The order of indeterminate moment problem

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Consider a probability measure μ on $[a, b]$. The quantities

$$s_n = \int_a^b x^n \, d\mu(x), \qquad n \geq 0$$

are called the **moments** of μ. The sequence s_n determines the measure μ as by Weierstrass theorem the monomials x^n are linearly dense in $C[a, b]$.

The situation is entirely different when we consider probability measures with unbounded support in $\mathbb{R}$. In that case we assume that

$$\int_{-\infty}^{\infty} x^{2n} d\mu(x) < \infty, \qquad n \geq 0.$$

Then the quantities

$$s_n = \int_{-\infty}^{\infty} x^n \, d\mu(x), \qquad n \geq 0$$

are well defined. Moreover the set of all polynomials $\mathcal{P}(x)$ is contained in $L^2(\mu)$.

For example

$$d\mu(x) = e^{-x}\, dx, \qquad x \geq 0$$

$$s_n = \int_0^\infty x^n e^{-x}\, dx = n!.$$

Sequences s_n for which there exists a measure μ with infinite support so that

$$s_n = \int_{-\infty}^\infty x^n\, d\mu(x),$$

where described by Hamburger. A necessary and sufficient condition is that the Hankel matrix

$$\{s_{i+j}\}_{i,j=0}^\infty$$

is strictly positive definite, i.e.

$$\sum_{i,j=0}^\infty s_{i+j} t_i t_j > 0, \quad \text{when } \sum_{i=0}^\infty t_i^2 > 0,$$

for all sequences of real numbers t_i vanishing for large i.

By Sylvester's criterion the latter is equivalent to positivity of the leading principal minors of the Hankel matrix, i. e.

$$\Delta_n := \begin{vmatrix} s_0 & s_1 & \cdots & s_n \\ s_1 & m_2 & \cdots & s_{n+1} \\ \vdots & \vdots & \cdots & \vdots \\ s_n & s_{n+1} & \cdots & s_{2n} \end{vmatrix} > 0, \quad n \geq 0.$$

A natural problem arises: do the moments s_n determine the measure μ ? The first obstacle is that the polynomials may not be linearly dense in $L^2(\mu)$, unlike in the case of measures with bounded support.

A measure μ is called **determinate** if the sequence of its moments s_n determines μ. Otherwise the measure μ (or the sequence s_n) is called **indeterminate**.

A measure μ is called **N-extreme** if the polynomials are dense in $L^2(\mu)$. It can be shown that determinate measures are N-extreme.

In the indeterminate case there are many measures on the real line with moments s_n. Indeed, if two measures μ_1 and μ_2 have moments equal s_n, then the moments of every convex combination

$$\alpha\mu_1 + (1-\alpha)\mu_2$$

are equal s_n. It turns out that in the indeterminate case there are much more solutions of the equations

$$s_n = \int_{-\infty}^{\infty} x^n \, d\mu(x), \quad n \geq 0,$$

(s_n-given, μ-unknown). The full description is due to Rolf Nevanlinna. Part of these measures are N-extreme. Moreover in the indeterminate case every N-extreme solution is a measure with discrete support on the real line. Every two different N-extreme solutions have disjoint supports and the union of all supports of N-extreme solutions cover the real line, i.e. every real number belongs to the support of exactly one N-extreme solution. **All these measures have the same sequence of moments**.

Moreover if μ is an N-extreme indeterminate measure and $a \in \text{supp}\mu$, then the measure

$$\widetilde{\mu} = \mu - \mu(\{a\})\delta_a$$

is determinate. It means that after removing one point from the support of the measure may entirely change its properties. One of the N-extreme solutions contains $a = 0$ in its support. Then the moments of the measures $\widetilde{\mu}$ and μ are equal for $n \geq 1$. Only the moments of order 0, i.e. the total masses, are different.

Example (Stieltjes, 1894)

$$\int_0^\infty x^n[1+c\sin(2\pi\log x)]\, e^{-\log^2 x}\, dx = \sqrt{\pi}e^{(n+1)^2/4}.$$

The moments do not depend on the parameter c. Thus the measures

$$\mu_c(x) = [1 + c\sin(2\pi\log x)]\, e^{-\log^2 x}\, dx$$

for $|c| \leq 1$ have equal moments.

The main tool for studying the moment problem is orthogonal polynomial system. In the space $L^2(\mu)$ we apply the Gram-Schmidt procedure to the sequence of monomials x^n in order to obtain a sequence of polynomials $p_n(x)$ orthonormal with respect to

$$(f, g) = \int_{-\infty}^{\infty} f(x)g(x)\, d\mu(x).$$

For example, in case $d\mu(x) = \pi^{-1/2}e^{-x^2}\, dx$ we obtain the Hermite polynomials $h_n(x)$,

$$xh_n(x) = \sqrt{n+1}h_{n+1}(x) + \sqrt{n}h_{n-1}(x), \quad n \geq 1,$$

$h_0 = 1$, $h_1(x) = x$.

In general case (for measure μ) we obtain the sequence of polynomials $p_n(x)$ with properties

$$p_n \in \operatorname{span}\{1, x, \ldots, x^n\}, \text{ (or } \deg p_n \leq n)$$

$$p_n \perp \{1, x, \ldots, x^{n-1}\}.$$

Then $\deg p_n = n$ (i.e. the sequence p_n constitutes a linear basis for the space of all polynomials) thus

$$xp_n(x) = b_n p_{n+1}(x) + a_n p_n(x) + c_n p_{n-1}(x).$$

Lower order terms vanish as

$$(xp_n, p_k) = (p_n, xp_k) = 0, \quad k \leq n-2.$$

Moreover

$$b_n = (xp_n, p_{n+1})$$

and

$$c_n = (xp_n, p_{n-1}) = (p_n, xp_{n-1}) = b_{n-1}.$$

Summarizing we obtain the recurrence relation of order 2

$$\begin{aligned} xp_n(x) &= b_n p_{n+1}(x) + a_n p_n(x) + b_{n-1} p_{n-1}(x), \\ & \quad p_0 = 1, p_{-1} = 0. \end{aligned}$$

We can assume that the leading coefficient of each polynomial p_n is positive, hence $b_n > 0$ and $a_n \in \mathbb{R}$. In the space of all polynomials $\mathcal{P}(x) \subset L^2(\mu)$ the system p_n constitutes an orthonormal basis. Consider the linear operator of multiplication by x in the space $\mathcal{P}(x)$. This operator relative to the basis $\{p_n\}_{n=0}^{\infty}$ is described by the so called Jacobi matrix

$$J = \begin{pmatrix} a_0 & b_0 & 0 & 0 & 0 & \cdots \\ b_0 & a_1 & b_1 & 0 & 0 & \cdots \\ 0 & b_1 & a_2 & b_2 & 0 & \cdots \\ 0 & 0 & b_2 & a_3 & \ddots & \\ 0 & 0 & 0 & \ddots & \ddots & \\ \vdots & \vdots & \vdots & & & \end{pmatrix}.$$

The coefficients b_n and a_n can be given in terms of moments s_n, which follows from the Gram-Schmidt procedure. The matrix J is tridiagonal and symmetric, with real coefficients on the main diagonal and positive ones on the upper and lower diagonals. As an operator, J can be defined in the Hilbert space of square summable sequences $\ell^2(\mathbb{N}_0)$ with domain $D(J)$ consisting of sequences vanishing from certain place. If the sequences b_n and a_n are bounded, then J can be extended to a bounded selfadjoint operator on $\ell^2(\mathbb{N}_0)$. Otherwise J is an unbounded symmetric operator. It can be shown that J is bounded if and only if

$$\sup_n \sqrt[2n]{s_{2n}} < \infty.$$

Equivalently the support of the measure μ is bounded.

It turns out that studying the determinacy of the moment problem can be reduced to studying the properties of the matrix J. In particular the moment problem $\{s_n\}$ is determinate if and only if the operator J is essentially selfadjoint, i.e. its closure $\overline{J}$ is selfadjoint. Equivalently the spaces $\mathrm{Im}(J - zI)$ are dense in $\ell^2(\mathbb{N}_0)$ for any $z \notin \mathbb{R}$.

Starting with a measure on $\mathbb{R}$, having finite moments, we determined a Jacobi matrix J. Conversely, given a Jacobi matrix J, if J is essentially selfadjoint, there is a unique measure μ on the real line, which determines J in the way described earlier. Indeed, if $E(x)$ denotes the resolution of the identity associated with the operator $\overline{J}$ (closure of J), then

$$d\mu(x) = d(E(x)\delta_0, \delta_0)_{\ell^2}.$$

$$\delta_0 = (1, 0, 0, \ldots)^t.$$

Moreover $\sigma(\overline{J}) = \mathrm{supp}\mu$.

The relation between the operator J and the measure μ, even when the operator J is bounded, is highly implicit. Slight modification of the matrix J may cause essential change of the properties of the underlying measure. The following example can be found in the textbook of M. Reed and B. Simon. Let μ denote the arithmetic mean of the Lebesgue measure and the Cantor measure on the interval $[0, 1]$. Let J be the Jacobi matrix associated with μ. Set

$$J_\varepsilon = J + \varepsilon\delta_{0,0},$$

i.e. we modify by ε the upper left entry of the matrix J.

$$J_a = \begin{pmatrix} a_0 + \varepsilon & b_0 & 0 & 0 & 0 & \cdots \\ b_0 & a_1 & b_1 & 0 & 0 & \cdots \\ 0 & b_1 & a_2 & b_2 & 0 & \cdots \\ 0 & 0 & b_2 & a_3 & \ddots & \\ 0 & 0 & 0 & \ddots & \ddots & \\ \vdots & \vdots & \vdots & & & \end{pmatrix}$$

It turns out that the measure corresponding to J_ε is absolutely continuous on the interval $[0, 1]$. Since J_ε is a one-dimensional perturbation of the matrix J, the spectrum of J_ε is equal $[0, 1]$ plus eventually one point.

In case the matrix J is not essentially selfadjoint, the defect indices, i.e. the dimensions of

$$(\mathrm{Im}(J - zI))^{\perp}, \quad (\mathrm{Im}(J - \overline{z}I))^{\perp}, \quad \mathrm{Im} z > 0,$$

are equal 1. Then the selfadjoint extensions of the matrix J correspond to N-extreme solutions in the following way. Let $\tilde{J}$ be such extension of J. The operator $\tilde{J}$ provides the associated resolution of the identity $E(x)$. Then the measure

$$d\mu(x) = d(E(x)\delta_0, \delta_0)_{\ell^2},$$

where $\delta_0 = (1, 0, 0, \ldots)^t$ is N-extreme solution of the moment problem, i.e. the polynomials $\mathcal{P}(x)$ are dense in $L^2(\mu)$.

Jacobi matrices are building blocks for any self-adjoint operator (bounded or unbounded) in the Hilbert space. Indeed, consider a symmetric operator $A : D(A) \to \mathcal{H}$, $D(A) \subset \mathcal{H}$, with a cyclic vector v, i.e. $v \in D(A^n)$ for any $n \geq 1$ and the vectors

$$v, Av, A^2v, \ldots, A^nv, \ldots$$

are linearly dense in $\mathcal{H}$. We can apply the Gram-Schmidt procedure to the sequence of vectors $\{A^nv\}_{n=0}^{\infty}$ and obtain an orthonormal system $w_n = p_n(A)v$ similarly as in the case of monomials x^n in $L^2(\mu)$. It suffices to replace mechanically

$$x^n \leftrightarrow A^nv$$

Multiplication by x corresponds to applying the operator A. The representation of the operator A relative to the basis $\{w_n\}$ is a Jacobi matrix.

Thus a symmetric operator with cyclic vector can be identified with a Jacobi matrix.

Every selfadjoint operator in the Hilbert space can be decomposed into a direct sum of self-adjoint operators with cyclic vector.

Consider a moment problem associated with positive definite sequence $\{s_n\}_{n=0}^{\infty}$. Along with orthonormal polynomials $p_n(z)$ we will need the polynomials of the second kind $q_n(z)$. Both systems satisfy the same recurrence relation

$$zr_n(z) = b_n r_{n+1}(z) + a_n r_n(z) + b_{n-1} r_{n-1}(z),$$

$n \geq 1$, with

$$\begin{aligned} p_0(z) &= 1, & p_1(z) &= (z - a_0)/b_0, \\ q_0(z) &= 0, & q_1(z) &= 1/b_0. \end{aligned}$$

It is known that the sequence of moments $\{s_n\}_{n=0}^{\infty}$ is indeterminate if and only if

$$P(z) := \sum_{n=0}^{\infty} |p_n(z)|^2 < \infty,$$

for any complex (a nonreal) number z.

Equivalently

$$\sum_{n=0}^{\infty} p_n(0)^2 + q_n(0)^2 < \infty.$$

Moreover the function

$$P(z) = \sum_{n=0}^{\infty} |p_n(z)|^2$$

is of minimal exponential type, i.e. for any $\varepsilon > 0$

$$P(z) \leq C_\varepsilon e^{\varepsilon |z|}$$

We would like to study the order more precisely. For example under what assumptions may we expect

$$P(z) \leq C e^{K|z|^\alpha}$$

with $0 < \alpha < 1$?

Theorem. 1

(a) Assume that

$$\sum_{n=0}^{\infty} [p_n^{2\alpha}(0) + q_n^{2\alpha}(0)] < \infty,$$

for some number $0 < \alpha < 1$. Then the moment problem is of order at most α, i.e.

$$P(z) \leq C \exp(K|z|^{\alpha}).$$

(b) Let $\sum_{n=0}^{\infty} \frac{|a_n|}{b_{n-1}} < \infty$ (for instance $a_n \equiv 0$). Assume also that either $b_{n-1}b_{n+1} \leq b_n^2$ for any n large enough or $b_{n-1}b_{n+1} \geq b_n^2$ for any such n. If the moment problem has order α, with $0 < \alpha < 1$, then

$$|p_n(z)|^{2\alpha} = O(n^{-1})$$

for any $z \in \mathbb{C}$. In particular for any $\varepsilon > 0$

$$\sum_{n=0}^{\infty} |p_n(z)|^{2\alpha+\varepsilon} < \infty$$

The condition

$$\sum_{n=0}^{\infty} |p_n(z)|^{2\alpha} < \infty$$

for any (a nonreal) $z \in \mathbb{C}$, is equivalent to

$$\sum_{n=0}^{\infty} [p_n^{2\alpha}(0) + q_n^{2\alpha}(0)] < \infty$$

This follows from the invariance theorem.

Proposition. 1. *Let* $0 < \alpha < 1$ *and* $\{u_n\}_{n=1}^{\infty}$ *be a sequence of positive numbers such that*

$$K := \frac{1}{\alpha} \sum_{n=1}^{\infty} u_n^{\alpha} < \infty.$$

Then for $r > 0$

$$\prod_{n=1}^{\infty} (1 + u_n r) \leq \exp(K r^{\alpha}).$$

Proof. The conclusion follows immediately from the inequalities below.

$$1 + u_n r \leq (1 + u_n^{\alpha} r^{\alpha})^{\frac{1}{\alpha}} \leq \exp(\tfrac{1}{\alpha} u_n^{\alpha} r^{\alpha}).$$

$\diamondsuit$

Proof Consider (see Akhiezer, Theory of moments, 1962)

$$
\begin{aligned}
A_n(z) &= \phantom{-1 +{}} z \sum_{i=0}^{n-1} q_i(0) q_i(z), \\
B_n(z) &= -1 + z \sum_{i=0}^{n-1} q_i(0) p_i(z), \\
C_n(z) &= 1 + z \sum_{i=0}^{n-1} p_i(0) q_i(z), \\
D_n(z) &= \phantom{-1 +{}} z \sum_{i=0}^{n-1} p_i(0) p_i(z).
\end{aligned}
$$

These polynomials are associated with the Nevanlinna functions ($n = \infty$).

(a) By [B. Simon, Adv. Math. 1998] we have

$$\begin{pmatrix} A_{n+1}(z) & B_{n+1}(z) \\ C_{n+1}(z) & D_{n+1}(z) \end{pmatrix} = \left[I + z \begin{pmatrix} -p_n(0)q_n(0) & q_n^2(0) \\ -p_n^2(0) & p_n(0)q_n(0) \end{pmatrix} \right] \times \begin{pmatrix} A_n(z) & B_n(z) \\ C_n(z) & D_n(z) \end{pmatrix} \quad (1)$$

Therefore evaluating the operator norm and iterating gives

$$\left\| \begin{pmatrix} A_n(z) & B_n(z) \\ C_n(z) & D_n(z) \end{pmatrix} \right\| \leq \prod_{k=0}^{n-1} \left[1 + |z|(p_k^2(0) + q_k^2(0)) \right] \leq \prod_{k=0}^{\infty} \left[1 + |z| p_k^2(0) \right] \prod_{k=0}^{\infty} \left[1 + |z| q_k^2(0) \right].$$

In particular we have

$$\sqrt{|B_n(z)|^2 + |D_n(z)|^2}$$
$$\leq \prod_{k=0}^{\infty}\left[1 + |z|p_k^2(0)\right] \prod_{k=0}^{\infty}\left[1 + |z|q_k^2(0)\right]. \quad (2)$$

By our assumptions in view of Proposition 1 we obtain

$$\sqrt{|B_n(z)|^2 + |D_n(z)|^2} \leq \exp(K|z|^\alpha),$$

for some constant K. We also have (Akhiezer's textbook)

$$p_n(z) = -p_n(0)B_n(z) + q_n(0)D_n(z). \quad (3)$$

Therefore by the Schwarz inequality we obtain

$$P(z) = \sum_{n=0}^{\infty} |p_n(z)|^2$$
$$\leq \sum_{n=0}^{\infty} [p_n^2(0) + q_n^2(0)][|B_n(z)|^2 + |D_n(z)|^2]$$
$$\leq C\exp(2K|z|^\alpha) \quad (4)$$

Proposition (Berezanski). *Assume that*

$$\sum_{n=1}^{\infty} \frac{1+|a_n|}{b_{n-1}} < \infty.$$

Let either
(i) $b_{n-1}b_{n+1} \geq b_n^2$ *for all* $n \geq 1$ *or*
(ii) $b_{n-1}b_{n+1} \leq b_n^2$ *for all* $n \geq 1$.
Then

$$\sqrt{b_{n-1}}|r_n(z)| \leq (C|z|+D) \prod_{k=1}^{\infty} \left(1+\frac{|z|}{b_{k-1}}\right),$$

for some constants C *and* D.

In particular the sequence $\sqrt{b_{n-1}}|r_n(z)|$ is bounded, i.e. the sequence $r_n(z)$ is square summable. Therefore the moment problem is indeterminate. The assumptions made only on the growth of b_n may be insufficient to guarantee indeterminacy. For example $a_n \equiv 0$ and $b_{2n} = b_{2n+1}$ for every n correspond to the determinate case, independently of the growth of b_n.

(b) Both assumptions imply that b_n is an increasing sequence for large n. By Proposition (Berezansky) we get

$$\sqrt{b_{n-1}}|r_n(z)| \leq (C|z| + D) \prod_{k=1}^{\infty} \left(1 + \frac{|z|}{b_{k-1}}\right),$$

where either $r_n(z) = p_n(z)$ or $r_n(z) = q_n(z)$. In particular plugging in $z = 0$ gives

$$p_n^2(0) + q_n^2(0) \leq \frac{2D^2}{b_{n-1}}, \quad n \geq 1. \tag{5}$$

We want to estimate $1/b_{n-1}$ from above and make use of the assumptions made on the order. We have

$$\frac{1}{b_{n-1}^{2n}} \leq \frac{1}{b_0^2 b_1^2 \dots b_{n-1}^2}$$

Let

$$p_n(z) = \sum_{k=0}^{n} b_{k,n} z^k.$$

Then

$$b_{n,n} = \frac{1}{b_0 b_1 \dots b_{n-1}}$$

in view of

$$z p_n(z) = b_n p_{n+1}(z) + a_n p_n(z) + b_{n-1} p_{n-1}(z).$$

Hence

$$\frac{1}{b_{n-1}^{2n}} \le b_{n,n}^2 \le \sum_{k=n}^{\infty} b_{n,k}^2 =: c_n^2.$$

In a joint paper with C. Berg (2011) we have observed that

$$\sum_{n=0}^{\infty} c_n^2 r^{2n} = \frac{1}{2\pi} \int_0^{2\pi} P(re^{i\theta})\, d\theta.$$

Hence

$$\sum_{n=1}^{\infty}\left(\frac{r}{b_{n-1}}\right)^{2n} \le \frac{1}{2\pi}\int_0^{2\pi} P(re^{i\theta})\,d\theta \le C\exp[Kr^\alpha].$$

Therefore

$$\frac{r}{b_{n-1}} \le C^{1/2n}\exp[Kr^\alpha/2n]. \tag{6}$$

Set $r_n = (2n)^{1/\alpha}$. Then

$$\frac{1}{b_{n-1}} \le \frac{Ce^K}{n^{1/\alpha}}.$$

Now in view of

$$p_n(z) = -p_n(0)B_n(z) + q_n(0)D_n(z),$$

$$p_n^2(0) + q_n^2(0) \le \frac{2D^2}{b_{n-1}}$$

we get by the Schwarz inequality

$$|p_n(z)|^2 \le [p_n^2(0) + q_n^2(0)][|B_n(z)|^2 + |D_n(z)|^2] \le \frac{C}{b_{n-1}},$$

$$|p_n(z)|^{2\alpha} = O(n^{-1}).$$

$\diamondsuit$

We would like to study other (possibly slower) types of growth of the function

$$P(z) = \sum_{n=0}^{\infty} |p_n(z)|^2.$$

Observe that

$$P(z) \geq |p_n(z)|^2,$$

hence the growth of $P(z)$ is at least as $|z|^{2n}$ for any n. Thus we may expect that

$$\sup_{|z|\leq r} P(z) \leq Cr^{\alpha(r)} = Ce^{\alpha(r)\log r}, r \geq 1$$

with $\alpha(r) \to \infty$ as $r \to \infty$.

Typical examples are

- $\alpha(r) = \log^\alpha r$, with $\alpha > 0$.
- $\alpha(r) = \log^\alpha \log r$, with $\alpha > 0$.

We can as well consider the growth which is faster than r^α

$$\alpha(r) = \frac{r}{\log^\alpha r}, \quad \alpha > 1.$$

Can we find a method of dealing with different types of growth in one fashion ?

What assumptions should we impose on the summability of $p_n(z)$?

Let $\alpha(r)$ be a continuous, positive and increasing function on (r_0, ∞) such that the function $r/\alpha(r)$ is also increasing. The function

$$\beta(r) = \frac{1}{\alpha(r^{-1})}$$

will be called the **dual function**. This function is defined on the interval $(0, r_0^{-1})$. It is also increasing and $r/\beta(r)$ is increasing. Indeed

$$\frac{r}{\beta(r)} = \frac{\alpha(r^{-1})}{r^{-1}}.$$

Examples.

1. The function $\alpha(r) = r^\alpha$ with $0 < \alpha \le 1$ satisfies the assumptions and

$$\beta(r) = (\alpha(r^{-1}))^{-1} = r^\alpha.$$

2. The function $\alpha(r) = \log^\alpha r$ for $r > 1$ with $\alpha > 0$ also satisfies the assumptions and

$$\beta(r) = (\alpha(r^{-1}))^{-1} = \frac{1}{(-\log r)^\alpha}.$$

Definition. *A complex function $F(z)$ on $\mathbb{C}$ of subexponential growth is said to have order bounded by $\alpha(r)$ if*

$$\max_{|z|\le r} |F(z)| \le C e^{K\alpha(r)\log r}, \quad r \ge r_0,$$

for some constants C and K.

Theorem. 2.

(a) Assume that

$$\sum_{n=0}^{\infty} [\beta(p_n^2(0)) + \beta(q_n^2(0))] < \infty.$$

Then the function $P(z)$ has order bounded by $\alpha(r)$.

(b) Let $\sum \frac{|a_n|}{b_{n-1}} < \infty$. Assume also that either $b_{n-1}b_{n+1} \leq b_n^2$ for any n large enough or $b_{n-1}b_{n+1} \leq b_n^2$ for any such n. Let the function $P(z)$ have the order bounded by $\alpha(r)$.

(i) If there is $0 < \alpha < 1$ so that $\alpha(r) \geq r^\alpha$ for large r, then

$$\beta(p_n^2(0)) + \beta(q_n^2(0)) \leq \frac{C \log n}{n}.$$

(ii) If $\alpha(r^2) \leq O(\alpha(r))$ when $r \to \infty$, then

$$\beta(p_n^2(0)) + \beta(q_n^2(0)) \leq \frac{C}{n}.$$

In particular for any $\varepsilon > 0$

$$\sum_{n=0}^{\infty} [\beta^{1+\varepsilon}(p_n^2(0)) + \beta^{1+\varepsilon}(q_n^2(0))] < \infty.$$

The following functions satisfy the assumptions (i) ($\alpha(r) \geq r^\alpha$)

$$\alpha(r) = r^\alpha,\ (0 < \alpha < 1), \qquad \alpha(r) = \frac{r}{\log^\alpha r}.$$

On the other hand the functions

$$\alpha(r) = \log^\alpha r, \quad \alpha(r) = \log^\alpha \log r$$

satisfy (ii) $\alpha(r^2) \leq O(\alpha(r))$.

The method goes along similar lines as in the proof of the previous theorem.

Where does the dual function come from ?

The key element is to find an analog of Proposition 1.

Proposition. 1. *Let* $0 < \alpha < 1$ *and* $\{u_n\}_{n=1}^{\infty}$ *be a sequence of positive numbers such that*

$$K := \frac{1}{\alpha} \sum_{n=1}^{\infty} u_n^{\alpha} < \infty.$$

Then for $r > 0$

$$\prod_{n=1}^{\infty} (1 + u_n r) \leq \exp(K r^{\alpha}).$$

Proof.

$$1 + u_n r \leq (1 + u_n^{\alpha} r^{\alpha})^{\frac{1}{\alpha}} \leq \exp(\tfrac{1}{\alpha} u_n^{\alpha} r^{\alpha}).$$

$\diamondsuit$

For example let $u_n = 2^{-n}$. Find an estimate for

$$\prod_{n=1}^{\infty}\left(1+\frac{r}{2^n}\right)$$

$$\prod_{n=1}^{\infty}\left(1+\frac{r}{2^n}\right) \le C_k\, e^{k(\log r)^2}$$

where

$$k > \frac{1}{2\log 2}.$$

Lemma. 1. *Let* $\{u_n\}_{n=1}^{\infty}$ *be a sequence of positive numbers such that* $u_n \to 0$*. For any number* $r \geq 1$ *let* $N_r = \#\{n : u_n \geq 1/r\}$.

(a) Assume $\sum_{n=1}^{\infty} \beta(u_n) < \infty$*. Then* $N_r \leq K\alpha(r)$ *for some constant* K.

(b) Assume $N_r \leq K\alpha(r)$ *for some constant* K*. Then for any* $\varepsilon > 0$

$$\sum_{n=1}^{\infty} \beta(u_n)^{1+\varepsilon} < \infty.$$

Proof Let v_n be the nonincreasing rearrangment of the sequence u_n. Then

$$N_r = \#\{n : v_n \ge r^{-1}\}.$$

Since $\beta(r)$ is increasing we have

$$N_r \le \#\{n : \beta(v_n) \ge \beta(r^{-1})\}.$$

(a) We have $\sum_{n=1}^{\infty} \beta(v_n) < \infty$, hence $n\beta(v_n) \to 0$. Thus $n\beta(v_n) \le K$ for a constant K, i.e.

$$\beta(v_n) \le K/n.$$

We obtain

$$N_r \le \# \left\{n : \frac{K}{n} \ge \beta(r^{-1})\right\} = \# \left\{n : n \le \frac{K}{\beta(r^{-1})}\right\}$$
$$= \# \{n : n \le K\alpha(r)\} \le K\alpha(r).$$

(b) Assume $N_r \leq K\alpha(r)$, where

$$N_r = \#\{k : v_k \geq r^{-1}\}.$$

Observe that $N_{v_n^{-1}} \geq n$, because

$$v_1, v_2, \ldots, v_n \geq (v_n^{-1})^{-1}.$$

Hence $n \leq N_{v_n^{-1}} \leq K\alpha(v_n^{-1})$, i.e.

$$\beta(v_n) = \frac{1}{\alpha(v_n^{-1})} \leq \frac{K}{n}.$$

Therefore

$$\sum_{n=0}^{\infty} \beta(u_n)^{1+\varepsilon} = \sum_{n=0}^{\infty} \beta(v_n)^{1+\varepsilon} < \infty.$$

$\diamondsuit$

Lemma. 2. *Assume* $\sum_{n=0}^{\infty} \beta(u_n) < \infty$. *Then*

$$\log \prod_{n=1}^{\infty} (1+ru_n) \leq N_r[\log r+C]+\alpha(r) \sum_{n=1}^{\infty} \beta(u_n),$$

where $C = \max\{\log(2u_n)\}$ *and*
$N_r = \#\{n : u_n \geq r^{-1}\}$.

By Lemma 1 we have $N_r \leq K\alpha(r)$. Thus Lemma 2 gives

$$\log \prod_{n=1}^{\infty} (1 + ru_n) \leq K\alpha(r) \log r$$

or

$$\prod_{n=1}^{\infty} (1 + ru_n) \leq e^{K\alpha(r) \log r}.$$

Proof. Let $A_r = \{n : u_n \geq r^{-1}\}$. For $n \in A_r$ we have $ru_n \geq 1$ hence

$$\begin{aligned}\log(1 + ru_n) &\leq \log 2ru_n \\ &= \log r + \log(2u_n) \leq \log r + C.\end{aligned}$$

Furthermore for $n \notin A_r$ we have $u_n < r^{-1}$ hence

$$ru_n = \frac{u_n}{r^{-1}} \leq \frac{\beta(u_n)}{\beta(r^{-1})} = \alpha(r)\beta(u_n).$$

because $x/\beta(x)$ is nondecreasing. Thus

$$\begin{aligned}&\log \prod_{n=1}^{\infty} (1 + ru_n) \\ &= \sum_{n \in A_r} \log(1 + ru_n) + \sum_{n \notin A_r} \log(1 + ru_n) \\ &\leq N_r[\log r + C] + \sum_{n \notin A_r} \alpha(r)\beta(u_n) \\ &\leq N_r[\log r + C] + \alpha(r) \sum_{n=1}^{\infty} \beta(u_n).\end{aligned}$$

$\diamondsuit$

By analyzing the proof of Theorem 2 we can obtain the following.

Corollary 1. *Let*

$$\sum_{n=1}^{\infty} \frac{|a_n|}{b_{n-1}} < \infty.$$

Assume also that either $b_{n-1}b_{n+1} \leq b_n^2$ *for any* n *large enough or* $b_{n-1}b_{n+1} \leq b_n^2$ *for any such* n.

(i) If the sequence $1/\alpha(b_n)$ *is summable, the function* $P(z)$ *has the order bounded by* $\alpha(r)$.

(ii) If the function $P(z)$ *has the order bounded by* $\alpha(r)$ *then*

$$\frac{1}{\alpha(b_n)} = O(\log n/n).$$

This enables us to determine the growth of the function $P(z)$ just by the behaviour of the coefficients b_n.

Example Let $b_n = q^{n^{1/\alpha}}$, with $q > 1$ and let $|a_n| \le q^{\frac{1}{2}n^{1/\alpha}}$. Then for

$$\alpha(r) = \log^{\alpha+\varepsilon} r$$

we have

$$\frac{1}{\alpha(b_n)} = \frac{C}{n^{1+\varepsilon/\alpha}}.$$

Hence

$$P(z) \le C \exp(K \log^{\alpha+1+\varepsilon} |z|).$$

Since ε is arbitrary the logarithmic order of the function $P(z)$ is not greater than $\alpha + 1$. We will show that $\alpha + 1$ is the logarithmic order of $P(z)$.

Assume that the logarithmic order of $P(z)$ is equal $\gamma + 1$. Then

$$P(z) \leq C \exp(K \log^{\gamma+1+\varepsilon} |z|),$$

for some constants C and K. Let

$$\alpha(r) = \log^{\gamma+\varepsilon} r.$$

By Corollary 1(ii) we get that the sequence

$$\frac{1}{\alpha(b_n)} = \frac{1}{\log q} n^{-(\gamma+\varepsilon)/\alpha}, \qquad b_n = q^{n^{1/\alpha}},$$

is summable after raising to any power greater than 1. Thus $\gamma + \varepsilon \geq \alpha$ for any $\varepsilon > 0$, i.e. $\gamma \geq \alpha$. Hence the logarithmic order is at least $\alpha + 1$.

Definition. *We say that the function $P(z)$ has double logarithmic order equal to α if*

$$P(z) \le C \exp\{K \log^{\alpha+\varepsilon}(\log r) \log r\}, \quad |z| \le r,$$

for any $\varepsilon > 0$ and r large enough.

Example. Let $b_n = q^{q^{n^{1/\alpha}}}$, with $q > 1$ and let $|a_n| \le q^{\frac{1}{2}q^{n^{1/\alpha}}}$. Similarly as in the previous example we can show the double logarithmic order of $P(z)$ is equal α.

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2. C. Berg, R. Szwarc, Symmetric moment problems and a conjecture of Valent, *Mat. Sb.* **208** (2017) 28-53.