Quantum Theory Group

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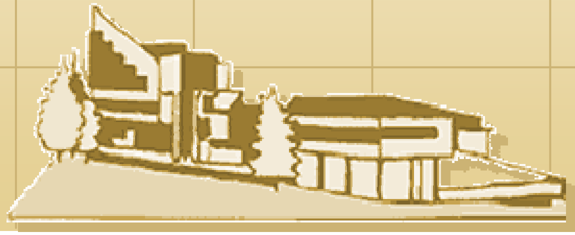
Main lines of research

- Qualification and quantification of entanglement in continuous variable systems
- Statics and dynamics of information in quantum spin systems
- Production of entangled states of atomic samples and multiphoton systems

PISA, December 16, 2004



Entanglement Scaling, Localization and Sharing in Continuous Variable Systems

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in collaboration with

Gerardo Adesso
Alessio Serafini

Outline

- Gaussian states
  of continuous variable (CV) systems
- Entanglement and purities
- Unitary localization and scaling
  of multimode bipartite entanglement
- Genuine multipartite entanglement:
  the continuous variable tangle
- Sharing *(polygamy)* of CV entanglement
- Optimal use of entanglement for
  CV teleportation

Continuous variable systems

- Quantum systems such as harmonic oscillators, light modes, or cold bosonic gases
- Infinite-dimensional Hilbert spaces $\mathcal{H} = \bigotimes_{i=1}^{N} \mathcal{H}_i$ for *N* modes
- Quadrature operators $\hat{X} = (\hat{q}_1, \hat{p}_1, \ldots, \hat{q}_N, \hat{p}_N)$

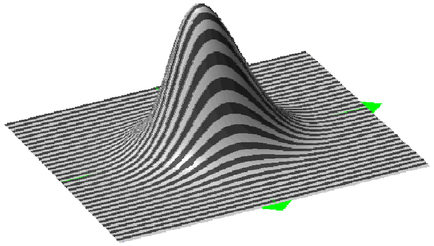
$$\hat{q}_j = \hat{a}_j + \hat{a}_j^\dagger , \quad \hat{p}_j = (\hat{a}_j - \hat{a}_j^\dagger)/i$$

- Canonical commutation relations

$$[\hat{X}_i, \hat{X}_j] = 2i\,\Omega_{ij} \ , \quad \Omega = \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & 0 & 1 & \\ & & -1 & 0 & \\ & & & & \ddots \end{pmatrix}$$

- Described in phase space by quasiprobability distributions, such as Wigner function, Glauber *P*-function, Husimi *Q*-function

Gaussian states



- *states whose Wigner function is Gaussian*
- *fully determined by*
  - Vector of first moments $\bar{X} \equiv (\langle \hat{X}_1 \rangle, \ldots, \langle \hat{X}_N \rangle)$

    (arbitrarily adjustable by local displacements)
  - Second moments encoded in the **Covariance Matrix** (CM) $\boldsymbol{\sigma}$

    $\sigma_{ij} \equiv \langle \hat{X}_i \hat{X}_j + \hat{X}_j \hat{X}_i \rangle / 2 - \langle \hat{X}_i \rangle \langle \hat{X}_j \rangle$ *(real, symmetric, 2N x 2N)*
- *Robertson-Schrödinger uncertainty principle*

$$\boldsymbol{\sigma} + i\Omega \geq 0$$

(bona fide condition for any physical CM)

- *can be realized experimentally with current technology*
- *thermal, coherent, squeezed states are Gaussian*
- *implemented in CV quantum information processes*

Phase space and symplectics

Hilbert space $\mathcal{H}$ ⟺ Phase space Γ
Density matrix ρ ⟺ Covariance matrix σ
Unitary operations U ⟺ Symplectic operations S

...so we move into phase space...

- Symplectic 'Williamson' diagonalization of a CM: *normal mode decomposition*

$$\begin{pmatrix} \sigma_1 & \varepsilon_{12} & \cdots & \varepsilon_{1N} \\ \varepsilon_{12}^T & \sigma_2 & & \varepsilon_{2N} \\ \vdots & & \ddots & \vdots \\ \varepsilon_{1N}^T & \varepsilon_{2N}^T & \cdots & \sigma_N \end{pmatrix} \xrightarrow{S} \begin{pmatrix} \nu_1 & & & & & 0 \\ & \nu_1 & & & & \\ & & \nu_2 & & & \\ & & & \nu_2 & & \\ & & & & \ddots & \\ 0 & & & & \nu_N & \\ & & & & & \nu_N \end{pmatrix}$$

the ν_i's are the **symplectic eigenvalues**

computable as the standard eigenvalues of the matrix $|i\Omega\sigma|$

⇨ determined by N symplectic invariants, including...

- Determinant $\text{Det}\,\sigma = \prod_i \nu_i^2$ *(Purity = [Det σ]* $^{-1/2}$ *)*
- Seralian $\Delta = \sum_i \nu_i^2$ *(sum of* 2x2 *sub-determinants)*

Entanglement properties

(Simon 2000) Transposition ⟺ Time reversal in phase space

Partial transposition $\boldsymbol{\sigma} \to \tilde{\boldsymbol{\sigma}}$ ⟺ Inversion of the *p* operator of a mode

PPT: $\boldsymbol{\sigma}$ *separable iff* $\tilde{\boldsymbol{\sigma}}$ *bona fide* i.e. $\tilde{\boldsymbol{\sigma}} + i\Omega \geq 0$ ***for 1xN partitions***

$\{\nu_i\}$ and $\{\tilde{\nu}_i\}$ are the symplectic eigenvalues of $\boldsymbol{\sigma}$ and $\tilde{\boldsymbol{\sigma}}$

$\nu_i \geq 1 \Leftrightarrow$ **physical** state
full saturation: *pure state*
partial saturation:
minimum-uncertainty mixed state

$\tilde{\nu}_i \geq 1 \Leftrightarrow$ **separable** state
(only $\Leftarrow$ for M x N, M>1)
violation: **entanglement**

We can compute the **logarithmic negativity** to quantify the entanglement

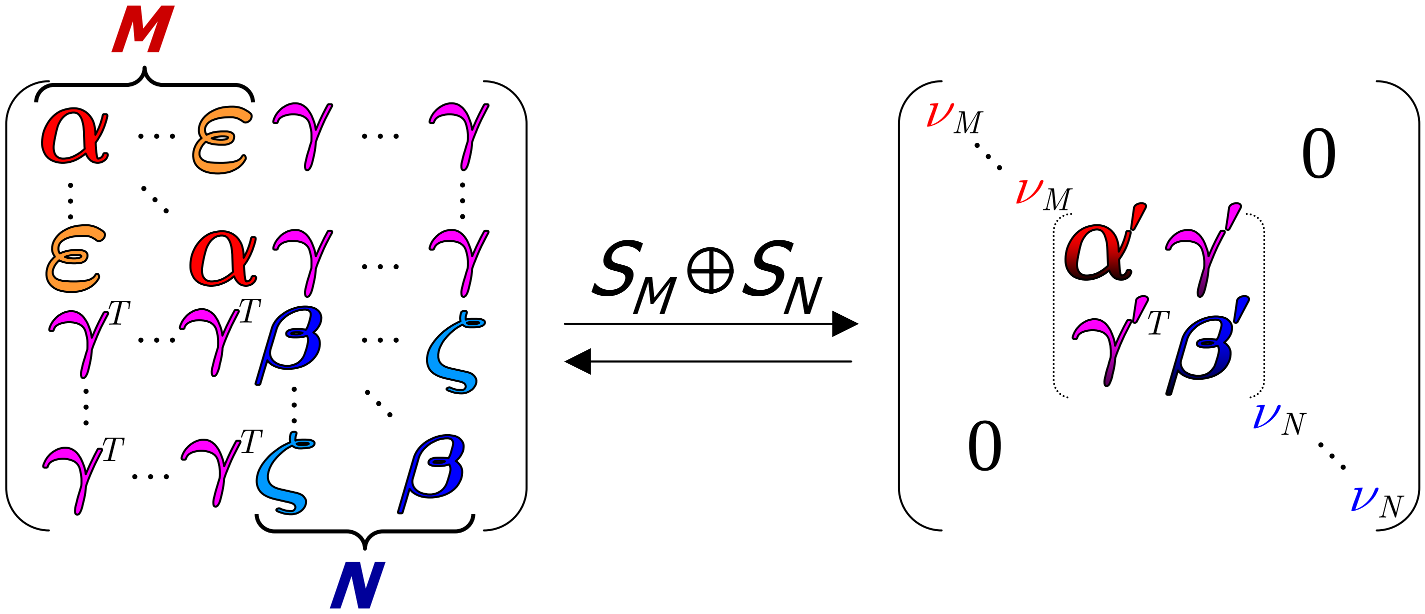
$$E_{\mathcal{N}}(\boldsymbol{\sigma}) = \begin{cases} 0, & \tilde{\nu}_i \geq 1 \; \forall \, i\,; \\ -\sum_{i:\, \tilde{\nu}_i < 1} \log \tilde{\nu}_i\,, & \text{else}\,. \end{cases}$$

The EoF is computable* for 1x1 symmetric states and it is completely equivalent

**Giedke et al., PRL 2003*

Unitary localization

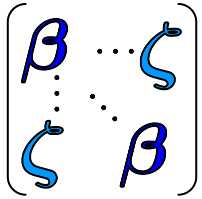
- Bisymmetric (*M*+*N*)-mode states

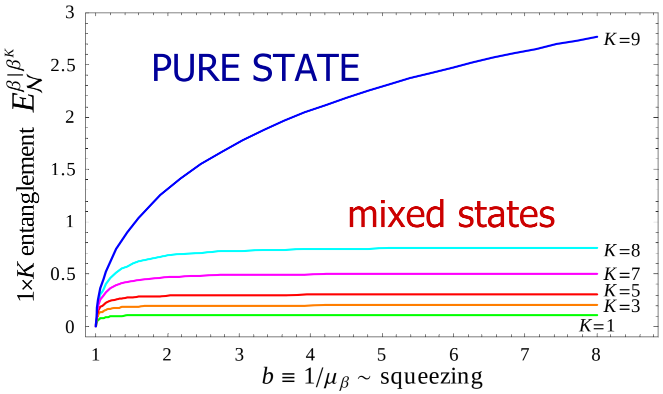


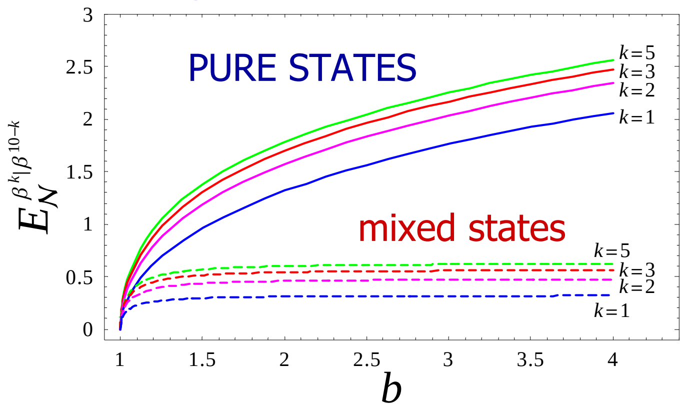
- PPT criterion holds: no bisymmetric bound entanglement
- Logarithmic negativity (also EoF if $\alpha'=\beta'$) computable
- Reversible multimode/two-mode entanglement switch

Entanglement scaling

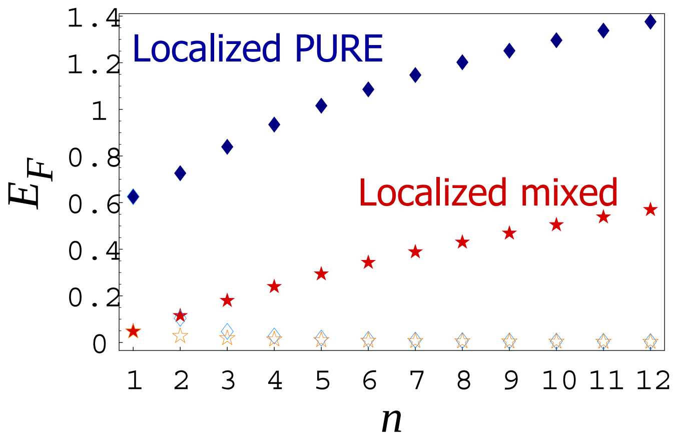
- We exploit the two-mode equivalence to investigate multimode entanglement. Example: fully symmetric *N*-mode states







- Best localization strategy: equal splitting between two parties



- Scaling with the number of modes
  - Bipartite two-mode entanglement (original) goes to zero
  - Bipartite two-mode entanglement (localized) = bipartite multimode entanglement increases (diverges only in pure states)

The Continuous Tangle

- The hierarchy of unitarily localizable bipartite entanglements gives a hint on the structure of the multipartite entanglement

what about the GENUINE 1x1x…1 entanglement?

- For 3 qubits: T[A(BC)] ≥ T[AB] + T[AC], with T: Tangle (CKW 2000)

Could the same hold for Gaussian states?... What measure?

Continuous Variable Tangle

Contangle $E_\tau \equiv (E_{\mathcal{N}})^2$

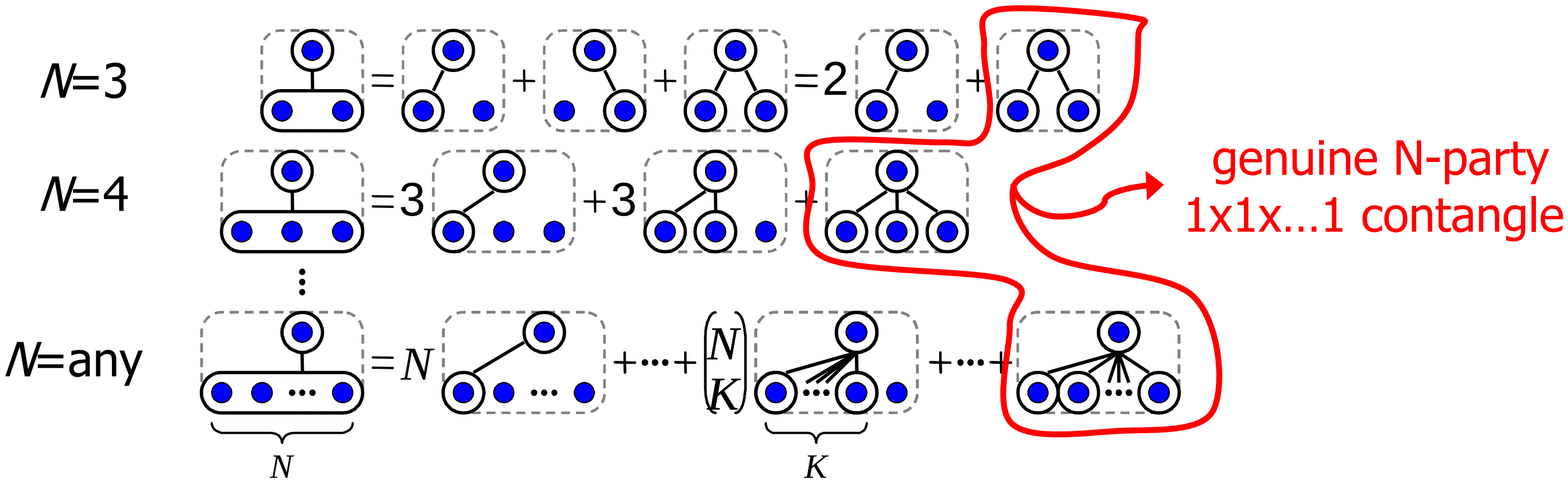
analogy with discrete systems

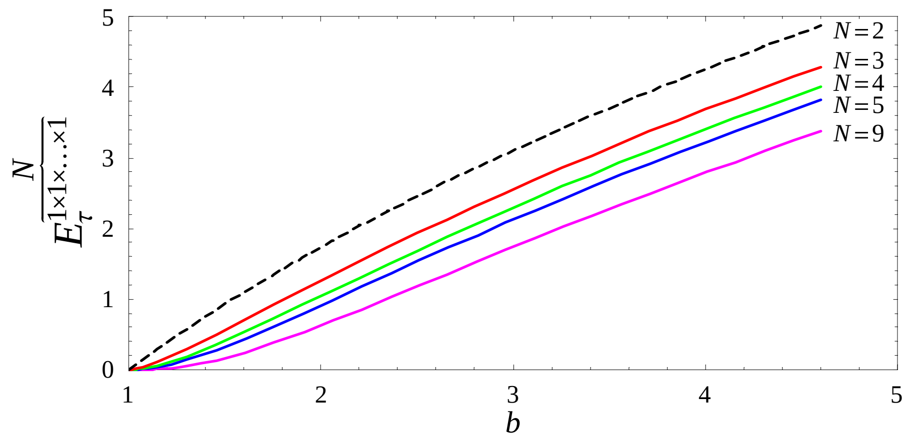
	DV	CV
bipartite	$\mathcal{C}$	$E_{\mathcal{N}}$
multipart	$\mathcal{C}^2$	$E_{\mathcal{N}}^2$

Multiparty entanglement

- Structure of multipartite entanglement

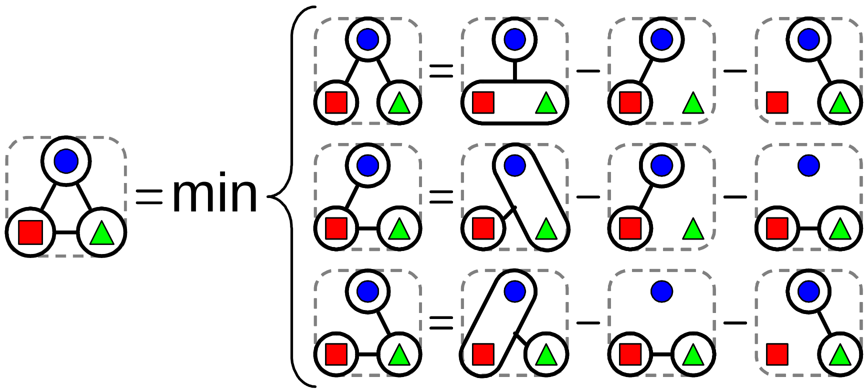
(example: fully symmetric pure *N*-mode states)

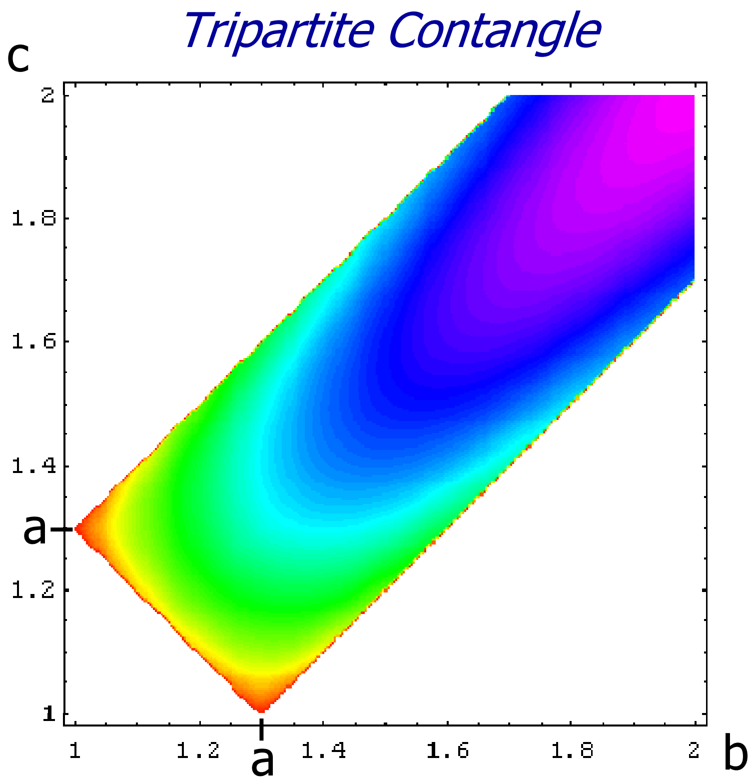




Contangle in generic states

beyond the symmetry…





- Generic three-mode pure states

only parametrized by the 3 local single-mode purities 1/a, 1/b, 1/c, with (|a-b| + 1) ≤ c ≤ (a+b-1) [triangle ineq]

Polygamous entanglement

Monogamy of quantum entanglement

3 qubits: two inequivalent families of tripartite entangled states
- **GHZ states**: no 1x1, max any 1x2 ⇨ max 1x1x1 three-tangle
- **W states**: max 1x1 between any couple ⇨ (1x2)=2(1x1)
  ⇨ zero 1x1x1 three-tangle

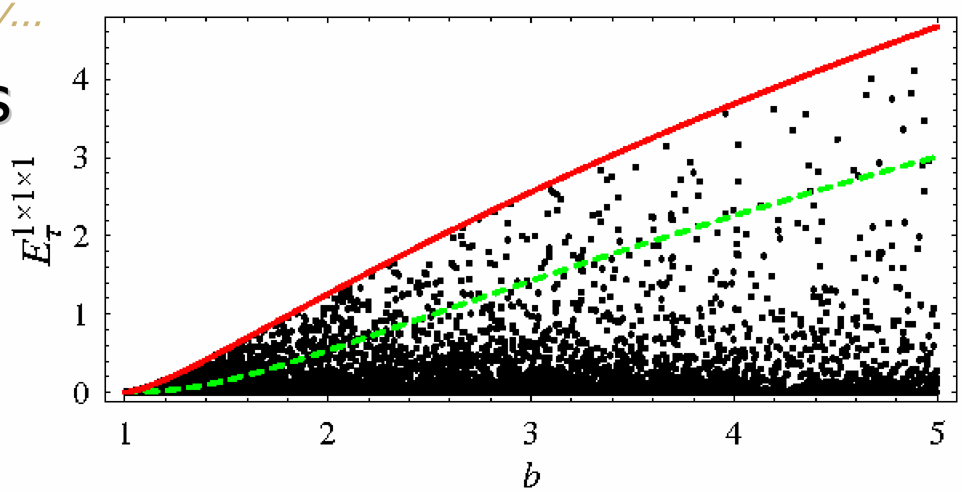
CV finite-squeezing analogy: Gaussian fully symmetric 3-mode states, based on the same bipartite properties…

W states: max 1x1, max 1x2 <u>AND max 1x1x1</u> !!! (**GHZ states**: lower 1x1x1)

…the more two-party, the more three-party…

Polygamy of CV systems

… when there is an 'harem' of infinitely many degrees of freedom available for the entanglement, its monogamy inevitably fails !

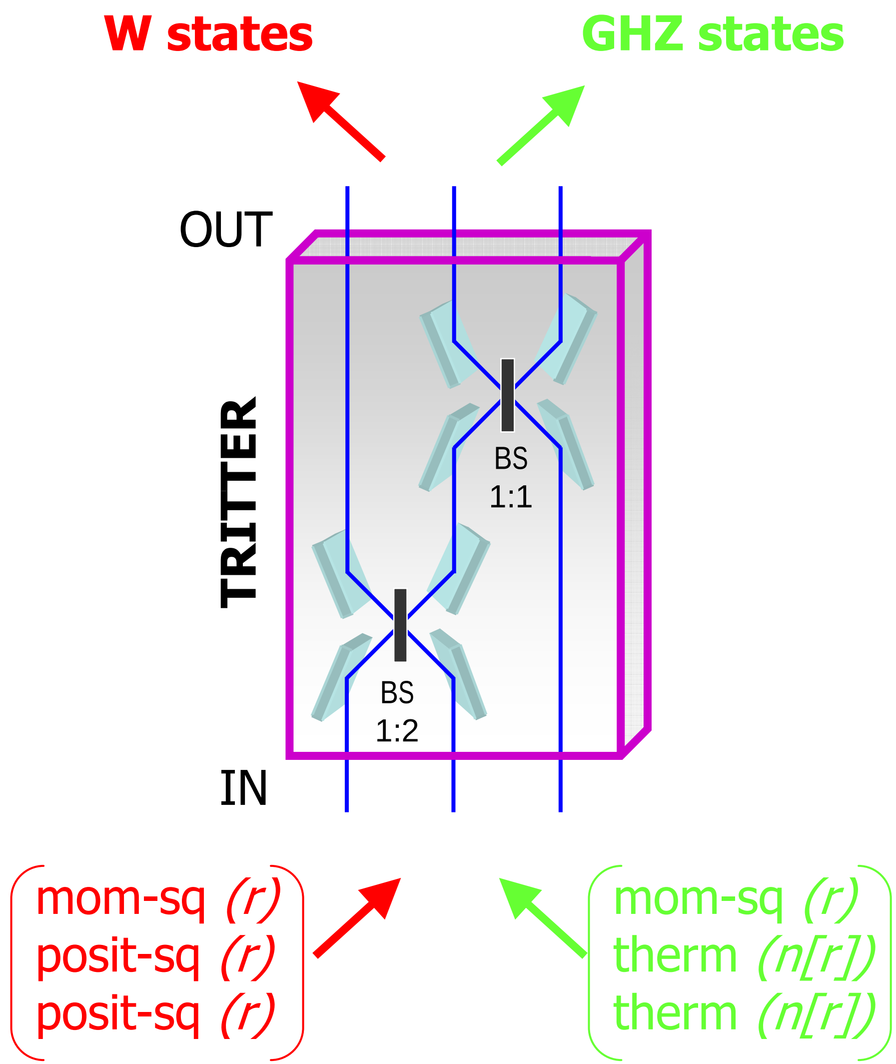


Rewind/1: W & GHZ states

- CV entanglement is polygamously shareable

  this follows by comparing the tri-contangle in the CV GHZ and W states: the latter maximize tripartite and any reduced bipartite entanglement.

- How can these states be produced?



Rewind/2: there's a multiparty

- The Contangle is a measure of genuine multipartite entanglement

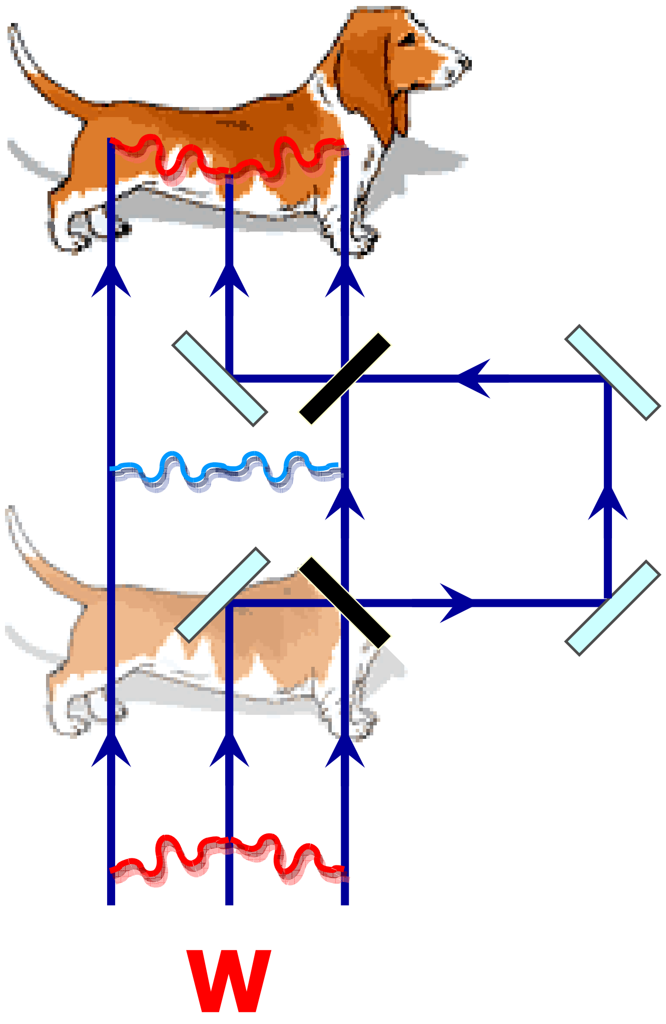
  it can be measured *e.g.* in three-mode pure states by measurements of local purities (diagonal elements of CM)

- The multimode entanglement under symmetry can be computed

  Its scaling can be investigated, and the MxN entanglement can be reversibly converted into 1x1 ('localized') by optical means.

  PPT criterion is necessary and sufficient for separability of MxN symmetric and bisymmetric Gaussian states

Rewind/3: *unitarily localizing*



When you cut the head of a basset hound…
…it will grow again!

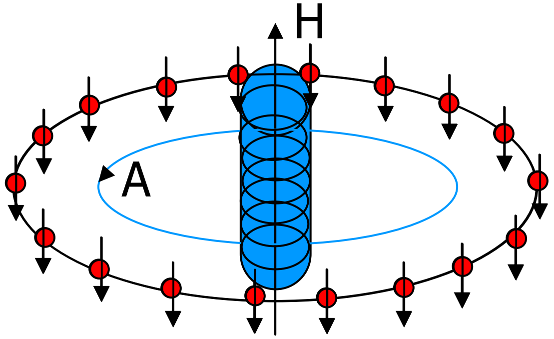
References

- Two-mode entanglement vs purity & entropic measures
  - G. Adesso, A. Serafini, and F. Illuminati, Phys. Rev. Lett. **92**, 087901 (2004)
  - G. Adesso, A. Serafini, and F. Illuminati, Phys. Rev. A **70**, 022318 (2004)
- 1xN and MxN multimode entanglement
  - G. Adesso, A. Serafini, and F. Illuminati, Phys. Rev. Lett. **93**, 220504 (2004)
  - A. Serafini, G. Adesso and F. Illuminati, quant-ph/0411109 (2004)
- Genuine multipartite entanglement
  - G. Adesso and F. Illuminati, quant-ph/0410050 (2004)
- Three-mode entanglement production and characterization
  - in preparation…
- See also the poster by Gerardo Adesso
  - Optimal use of multipartite entanglement for continuous variable teleportation

Storing massive information - 1

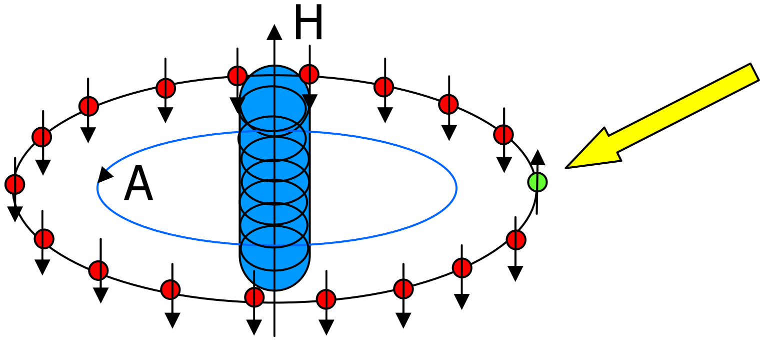
Quantum spin system on a ring with periodic boundary condition

Linked magnetic flux



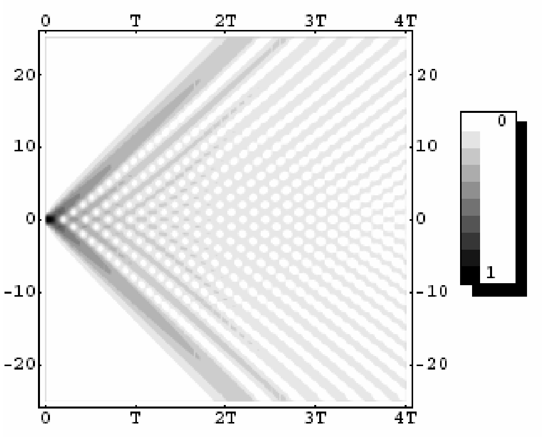
$$H = -\lambda \sum_{i=1}^{N} \left(e^{i\frac{\phi}{N}} S_i^{+} S_{i+1}^{-} + H.C \right) + B \sum_{i=1}^{N} S_i^{z}$$

Local perturbation (Spin Flip)

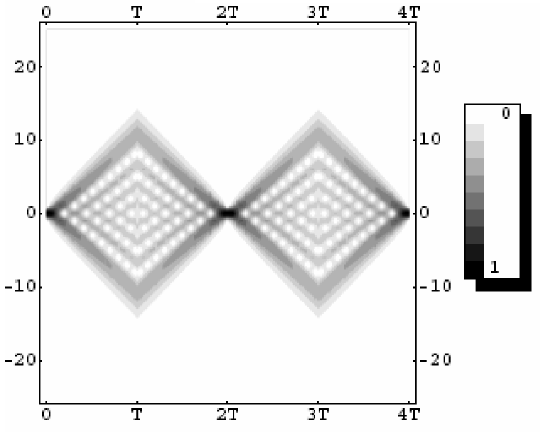


Storing massive information - 2

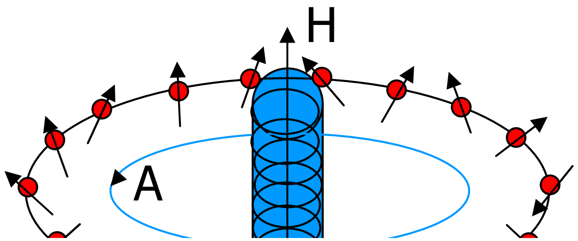
ϕ constant in time

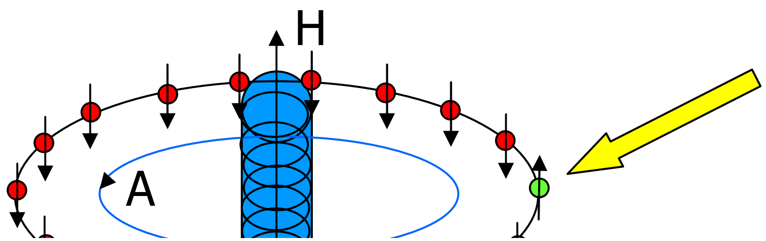


ϕ modulated: $\frac{\phi}{N} = \left[\alpha + \pi\theta\left(t - T\right)\right]$



Physical situation after a time t=2T





See also the poster by S. M. Giampaolo & A. Di Lisi

Storage of massive logical memory in a quantum spin ring with modulated magnetic flux

[redacted]

Two-mode Gaussian states

Standard form: 4 parameters

$$\boldsymbol{\sigma}_{sf} \equiv \begin{pmatrix} a & 0 & c_+ & 0 \\ 0 & a & 0 & c_- \\ c_+ & 0 & b & 0 \\ 0 & c_- & 0 & b \end{pmatrix}$$

4 symplectic invariants

local purities $\mu_1 = \frac{1}{a}, \quad \mu_2 = \frac{1}{b},$

global purity $\frac{1}{\mu^2} = \mathrm{Det}\boldsymbol{\sigma} = (\mathrm{ab})^2 - \mathrm{ab}(\mathrm{c}_+^2 + \mathrm{c}_-^2) + (\mathrm{c}_+\mathrm{c}_-)^2,$

seralian $\Delta = a^2 + b^2 + 2c_+c_- .$

Partial transposition flips the sign of c_- ⇨ $\tilde{\Delta} = a^2 + b^2 - 2c_+c_- = -\Delta + 2/\mu_1^2 + 2/\mu_2^2$

Symplectic eigenvalues : $2\nu_\mp^2 = \Delta \mp \sqrt{\Delta^2 - \frac{4}{\mu^2}}, \quad 2\tilde{\nu}_\mp^2 = \tilde{\Delta} \mp \sqrt{\tilde{\Delta}^2 - \frac{4}{\mu^2}}.$

- ***The entanglement is fully determined by $\tilde{\nu}_-$!***

Logarithmic negativity $E_{\mathcal{N}} = \max\{0, -\log \tilde{\nu}_-\}$

Symplectic parametrization

We choose this parametrization: $\{a,\, b,\, c_+,\, c_-\} \rightarrow \{\mu_1,\, \mu_2,\, \mu,\, \Delta\}$

*we know the purities, but **who is Δ ???***

- $\left.\dfrac{\partial\, \tilde{\nu}_-^2}{\partial\, \Delta}\right|_{\mu_1,\, \mu_2,\, \mu} > 0 \quad \Rightarrow$ the seralian regulates the entanglement of a generic Gaussian state with given purities

We have some constraints on the symplectic invariants…

$$\underbrace{\frac{2}{\mu} + \frac{(\mu_1 - \mu_2)^2}{\mu_1^2 \mu_2^2} \leq \Delta \leq 1 + \frac{1}{\mu^2}}$$

Maximally and minimally entangled Gaussian states for fixed global *and* marginal degrees of purity

GMEMS

Gaussian Max. Entangled Mixed States

two-mode squeezed thermal states

squeezing parameter

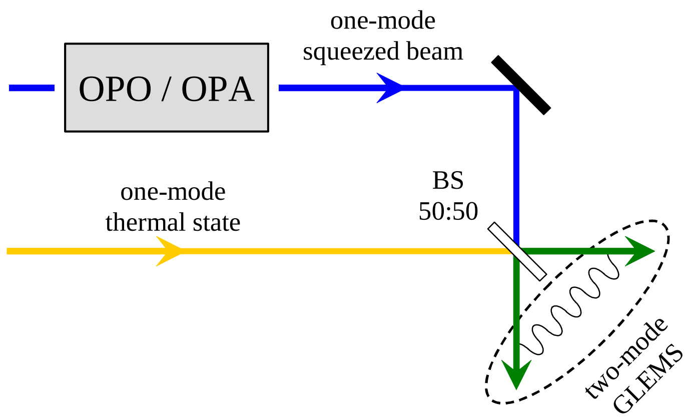
$$\tanh 2r = 2(\mu_1\mu_2 - \mu_1^2\mu_2^2/\mu)^{1/2}/(\mu_1+\mu_2)$$

pure GMEMS are a good approximation of EPR beams (infinitely entangled)

GLEMS

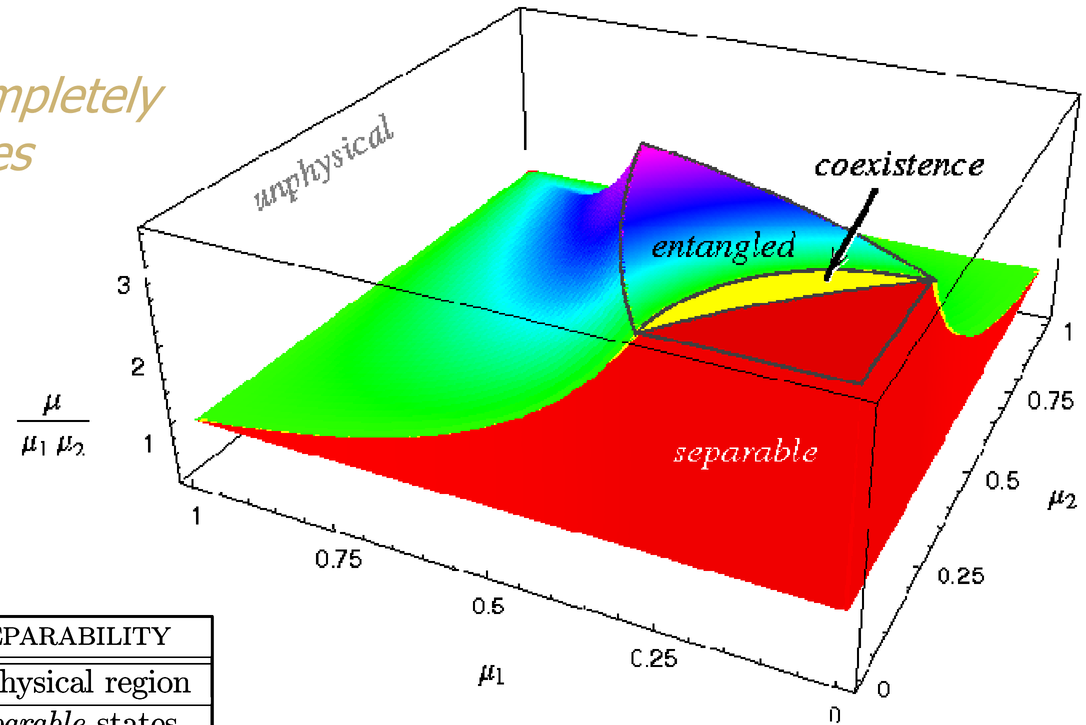
Gaussian Least Entangled Mixed States

minimum-uncertainty mixed states



Entanglement vs purities

The separability is completely qualified by the purities except for a narrow coexistence region



Degrees of Purity	Separability
$\mu < \mu_1\mu_2$	unphysical region
$\mu_1\mu_2 \le \mu \le \frac{\mu_1\mu_2}{\mu_1+\mu_2-\mu_1\mu_2}$	*separable* states
$\frac{\mu_1\mu_2}{\mu_1+\mu_2-\mu_1\mu_2} < \mu \le \frac{\mu_1\mu_2}{\sqrt{\mu_1^2+\mu_2^2-\mu_1^2\mu_2^2}}$	*coexistence* region
$\frac{\mu_1\mu_2}{\sqrt{\mu_1^2+\mu_2^2-\mu_1^2\mu_2^2}} < \mu \le \frac{\mu_1\mu_2}{\mu_1\mu_2+\mu_1-\mu_2}$	*entangled* states
$\mu > \frac{\mu_1\mu_2}{\mu_1\mu_2+\mu_1-\mu_2}$	unphysical region

Entanglement estimation

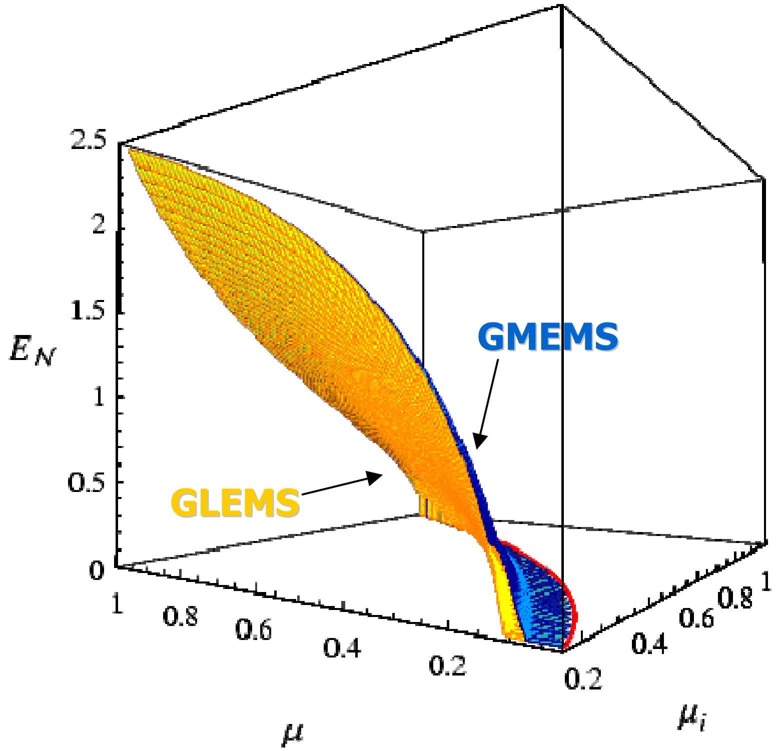
*A more **quantitative** look…*

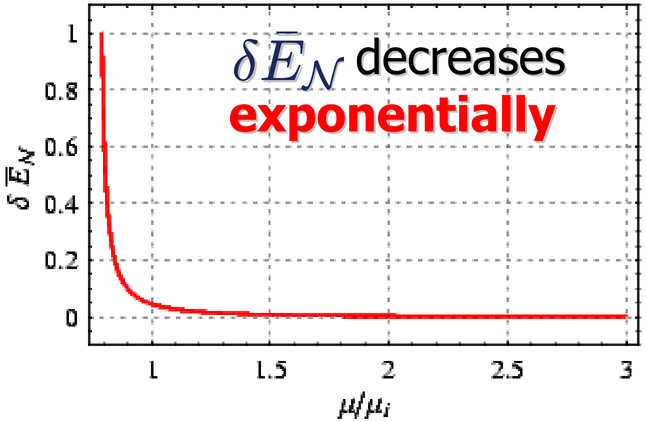
- 'Average Logarithmic Negativity'

$$\bar{E}_{\mathcal{N}}(\mu_{1,2},\mu) \equiv \frac{E_{\mathcal{N}max}(\mu_{1,2},\mu)+E_{\mathcal{N}min}(\mu_{1,2},\mu)}{2}$$

- Relative error on the estimate

$$\delta\bar{E}_{\mathcal{N}}(\mu_{1,2},\mu) \equiv \frac{E_{\mathcal{N}max}(\mu_{1,2},\mu)-E_{\mathcal{N}min}(\mu_{1,2},\mu)}{E_{\mathcal{N}max}(\mu_{1,2},\mu)+E_{\mathcal{N}min}(\mu_{1,2},\mu)}$$





We can estimate entanglement by measurements of purity

…the estimate is reliable !