The Painlevé II equation from the isomonodromic point of view

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Joint work with J.P. Ramis.

1. Painlevé equations as isomonodromic deformations

The fuchsian case (Painlevé VI equation):
The moduli space:

$$\mathcal{S} = \{Y' = A(\lambda) d\lambda \cdot Y\}$$

$$\text{with } A(\lambda) = \sum_{i=1}^{4} \frac{A_i}{\lambda - s_i}, \; \lambda \in \mathbb{P}^1(\mathbb{C}), \; A_i \in sl(2, \mathbb{C}).$$

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$\pi_1(T, t_0) = Out(\pi_1(\mathbb{P}^1 \setminus S, a))$.
One can study the dynamic of the Painlevé VI equation through the dynamic on the character variety: (Cantat-Loray, 2009)

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$\rightarrow$ *consider the Poisson structures on $\mathcal{M}$.*

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$$\mathcal{S} = \{Y' = A(\lambda)d\lambda \cdot Y\}$$

$$\text{with } A(\lambda) = -(A_3\lambda^3 + A_2\lambda^2 + A_1\lambda)\frac{d\lambda}{\lambda})$$

$$= (A_3\mu^{-3} + A_2\mu^{-2} + A_1\mu^{-1})\frac{d\mu}{\mu} \quad (\mu = \lambda^{-1}).$$

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$\mathcal{S}^* = \{(A) \in \mathcal{S},\ A_3 \neq 0,\ A_3 \text{ is semi-simple}\}$.
The connections A in $\mathcal{S}^*$ have a unique irregular singular point at $\lambda = \infty$ $(\mu = 0)$ of rank 3.

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$$(\mathcal{T}^*/(L \rtimes W))/(\lambda \to \lambda + \beta) = \mathcal{U}^*/(L \rtimes W)$$
$$\textit{with } \mathcal{U}^* = \{A \in \mathcal{T}^*,\ a_2 = 0\}.$$

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Conclusion: $\mathcal{M}^*$ is the first chart of a weighted projective space $\mathcal{M}$ (an orbifold):
$\mathcal{M}^* = \{(\alpha_6 : \alpha_4 : \alpha_3 : \beta_6 : \beta_4) \in \mathcal{M} = \mathbb{P}^4_\omega(\mathcal{U}^*/(L \rtimes W)),\ \alpha_6 \neq 0\}.$

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Orbifold structure of $\mathcal{M}$: atlas of 5 orbifold-charts.

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$$(A) \,\widehat{\approx}\, (D): \quad \widehat{Y} = (D_3\mu^{-3} + D_1\mu^{-1} + D_0)\frac{d\mu}{\mu} \cdot \widehat{Y}, \;\; D_i = diag(\pm d_i).$$

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$\mathcal{B}^* := \mathcal{D}^*$ quotiented by gauge (W) and $Aut(\mathbb{P}^1, 0, \infty)$.

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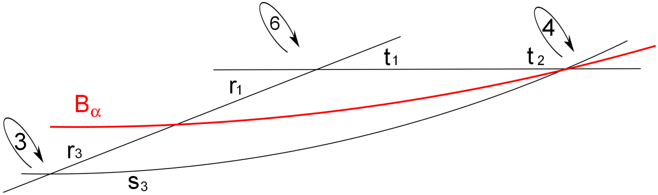

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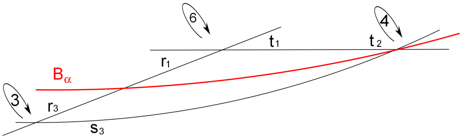

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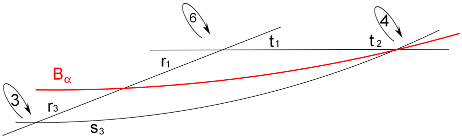

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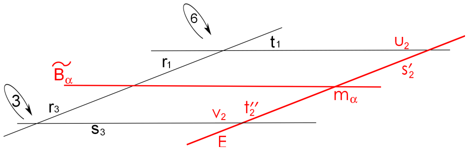

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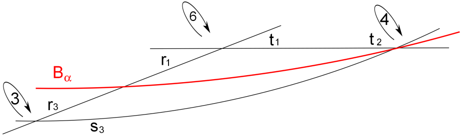

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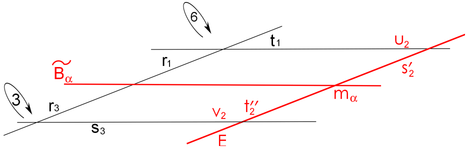

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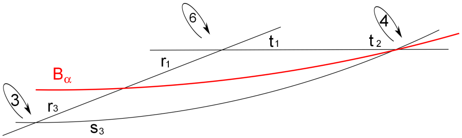

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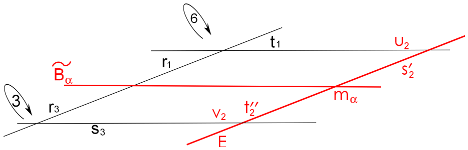

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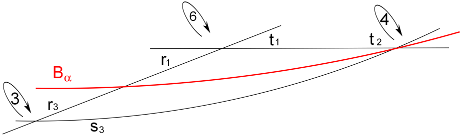

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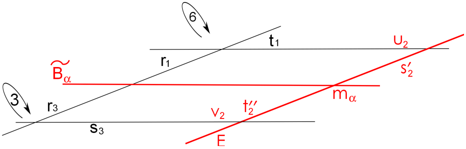

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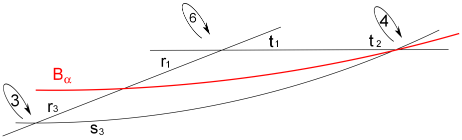

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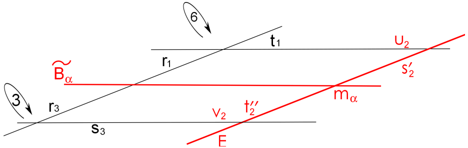

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create Poisson-Darboux coordinates: $\omega = dq_1 \wedge dp_1 = dq_2 \wedge dp_2$.
In the Poisson-Darboux coordinates:
$h_1 = 4[(\frac{t_1}{2} - p_1)^2 - 2p_1 q_1^2 - r_1 q_1]$
$h_2 = (1 - u_2^4 q_2)^2 - 8p_2 q_2^2 - 4q_2 s_2'$.

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We consider the link $t = t_1$, i.e. the (non automous) hamiltonian vector field X_{II}:

$$\begin{cases} \dot{p}_1 = -\dfrac{\partial h_1}{\partial q_1} \\ \dot{q}_1 = \dfrac{\partial h_1}{\partial p_1} \\ \dot{t}_1 = 1 \end{cases}$$

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Direct proof (following H. Chiba): The link $t = t_1$ transforms the isospectral Lax pair related to X_{α_2} in an isomonodromic Lax pair.

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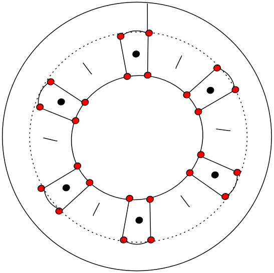

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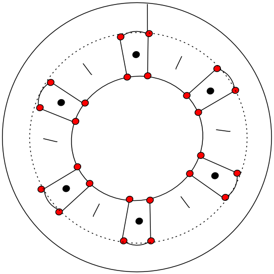

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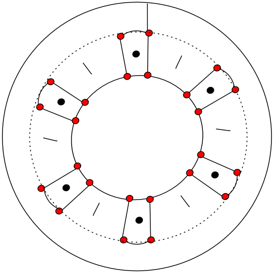

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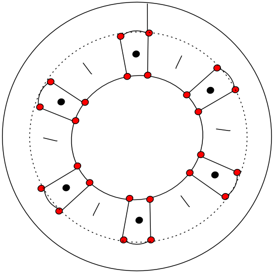

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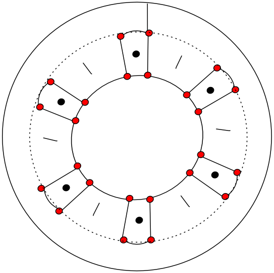

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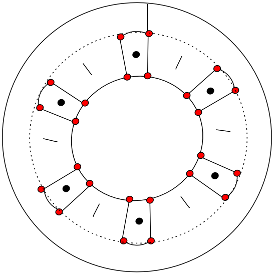

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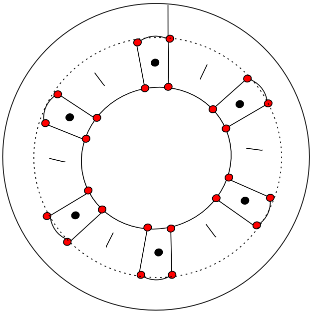

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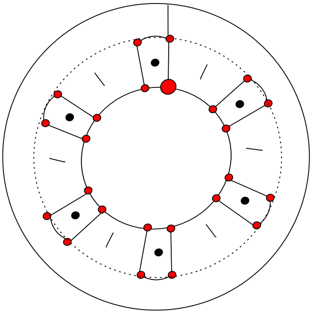

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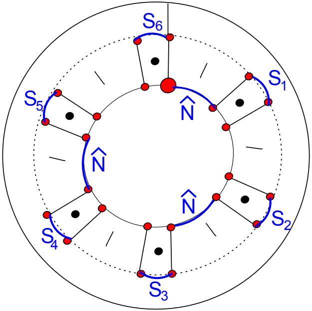

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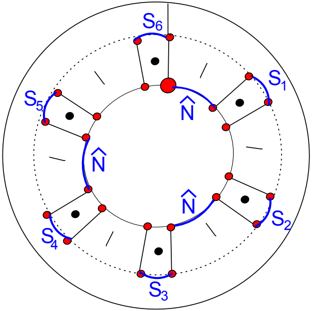

$$\chi = \{(S_1, \cdots, S_6, \widehat{N}),\ \widehat{N} S_1 S_2 \widehat{N} S_3 S_4 \widehat{N} S_5 S_6 = I,\ \widehat{N} = diag(\widehat{\alpha}, \widehat{\alpha}^{-1})\}/D.$$

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$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} z \\ -\frac{\widehat{\alpha}^{-1} xyz + (\widehat{\alpha}^{-3} - 1) xz}{\widehat{\alpha} xy + \widehat{\alpha} xyz - \widehat{\alpha}^{-1}) xz} \\ \widehat{\alpha}^{-1} y + (\widehat{\alpha}^{-3} - 1) \end{pmatrix}$$

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c induces an automorphism of the wild fundamental groupoid whose action on $F_{\widehat{\alpha}}$ is h.

Idea of the proof. If c is parametrized by θ: $Y_{\widehat{a}}(\mu, \theta)$. At the end of the orbifold loop: $\widehat{Y}(\mu, 2\pi/3)$. We have:

$$\widehat{Y}(\mu, 2\pi/3) = \widehat{Y}(\mu e^{-2i\pi/3}, 0)e^{2i\pi D_0/3}.$$

The automorphism of the fundamental groupoid induced by c is just a rotation of angle $2\pi/3$: $S_i \to S_{i+2}$: h.

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Work in progress...