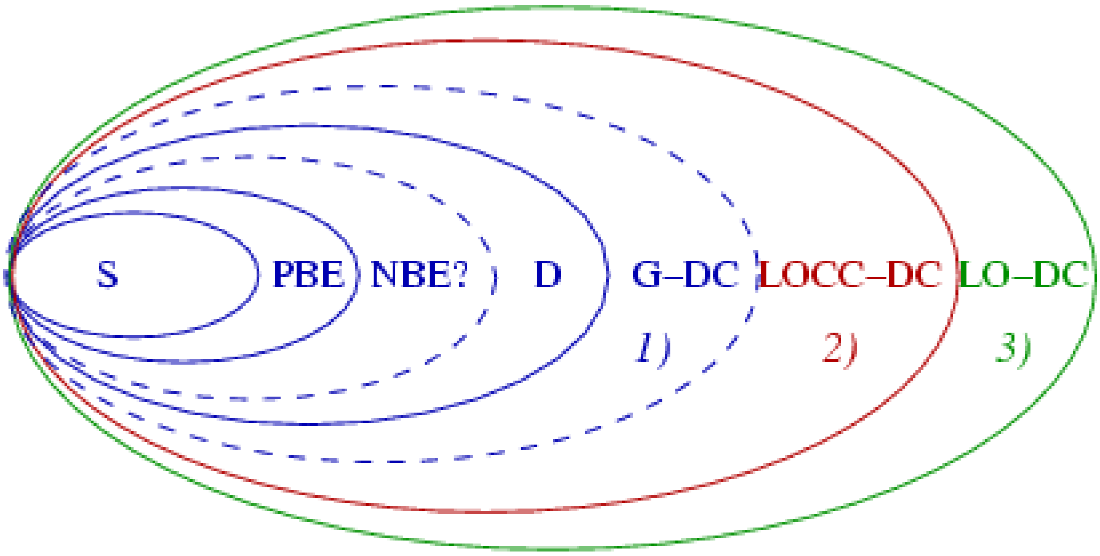
Bounds on LOCC-capacity of DC with 2 receivers:

$$\chi^{LOCC} \geq \chi^{B_1} + \chi^{B_2} = \chi^{B_1 B_2}$$

$$\chi^{LOCC} \leq \log d_{A_1} + \ldots + \log d_{A_N} + S(\rho^{B_1 B_2}) - S(\rho) = \chi^{glob}$$

$$\chi^{LOCC} \leq \log d_{A_1} + \ldots + \log d_{A_N} + S(\rho^{B_1}) + S(\rho^{B_2}) - \max_{x=1,2} S(\rho^x) \equiv \mathcal{B}^{LOCC}$$

Classification scheme: Dense codeability



Examples:

1) G-DC, but not LOCC-DC:

$$|\psi_{VDDV}\rangle = \frac{1}{2}\left(|0000\rangle + |0101\rangle + |1000\rangle + |1110\rangle\right)$$

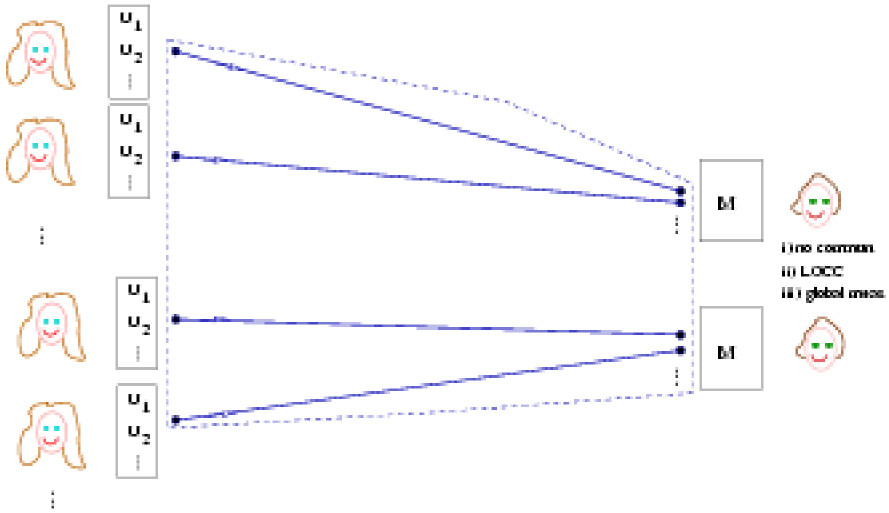
2) LOCC-DC, but not LO-DC:

$$|\psi_{GHZ}\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle) \quad \rightsquigarrow \chi^{LOCC} = 3$$

3) LO-DC:

$$|\psi_{2Bell}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \otimes \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

Summary



* Dense coding capacity for bipartite systems

* DC capacity for N senders, 1 receiver

* DC capacity for 2 receivers: upper bound

* DC is not equivalent to quantum teleportation

* Classification scheme: Dense codeability

Open problems

* More than 2 receivers?

* Convexity of G-DC border?

* $\exists$ multipartite bound entangled states for DC?

* is $|W\rangle_{A_1A_2B_1B_2}$ in LOCC-DC?