# A two parameter extension of the Urbanik semigroup

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# The extended Urbanik semigroup

**Theorem (Berg 2018, ArXiv 1802.00993)**

*For* $a, b > 0$
*(i)* $s_n(a,b) = \Gamma(na+b)/\Gamma(b)$ *is an infinitely divisible Stieltjes moment sequence.*
*(ii) There exists a uniquely determined convolution semigroup*[a] $(\tau_c(a,b))_{c>0}$ *of probability measures on the multiplicative group* $(0,\infty)$ *such that*

$$\int_0^\infty t^z\, d\tau_c(a,b)(t) = [\Gamma(az+b)/\Gamma(b)]^c, \quad \mathrm{Re}\, z > -b/a,$$

*and in particular* $(s_n(a,b)^c)$ *is the moment sequence of* $\tau_c(a,b)$.

[a] which we call the two parameter extension of the Urbanik semigroup

# Continuation

## Theorem

*(iii)* $\tau_c(a,b) = e_c(a,b)(t)\,dt$ *on* $(0,\infty)$, *where*

$$e_c(a,b)(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} t^{ix-1}[\Gamma(b-iax)/\Gamma(b)]^c\,dx, \quad t>0$$

*is a probability density belonging to* $C^\infty(0,\infty)$.
*(iv)* $(s_n(a,b)^c)$ *is S-determinate if and only if* $ac \le 2$, *hence independent of* $b>0$.

For $a=b=1$ we have: $\Gamma(na+b)/\Gamma(b) = n!$. This case was discussed by Urbanik in 1992, except that he did not discuss the S-determinacy.
Let us explain the concepts involved.

# Basic definitions

A Stieltjes moment sequence is a sequence of non-negative numbers of the form

$$s_n = \int_0^\infty t^n \, d\mu(t), \quad n \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}, \quad (*)$$

where $\mu$ is a positive measure on $[0, \infty)$ such that $t^n \in L^1(\mu)$ for all $n \in \mathbb{N}_0$.
$(s_n)$ is called

- normalized, if $s_0 = \mu([0, \infty)) = 1$
- S-determinate, if (*) has exactly one solution $\mu$ as positive measure on $[0, \infty)$
- S-indeterminate, if there are different solutions $\mu_1 \neq \mu_2$ to (*). Then infinitely many: $c\mu_1 + (1-c)\mu_2$, $0 < c < 1$

All these concepts go back to a fundamental memoir of Stieltjes from 1894, where the Stieltjes integral was introduced.
$\mathcal{S}$ denotes the set of Stieltjes moment sequences.

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$\mathcal{S}$ denotes the set of Stieltjes moment sequences.

# Stability properties of $\mathcal{S}$

$$(s_n), (t_n) \in \mathcal{S}, \lambda \geq 0 \implies (s_n + t_n), (s_n t_n), (\lambda s_n) \in \mathcal{S}.$$

If $(s_n), (t_n)$ are the moments of $\mu, \nu$, then

$$s_n + t_n = \int_0^\infty t^n \, d(\mu + \nu)(t), \quad s_n t_n = \int_0^\infty t^n \, d\mu \diamond \nu(t)$$

The product convolution $\mu \diamond \nu$ is the measure such that

$$\int_0^\infty f(t) \, d\mu \diamond \nu(t) = \int_0^\infty \int_0^\infty f(st) \, d\mu(s) \, d\nu(t)$$

In particular:

$$(s_n) \in \mathcal{S} \implies (s_n^2), (s_n^3), \ldots \in \mathcal{S}.$$

$(s_n) \in \mathcal{S}$ is called infinitely divisible, if $(s_n^c)$ is a Stieltjes moment sequence for any $c > 0$. (Enough for $0 < c < 1$)

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# Examples of Stieltjes moment sequences

$$1/(n+1), \quad n!, \quad (2n-1)!!, \quad C_n = \binom{2n}{n}/(n+1), \quad \frac{2^{n+1}-1}{n+1}$$

$$1/(n+1)^c = \frac{1}{\Gamma(c)} \int_0^1 t^n \log(1/t)^{c-1}\,dt, \quad c>0$$

$$n! = \int_0^\infty t^n e^{-t}\,dt, \qquad (2n-1)!! = \frac{1}{\sqrt{2\pi}} \int_0^\infty t^n t^{-1/2} e^{-t/2}\,dt$$

$$C_n = \frac{1}{2\pi} \int_0^4 t^n \sqrt{(4-t)/t}\,dt,$$ Catalan numbers

$$\frac{2^{n+1}-1}{n+1} = \int_1^2 t^n\,dt$$

The first four are infinitely divisible. The 5th not.

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# How to characterize $\mathcal{S}$ ?

To any real sequence $s = (s_n)$ attach the Hankel matrices

$$\mathcal{H}_n = \left(s_{j+k}\right)_{j,k=0}^{n}, \quad n = 0, 1, \ldots$$

The Hankel matrices attached to $(Es)_n := s_{n+1}$ are

$$\mathcal{H}'_n = \left(s_{j+k+1}\right)_{j,k=0}^{n}, \quad n = 0, 1, \ldots$$

**Theorem (Stieltjes 1894)**

*A real sequence $s = (s_n)$ belongs to $\mathcal{S}$ iff all $\mathcal{H}_n, \mathcal{H}'_n$ are positive semidefinite.*
*$(s_n) \in \mathcal{S}$ with a representing measure with infinite support iff all $\mathcal{H}_n, \mathcal{H}'_n$ are positive definite,*
*iff*

$$\det \mathcal{H}_n > 0, \quad \det \mathcal{H}'_n > 0, \quad n = 0, 1, \ldots$$

# Stieltjes moment sequences of exponential growth

**Theorem**

*For $(s_n) \in \mathcal{S}$ the following are equivalent:*
*(i) There exist $a \geq 0, C > 0$ such that $s_n \leq Ca^n$, $n \geq 0$*
*(ii) There exists a positive measure $\mu$ on $[0, a]$ such that*

$$s_n = \int_0^a t^n \, d\mu(t), n \geq 0$$

*If (i) and (ii) hold, then $(s_n)$ is S-determinate.*

**Proof.**

$(ii) \implies (i)$ is easy. Next assume (i) and $\varepsilon > 0$. Then

$$Ca^n \geq s_n = \int_0^\infty t^n \, d\mu(t) \geq \int_{a+\varepsilon}^\infty t^n \, d\mu(t) \geq (a+\varepsilon)^n \mu([a+\varepsilon, \infty[)$$

hence $\mu([a+\varepsilon, \infty[) = 0$. Next apply Weierstrass' Theorem. $\square$

# Carleman's sufficient condition for S-determinacy, 1922

Assume $(s_n) \in \mathcal{S}$ and that

$$\sum_{n=0}^{\infty} \frac{1}{\sqrt[2n]{s_n}} = \infty, \qquad (C)$$

then $(s_n)$ is S-determinate.

Note that if $s_n \leq Ca^n$, then $\sqrt[2n]{s_n} \leq 2\sqrt{a}$ for $n$ large, so the Carleman series clearly diverges.
Carleman says: $(s_n)$ of moderate growth is S-determinate

**Warning** There are S-determinate $(s_n)$ such that (C) does not hold

# Krein's sufficient condition for S-indeterminacy, 1945

Let $(s_n) \in \mathcal{S}$ have a representing measure $\mu = \varphi(t)\,dt$ on $[0, \infty[$ with a density $\varphi$. If there exists a constant $C > 0$ such that

$$\int_C^\infty \frac{\log \varphi(t^2)}{1+t^2}\,dt > -\infty, \qquad \text{(Krein)}$$

then $(s_n)$ is S-indeterminate.

**Warning** There are S-indeterminate $(s_n)$ such that (Krein) does not hold

For $\alpha > 0$ consider

$$s_n(\alpha) = \frac{1}{\Gamma(1+1/\alpha)} \int_0^\infty t^n e^{-t^\alpha}\, dt = \frac{\Gamma((n+1)/\alpha)}{\alpha\Gamma(1+1/\alpha)}.$$

**Claim:** $(s_n(\alpha))$ is S-determinate iff $\alpha \geq 1/2$.

Stirling: $\Gamma(z) \sim \sqrt{2\pi} z^{z-1/2} e^{-z}$,

$$\sqrt[2n]{s_n(\alpha)} \sim \left(\frac{n+1}{\alpha e}\right)^{1/(2\alpha)}, \quad \text{(C) holds iff } \alpha \geq 1/2.$$

$$\int_C^\infty \frac{-(t^2)^\alpha}{1+t^2}\, dt > -\infty \;\text{ iff } 2-2\alpha > 1 \;\text{ iff } \alpha < 1/2.$$

# The log-normal moments–Stieltjes' approach

$0 < q < 1$: log-normal moments are $s_n = q^{-n(n+2)/2}$ given by

$$\frac{\sqrt{q}}{\sqrt{2\pi \log(1/q)}} \int_0^\infty x^n \exp\left(-\frac{(\log x)^2}{2\log(1/q)}\right) dx.$$

Defining

$$h(x) = \sin\left(\frac{2\pi}{\log(1/q)} \log x\right)$$

then the non-negative densities $(-1 \le r \le 1)$

$$\frac{\sqrt{q}}{\sqrt{2\pi \log(1/q)}} \exp\left(-\frac{(\log x)^2}{2\log(1/q)}\right) [1 + rh(x)]$$

all have the log-normal moments.

# Correspondence between Stieltjes and Hermite

Stieltjes to Hermite: January 30, 1892

"L'existence de ces fonctions $\varphi(x)$ qui, sans être nulles, sont telles que

$$\int_0^\infty x^n \varphi(x)\, dx = 0, \quad n = 0, 1, \dots,$$

me paraît très remarquable"

$$\varphi(x) = \sin\left(\frac{2\pi}{\log(1/q)} \log x\right) \exp\left(-\frac{(\log x)^2}{2\log(1/q)}\right)$$

is one of these functions.

# Hausdorff moment sequences

The bounded Stieltjes moment sequences are called Hausdorff moment sequences.

They are the moment sequences of the form

$$a_n = \int_0^1 t^n \, d\mu(t), \quad n \geq 0.$$

Hausdorff (1923) characterized the Stieltjes moment sequences for which the measure is concentrated on the unit interval $[0,1]$ by complete monotonicity, i.e.

$$(-1)^n \Delta^n s_k = \sum_{j=0}^{n} (-1)^j \binom{n}{j} s_{k+j} \geq 0$$

for all $n, k \geq 0$, where

$$\Delta a_k = a_{k+1} - a_k.$$

A Hausdorff moment sequence is either non-vanishing (i.e. $a_n > 0$ for all $n$) or of the form $a_n = (c, 0, 0, \ldots)$ with $c \geq 0$.

# A strange theorem inspired by mathematical finance

**Theorem (Berg-Durán 2004)**

*Let $(a_n)$ be a non-vanishing Hausdorff moment sequence. Then*

$$s_0 = 1, \quad s_n = 1/(a_1 a_2 \ldots a_n), n \geq 1$$

*is a normalized Stieltjes moment sequence.*

The proof is rather constructive:

• We find explicitly a Stieltjes measure for those sequences $(s_n)$, which are defined from the Hausdorff moment sequence of a finite linear combination of Dirac deltas.

• We use that the set of finite linear combinations of Dirac deltas is dense in the set of positive measures supported in $[0, 1]$.

# Completely monotonic functions–a continuous analogue

A function $\varphi : ]0, \infty[ \to \mathbb{R}$ is called completely monotonic denoted $\mathcal{C}$, if it is $C^\infty$ and

$$(-1)^n \varphi^{(n)}(s) \geq 0, \quad s > 0, n = 0, 1, 2, \dots$$

By a famous theorem of Bernstein from 1926, the $\mathcal{C}$ functions are precisely the functions

$$\varphi(s) = \int_0^\infty e^{-sx}\, d\sigma(x), \quad \sigma \geq 0.$$

Note that $\varphi(0^+) = \sigma([0, \infty[)$. If $\sigma([0, \infty[) < \infty$, then $a_n := \varphi(n)$ is a Hausdorff moment sequence of a measure $\mu$ given as the image measure of $\sigma$ under $x \to e^{-x}$ to get

$$\varphi(n) = \int_0^1 u^n\, d\mu(u).$$

The converse is almost true: Any Hausdorff moment sequence with representing measure $\mu$ such that $\mu(\{0\}) = 0$ has the form $\varphi(n)$.

# An equivalent formulation

**Theorem (Berg-Durán (2004))**

*Let $\varphi$ be a non-zero completely monotonic function. Then $(s_n)$ defined by $s_0 = 1$ and $s_n = 1/(\varphi(1) \cdot \ldots \cdot \varphi(n))$ for $n \geq 1$ is a normalized Stieltjes moment sequence.*

**Remark**

*The evaluation of $\varphi$ at the integers can be replaced by the evaluation at the sequence $p + nq, n = 1, 2, \ldots,$ where $p \geq 0, q > 0$ are real numbers. The conclusion is that $s_0 = 1$, $s_n = 1/(\varphi(p+q) \cdot \ldots \cdot \varphi(p+nq)), n \geq 1$ is a normalized Stieltjes moment sequence.*

# An example–coming back to log-normal moments

Let $0 < q < 1$. Then $a_n = q^n$ is a Hausdorff moment sequence. This implies that

$$s_n = 1/(qq^2q^3 \dots q^n) = q^{-n(n+1)/2}$$

is a Stieltjes moment sequence, and so is $s_n = a^n q^{-n(n+1)/2}$ for any $a > 0$.
For $a = q^{-1/2}$ we get the lognormal moments

$$s_n = q^{-n(n+2)/2}.$$

# Bernstein functions

A continuous function $f : [0, \infty[ \to [0, \infty[$ is called a Bernstein function if it is $C^\infty(]0, \infty[)$ and $f'$ is completely monotonic. Equivalently, Bernstein functions have the integral representation

$$f(s) = a + bs + \int_0^\infty (1 - e^{-sx})\, d\nu(x), \quad (*)$$

$a, b \geq 0$ and the Lévy measure $\nu$ on $]0, \infty[$ satisfies

$$\int \frac{x}{1+x}\, d\nu(x) < \infty.$$

A very simple example of a Bernstein function

$$f(s) = s$$

# Important properties of Bernstein functions

$\mathcal{B} := \{\text{Bernstein functions}\}, \quad \mathcal{C} := \{\text{completely monotonic functions}\}$

$$f \in \mathcal{B}, f \neq 0 \quad \Longrightarrow \begin{cases} e^{-tf(s)} \in \mathcal{C}, & t > 0 \\ f(s)/s \in \mathcal{C} \\ 1/f(s) \in \mathcal{C} \end{cases}$$

First assertion: Schoenberg's Theorem
Second assertion:

$$f(s)/s = a/s + b + \int_0^\infty (1 - e^{-sx})/s \, d\nu(x).$$

$$(1 - e^{-sx})/s = \int_0^x e^{-st} \, dt \ \in \mathcal{C}$$

Third assertion:

$$1/f(s) = \int_0^\infty e^{-tf(s)} \, dt.$$

# Bernstein functions and convolution semigroups

A convolution semigroup of sub-probability measures on $[0, \infty[$ is a family $(\eta_t)_{t>0}$ of positive measures on $[0, \infty[$ with $\eta_t([0, \infty[) \leq 1$, $\eta_t * \eta_s = \eta_{t+s}$ and $\eta_t \to \delta_0$ weakly for $t \to 0$.
There is a one-to-one correspondence between such convolution semigroups and Bernstein functions $f$ established via Laplace transformation by

$$\int_0^\infty e^{-sx}\, d\eta_t(x) = \exp(-tf(s)), \quad t > 0, s \geq 0.$$

In particular $\eta_t([0, \infty[) = \exp(-tf(0))$, so convolution semigroups of probability measures correspond to Bernstein functions satisfying $f(0) = 0$.
We shall now see the relation between Bernstein functions and Stieltjes moment sequences and later their relation to convolution semigroups for the multiplicative structure of positive real numbers.

# Two special cases of the theorem of Berg-Durán

Using the completely monotonic function $f(s)/s$:

**Corollary (Carmona, Petit, Yor (1994,1997), Urbanik (1992))**

*Let $f$ be a non-zero Bernstein function. Then*

$$s_n = n!/(f(1)\cdot\ldots\cdot f(n)),\ n \geq 1$$

*is an S-determinate Stieltjes moment sequence.*

Using the completely monotonic function $1/f(s)$:

**Corollary (Bertoin, Yor (2001))**

*Let $f$ be a non-zero Bernstein function. Then*

$$s_n = f(1)\cdot\ldots\cdot f(n),\ n \geq 1$$

*is an S-determinate Stieltjes moment sequence of a probability measure $\rho = \rho_f$.*

# Back to Infinitely divisible Stieltjes moment sequences

$$f \neq 0, f \in \mathcal{B} \quad \Longrightarrow \quad (1/f)^c \in \mathcal{C}, \quad c > 0.$$

Clear for $c \in \mathbb{N}$. For $c > 0, c \notin \mathbb{N}$ write $c = n + c'$ with $n \in \mathbb{N}_0, 0 < c' < 1$ so

$$(1/f)^c = (1/f)^n \cdot 1/(f^{c'}) \in \mathcal{C}$$

because

$$f \in \mathcal{B}, \quad 0 < c' < 1 \quad \Longrightarrow \quad f^{c'} \in \mathcal{B}.$$

By Berg-Durán we get:

**Corollary**

*For any non-zero Bernstein function $f$, the sequence*

$$s_n = (f(1) \cdot \ldots \cdot f(n))^c$$

*is a Stieltjes moment sequence for any $c > 0$, i.e. $f(1) \cdot \ldots \cdot f(n)$ is infinitely divisible.*

# A simple example

$f(s) = s$ is a Bernstein function.

Then $s_n = n!$ is an infinitely divisible Stieltjes moment sequence. By Carleman's criterion: $(n!)^c$ is S-determinate for $c \le 2$ because

$$\frac{1}{\sqrt[2n]{(n!)^c}} \approx \frac{1}{n^{c/2}}.$$

That $c = 2$ is sharp, i.e. $(n!)^c$ is S-indeterminate for $c > 2$ was an open question but settled first by Berg (2005) based on asymptotic results for one-sided stable distributions.

Later by Berg-López (2015) based on direct asymptotic analysis and the Krein criterion.

# Harmonic analysis of a special locally compact abelian group

$G = (0, \infty)$ is a locally compact abelian group under multiplication.
For each $x \in \mathbb{R}$:

$$j_x(t) := t^{ix} \text{ satisfies } j_x(t \cdot s) = j_x(t) \cdot j_x(s), \quad |j_x(t)| \equiv 1, t > 0,$$

i.e. $j_x$ is a continous character of the group $G$.
Each continous character of $G$ has this form for precisely one $x \in \mathbb{R}$, and in this way the dual group $\widehat{G}$ of continuous characters of $G$ can be identified with the additive group of real numbers.
The convolution between measures $\mu, \sigma$ on the group $G = (0, \infty)$, called product convolution and denoted $\mu \diamond \sigma$, is defined as

$$\int_0^\infty f(t)\, d\mu \diamond \sigma(t) = \int_0^\infty \int_0^\infty f(ts)\, d\mu(t)\, d\sigma(s)$$

for suitable classes of continuous functions $f$ on $(0, \infty)$, e.g. those of compact support.

# The Mellin transformation

The Mellin transform of a finite measure $\mu$ on $G = (0, \infty)$ is the function from $\mathbb{R}$ to $\mathbb{C}$

$$\mathcal{M}(\mu)(x) = \int_0^\infty t^{-ix}\, d\mu(t), \quad x \in \mathbb{R}.$$

A family $(\mu_c)_{c>0}$ of probability measures on the multiplicative group $G = (0, \infty)$ is called a convolution semigroup, if

$$\mu_c \diamond \mu_d = \mu_{c+d},\ c, d > 0, \quad \lim_{c \to 0} \mu_c = \varepsilon_1 \text{ weakly.}$$

Here $\varepsilon_1$ is the Dirac measure with total mass 1 concentrated in the neutral element 1 of the group.
The Mellin transform maps convolutions into products:

$$\mathcal{M}(\mu_c \diamond \mu_d)(x) = \mathcal{M}(\mu_c)(x)\mathcal{M}(\mu_d)(x).$$

# Continuation

Given a convolution semigroup $(\mu_c)_{c>0}$ on $(0,\infty)$, it is easy to see that if $\mu_1$ has moments of order $n$, then all the measures $\mu_c$ have moments of order $n$ and

$$\int_0^\infty t^n \, d\mu_c(t) = \left( \int_0^\infty t^n \, d\mu_1(t) \right)^c, \quad c > 0.$$

Therefore, if a probability measure $\mu$ on $G = (0,\infty)$ can be embedded as $\mu_1$ in a convolution semigroup $(\mu_c)_{c>0}$ of probability measures on $G$, then the Stieltjes moment sequence

$$s_n = \int_0^\infty t^n \, d\mu_1(t)$$

is infinitely divisible and

$$s_n^c = \int_0^\infty t^n \, d\mu_c(t), \quad c > 0, \, n = 0, 1, \ldots .$$

The converse is almost true and made precise in Berg (2007).

# A new result generalizing $(n!)^c$–back to first slide

**Theorem (Berg 2018, ArXiv 1802.00993)**

*For* $a, b > 0$
*(i)* $s_n(a,b) = \Gamma(na+b)/\Gamma(b)$ *is an infinitely divisible Stieltjes moment sequence.*
*(ii) There exists a uniquely determined convolution semigroup* $(\tau_c(a,b))_{c>0}$ *of probability measures on the multiplicative group* $(0,\infty)$ *such that*

$$\int_0^\infty t^z \, d\tau_c(a,b)(t) = [\Gamma(az+b)/\Gamma(b)]^c, \quad \operatorname{Re} z > -b/a,$$

*and in particular* $(s_n(a,b)^c)$ *is the moment sequence of* $\tau_c(a,b)$*.*

# Continuation

**Theorem**

*(iii)* $\tau_c(a,b) = e_c(a,b)(t)\,dt$ *on* $(0,\infty)$, *where*

$$e_c(a,b)(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} t^{ix-1}[\Gamma(b-iax)/\Gamma(b)]^c\,dx, \quad t>0$$

*is a probability density belonging to* $C^\infty(0,\infty)$.
*(iv)* $(s_n(a,b)^c)$ *is S-determinate if and only if* $ac \le 2$, *hence independent of* $b>0$.

For $b=1, a\in\mathbb{N}$ we have: $\Gamma(na+b)/\Gamma(b) = (na)!$. The results of the Theorem in these special cases were proved by G.D. Lin (2017), and he conjectured the general results. See G. D. Lin, *On powers of the Catalan number sequence*. ArXiv:1711.01536.

# Asymptotic results at infinity

The proof of the characterization of the S-indeterminacy is based on

**Theorem**

*For $c > 0$ we have for $t \to \infty$*

$$e_c(a,b)(t) = \frac{(2\pi)^{(c-1)/2}}{a\sqrt{c}\Gamma(b)^c} \frac{\exp(-ct^{1/(ac)})}{t^{1-(b-1/2+1/(2c))/a}} \left[1 + \mathcal{O}\left(t^{-1/(ac)}\right)\right].$$

This is a delicate asymptotic analysis along the lines of the paper by Berg-López.

For proofs of the asymptotic results and the main theorem see C. Berg, *A two-parameter extension of the Urbanik semigroup*, ArXiv :1802.00993.

# Asymptotic results at zero

## Theorem

*For $c > 0$ and $0 < t < 1$ we have for $t \to 0$*

$$e_c(a,b)(t) = \frac{t^{b/a-1}}{[a\Gamma(b)]^c} \frac{[\log(1/t)]^{c-1}}{\Gamma(c)} + \mathcal{O}\left(t^{b/a-1}[\log(1/t)]^{c-2}\right).$$

## Remark

The formula shows that $e_c(a,b)(t)$ tends to 0 for $t \to 0$ if $b/a > 1$, and to infinity if $b/a < 1$, independent of $c$. If $b/a = 1$ then $e_c(a,b)(t)$ tends to 0 for $c < 1$ and to infinity as a power of $\log(1/t)$ when $c > 1$.

Thank you for your attention