Non-Abelian integrable hierarchies: Matrix biorthogonal polynomials and perturbations

Francisco Marcellán

Instituto de Ciencias Matemáticas ICMAT-UC3M, Madrid, Spain

Complex ODEs: Asymptotics, Orthogonal Polynomials and Random Matrices

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Outline

Definition

A sequence of monic polynomials $(P_n(x))_{n\in\mathbb{N}}$ is said to be the SMOP with respect to a linear functional u in the linear space of polynomials if $\langle u, P_n(x)P_m(x)\rangle = 0, m \neq n$, and $\langle u, P_n(x)P_n(x)\rangle \neq 0, n \geq 0$. The linear functional u is said to be quasi-definite.

Proposition

The SMOP satisfies a three term recurrence relation (TTRR)
$xP_n(x) = P_{n+1}(x) + b_nP_n(x) + c_nP_{n-1}(x), n \geq 0$, with $c_n \neq 0, n \geq 1$.

Converse result: Favard's Theorem.

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Converse result: Favard's Theorem.

The Gram matrix associated with u is a Hankel matrix $H = (h_{i,j})_{i,j=0}^{\infty}$ such that $h_{i,j} = \langle u, x^{i+j} \rangle$. Moreover, if Δ_n= det H_n, $n \geq 0$, then

Proposition

- $c_n = \frac{\Delta_{n+1}\Delta_{n-1}}{\Delta_n^2}$.
- $b_n = \frac{\Delta_{n+1}^*}{\Delta_{n+1}} - \frac{\Delta_n^*}{\Delta_n}$.

The matrix counterpart of the TTRR is a tridiagonal matrix J. J is said to be a monic Jacobi matrix.

Notice that the eigenvalues of the n-th leading principal submatrix of J are the zeros of $P_n(x)$.

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If we assume $\langle u, 1\rangle = 1$, then the sequence of monic polynomials $(Q_n(x))_{n\in\mathbb{N}}$ such that $Q_n(x) = \langle u, \frac{P_{n+1}(x)-P_{n+1}(y)}{x-y}\rangle$, $n \geq 0$, is said to be sequence of polynomials of the first kind associated with u.

$(Q_n(x))_{n\in\mathbb{N}}$ is also called the numerator sequence of the diagonal Padé approximation to the Stieltjes function associated with the linear functional u. Indeed, if $S_u(z) = \sum_{n=0}^{\infty} \frac{\langle u, x^n\rangle}{z^{n+1}}$, then $P_n(z)S_u(z) - Q_{n-1}(z) = O(\frac{1}{z^{n+1}})$.

Notice that $(Q_n(x))_{n\in\mathbb{N}}$ is, again, an SMOP with respect to a linear functional v with Stieltjes function $S_v(z)$ such that $S_u(z) = \frac{1}{x-b_0-c_1 S_v(z)}$.

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Definition

A linear functional v is said to be a linear spectral transform of the linear functional u if there exist polynomials $A(z), B(z), C(z)$ such that $S_v(z) = \frac{A(z)S_u(z)+B(z)}{C(z)}$

Proposition(A. Zhedanov 1995)

Every linear spectral transformation can be represented as the composition of a finite number of canonical transformations

- Christoffel transformation: $v = C_a(u)$, where $v = (x-a)u$.
- Geronimus transformation: $w = G_{a,M}(u)$, where $w = (x-a)^{-1}u + M\delta(x-a)$.

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Proposition

Let J, J_C, and J_G be the monic Jacobi matrices associated with u, $C_a(u)$, and $G_{a,M}(u)$. If $J - aI = L_1 U_1$ and $J - aI = U_2 L_2$, then

- Christoffel transformation: $J_C - aI = U_1 L_1$.
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Such factorizations of matrices are called the discrete Darboux transformations.

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Outline

ODE and orthogonal polynomials

Orthogonal polynomials as eigenfunctions of a second order linear differential operator

Proposition (S. Bochner 1929)

- Hermite polynomials (normal distribution)
- Laguerre polynomials (Gamma distribution)
- Jacobi polynomials (Beta distribution)
- Bessel polynomials. An example of a quasi-definite linear functional.

Extension of this result to other linear operators: $\triangle_w$ and D_q.
The hypergeometric character of these "classical" orthogonal polynomials.
Askey tableau.

Orthogonal polynomials as eigenfunctions of higher order differential operators and the connection with linear spectral transformations.

Proposition (H. L. Krall 1940)

For fourth order linear differential operators with orthogonal polynomials as eigenfunctions, you get the classical ones as well as the following Uvarov perturbations of them.

- Laguerre-type polynomials. Orthogonality with respect to the measure $d\mu(x) = \chi_{(0,+\infty)} e^{-x} dx + M\delta(x)$.
- Legendre type polynomials. Orthogonality with respect to the measure $d\mu(x) = \frac{a}{2}\chi_{(-1,+1)} dx + \frac{1}{2}\delta(x-1) + \frac{1}{2}\delta(x+1)$.
- Jacobi type polynomials. Orthogonality with respect to the measure $d\mu(x) = \chi_{(0,1)}(1-x)^{\alpha} dx + M\delta(x)$.

For higher order linear differential operators you have Krall orthogonal polynomials and bispectrality as a natural framework.

Proposition

If $\alpha \geq 1$ is an integer number, orthogonal polynomials with respect to the measure $d\mu(x) = \chi_{(0,+\infty)} x^{\alpha-1} e^{-x} dx + M\delta(x)$ (Laguerre-Krall) are eigenfunctions of a linear differential operator of order $2\alpha + 2$.

Notice that the above measure is a Geronimus transformation of the Gamma distribution.

Definition

A linear differential operator $\mathfrak{L}$ is said to be exceptional if it has polynomial eigenfunctions of all but finitely many degrees, i.e. there exists a finite subset $\mathfrak{N}$ of non negative integer numbers such that there exists a polynomial of degree n as eigenfunction of $\mathfrak{L}$ if and only if does not belong to $\mathfrak{N}$.

Definition

A linear differential operator $\tilde{\mathfrak{L}}$ is said to be a Darboux conjugate of $\mathfrak{L}$ if there exists a differential operator $\mathfrak{H}$ such that $\tilde{\mathfrak{L}}\mathfrak{H} = \mathfrak{L}\mathfrak{H}$.

Proposition

If $\tilde{\mathfrak{L}}$ is a Darboux conjugate of $\mathfrak{L}$, then $\mathfrak{L}$ is a Darboux conjugate of $\tilde{\mathfrak{L}}$.

Proposition

If $\mathfrak{L}$ is an exceptional differential operator such that $\mathfrak{L}$ is a Darboux conjugate of $\tilde{\mathfrak{L}}$. If $\tilde{\mathfrak{L}}\mathfrak{H} = \mathfrak{L}\mathfrak{H}$ for a differential operator $\mathfrak{H}$, then $\tilde{\mathfrak{L}}$ is also exceptional.

Definition

A linear differential operator $\mathfrak{L}$ is said to be degree-filtration preserving if for every polynomial $p(x)$, $\mathfrak{L}(p)$ is a polynomial of degree at most the degree of $p(x)$.

Proposition (García Ferrero- Gómez-Ullate, Milson 2016)

If $\mathfrak{L}$ is an exceptional second order linear differential operator, then it is a Darboux conjugate to a degree filtration preserving differential operator.

Outline

Definition

Let $A_0, A_1 \cdots, A_N \in \mathbb{R}^{p\times p}$ *be square matrices with real entries . Let us assume that* A_N *is a non singular matrix. Then*

$$P_N(x) = A_N x^N + A_{N-1} x^{N-1} + \cdots + A_1 x + A_0$$

is said to be a matrix polynomial of degree N. The matrix polynomial is said to be monic when $A_N = I_p$*, the identity matrix. The set of matrix polynomials with coefficients in* $\mathbb{R}^{p\times p}$ *will be denoted by* $\mathbb{R}^{p\times p}[x]$.

Definition

x_0 *is said to be a zero of a matrix polynomial* $P_N(x)$ *if* $\det[P_N(x_0)] = 0$*. We denote the set of the zeros of matrix polynomial W by* $\sigma(W(x))$ *.*

A matrix of generalized kernels,

$$u_{x,y} := \begin{bmatrix} (u_{x,y})_{1,1} & \dots & (u_{x,y})_{1,p} \\ \vdots & & \vdots \\ (u_{x,y})_{p,1} & \dots & (u_{x,y})_{p,p} \end{bmatrix}$$

with $(u_{x,y})_{k,l} \in (\mathbb{C}[x,y])'$ provides a continuous sesquilinear form with entries given by

$$\Big(\langle P(x), Q(y)\rangle_u\Big)_{i,j} = \sum_{k,l=1}^{p} \Big\langle (u_{x,y})_{k,l}, P_{i,k}(x) \otimes Q_{j,l}(y)\Big\rangle,$$

$$\langle P(x), Q(y)\rangle_u = \langle u_{x,y}, P(x) \otimes Q(y)\rangle.$$

Proposition

If the sesquilinear form u is quasi-definite, then the block moment matrix M has a unique Gauss-Borel factorization

$$M = S_1^{-1} H S_2^{-\dagger}$$

where S_1 and S_2 are a lower unitriangular block matrices and H is a diagonal block matrix

Matrix bi-orthogonal polynomials

Let $\chi(x) = [I_p, xI_p, x^2 I_p \cdots]^\dagger$. We define the vectors of matrix polynomials $P^{[1]} = [P_0^{[1]\dagger}, P_1^{[1]\dagger}, \cdots]^\dagger$ and $P^{[2]} = [P_0^{[2]\dagger}, P_1^{[2]\dagger}, \ldots]^\dagger$, where

$$P^{[1]} := S_1\chi(x), \quad P^{[2]} := S_2\chi(y).$$

Notice that the matrix polynomials $P_n^{[i]}$ are monic with $\deg P_n^{[i]} = n$, $i = 1, 2$.

Definition

The families of monic matrix polynomials $(P_n^{[1]}(x))_{n\in\mathbb{N}}$ and $(P_n^{[2]}(y))_{n\in\mathbb{N}}$ are said to be bi-orthogonal with respect to u in the sense that

$$\left\langle P_n^{[1]}(x), P_m^{[2]}(y)\right\rangle_u = \delta_{n,m} H_n, \quad n, m \in \mathbb{N}.$$

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Given two sequences of matrix monic bi-orthogonal polynomials $(P_n^{[1]}(x))_{n\in\mathbb{N}}$ and $(P_n^{[2]}(y))_{n\in\mathbb{N}}$, with respect to $\langle\cdot,\cdot\rangle_u$, we define the *$n$-th Christoffel–Darboux kernel matrix polynomial*

$$K_n(x,y) := \sum_{k=0}^{n} (P_k^{[2]}(y))^{\dagger} H_k^{-1} P_k^{[1]}(x).$$

Proposition (Christoffel-Darboux formula)

The Christoffel-Darboux kernel satisfies

$$(x-y)K_n(x,y) = (P_n^{[2]}(y))^{\dagger}(H_n)^{-1}P_{n+1}^{[1]}(x) - (P_{n+1}^{[2]}(y))^{\dagger}(H_n)^{-1}P_n^{[1]}(x).$$

.

Second kind functions

Given a quasidefinite matrix of generalized kernels $u_{x,y}$ and $(P_n^{[1]}(x))_{n\in\mathbb{N}}$ and $(P_n^{[2]}(x))_{n\in\mathbb{N}}$ its sequences of matrix bi-orthogonal polynomials. For $z \notin supp(u_{x,y})$ we define the first and second families of second kind functions $(C_n^{[1]}(z))_{n\in\mathbb{N}}$ and $(C_n^{[1]}(z))_{n\in\mathbb{N}}$ where for each $n \in \mathbb{N}$,

$$C_n^{[1]}(z) = \left\langle P_n^{[1]}(x), \frac{I_p}{z-y} \right\rangle_u, \quad \text{and} \quad C_n^{[2]}(z) = \left\langle \frac{I_p}{z-x}, P_n^{[2]}(y) \right\rangle_u^\top.$$

We also define the mixed Christoffel–Darboux kernel

$$K_n^{(pc)}(x,y) := \sum_{k=0}^{n} \left(P_k^{[2]}(y)\right)^\top H_k^{-1} C_k^{[1]}(x).$$

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$$(x-y)K_n^{(pc)}(x,y) = (P_n^{[2]}(y))^\top H_n^{-1} C_{n+1}^{[1]}(x) - (P_{n+1}^{[2]}(y))^\top H_n^{-1} C_n^{[1]}(x) + I_p.$$

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Definition (The last quasideterminant)

Let $A \in \mathbb{C}^{m\times m}$, $B \in \mathbb{C}^{m\times p}$, $C \in \mathbb{C}^{p\times m}$ and $D \in \mathbb{C}^{p\times p}$, with A a non-singular matrix. If we take the block matrix $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, then

$$\Theta_* \begin{bmatrix} A & B \\ C & D \end{bmatrix} = D - CA^{-1}B \quad \textit{The last quasideterminant of the matrix} \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$$

Outline

Matrix Geronimus–Uvarov transformations

Definition

Given two matrix polynomials $W_C(x), W_G(y)$ of degrees N_C, N_G, respectively, and a matrix of generalized kernels $u_{x,y} \in (\mathcal{O}'_c)^{p\times p}$ such that $\sigma(W_G(y)) \cap \operatorname{supp}_y(u) = \varnothing$, a matrix Geronimus–Uvarov transformation $\hat{u}_{x,y}$ of $u_{x,y}$ is a matrix of generalized kernels such that

$$\hat{u}_{x,y} W_G(y) = W_C(x) u_{x,y}.$$

Outline

Jordan Chains

Let $x_1, \ldots, x_q$ be the zeros of $W(x)$ and $\alpha_1, \ldots, \alpha_q$ be their corresponding algebraic multiplicities. Notice that $\sum_{i=1}^{q} \alpha_i = Np$.

- **Definition.** If for x_a there exists a nonzero vector $r_0^{(a)}$ (covector $l_0^{(a)}$) such that $W(x_a) r_0^{(a)} = 0_p$ ($l_0^{(a)} W(x_a) = 0_p^{\dagger}$), then $r_0^{(a)}$ ($l_0^{(a)}$) is said to be a right (left) eigenvector of $W(x)$ corresponding to x_a.
- **Definition.** A sequence of vectors $\{r_0^{(a)}, \ldots, r_{m_a-1}^{(a)}\}$ ($\{l_0^{(a)}, \ldots, l_{m_a-1}^{(a)}\}$) is said to be a right (left) Jordan chain of length m_a corresponding to x_a if

  $r_0^{(a)}$ (*resp.* $l_0^{(a)}$) is a right (left) eigenvector of $W(x)$ corresponding to x_a

  and

$$\sum_{t=0}^{j} \frac{1}{t!} W^{(t)}(x_a) r_{j-t}^{(a)} = 0_p, \quad \left(\sum_{t=0}^{j} \frac{1}{t!} l_{j-t}^{(a)} W^{(t)}(x_a) = 0_p^{\dagger} \right), \; j = 0, \ldots, m_a - 1.$$

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The maximal length of a Jordan chain corresponding to the zero x_a is called the partial multiplicity and it is denoted by $m(r_0^{(a)}) = \kappa^{(a)}$.

Definition. Let $\{r_{1,0}^{(a)}, r_{2,0}^{(a)}, \ldots, r_{s_a,0}^{(a)}\}$ be a basis of the space $Ker(W(x_a))$ where $\dim(Ker(W(x_a))) = s_a$. For each vector $r_{j,0}^{(a)}$ of the basis, we consider its associated Jordan chains with maximal length

$$\left\{r_{j,0}^{(a)}, \ldots, r_{j,\kappa_j^{(a)}-1}^{(a)}\right\}, \quad j = 1, \ldots, s_a.$$

Proposition

The partial multiplicities $\{\kappa_1^{(a)}, \ldots, \kappa_{s_a}^{(a)}\}$ corresponding to the zero x_a satisfy that[a]

$$\sum_{j=1}^{s_a} \kappa_j^{(a)} = \alpha_a.$$

[a] A. S. Markus, *Introduction to the spectral theory of polynomials operator pencil. Translated from the Russian by H. H. McFaden. Translation edited by Ben Silver. With an appendix by M. V. Keldysh. Translations of Mathematical*

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$$\left\{r_{j,0}^{(a)}, \ldots, r_{j,\kappa_j^{(a)}-1}^{(a)}\right\}, \quad j = 1, \ldots, s_a.$$

Proposition

The partial multiplicities $\{\kappa_1^{(a)}, \ldots, \kappa_{s_a}^{(a)}\}$ corresponding to the zero x_a satisfy that[a]

$$\sum_{j=1}^{s_a} \kappa_j^{(a)} = \alpha_a.$$

[a] A. S. Markus, *Introduction to the spectral theory of polynomials operator pencil. Translated from the Russian by H. H. McFaden. Translation edited by Ben Silver. With an appendix by M. V. Keldysh. Translations of Mathematical*

For each eigenvalue $x_a \in \sigma(W)$, with multiplicity α_a and $s_a = \dim \operatorname{Ker} W(x_a)$, we choose a canonical set of Jordan chains

$$\left\{r_{j,0}^{(a)}, \dots, r_{j,\kappa_j^{(a)}-1}^{(a)}\right\}, \quad \left(\left\{l_{j,0}^{(i)}, \dots, l_{j,\kappa_j^{(a)}-1}^{(a)}\right\}\right) \quad j = 1, \dots, s_a.$$

We can consider the following right (left) adapted root polynomials

$$r_j^{(a)}(x) = \sum_{t=0}^{\kappa_j^{(a)}-1} r_{j,t}^{(a)}(x - x_a)^t, \quad \left(l_j^{(a)}(x) = \sum_{t=0}^{\kappa_j^{(a)}-1} l_{j,t}^{(a)}(x - x_a)^t \right).$$

$$\left.\frac{\mathrm{d}^t}{\mathrm{d}\,x^t}\right|_{x=x_a}\left(W(x) r_j^{(a)}(x)\right) = 0, \quad t = 0, \dots, \kappa_j^{(a)} - 1, \quad j = 1, \dots, s_a.$$

$$\left.\frac{\mathrm{d}^t}{\mathrm{d}\,x^t}\right|_{x=x_a}\left(l_j^{(a)}(x) W(x)\right) = 0 \quad t = 0, \dots, \kappa_j^{(a)} - 1, \quad j = 1, \dots, s_a.$$

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$$\left.\frac{\mathrm{d}^t}{\mathrm{d}\,x^t}\right|_{x=x_a}\left(W(x)r_j^{(a)}(x)\right) = 0, \quad t = 0, \dots, \kappa_j^{(a)} - 1, \quad j = 1, \dots, s_a.$$

$$\left.\frac{\mathrm{d}^t}{\mathrm{d}\,x^t}\right|_{x=x_a}\left(l_j^{(a)}(x)W(x)\right) = 0 \quad t = 0, \dots, \kappa_j^{(a)} - 1, \quad j = 1, \dots, s_a.$$

Given a matrix function $f(x)$ which is smooth in its domain of definition, we consider its root spectral jets vectors

$$\mathcal{J}_f^{(i)}(x_a) := \lim_{x \to x_a} \left[f(x_a) r_i^{(a)}(x_a), \dots, \frac{\left(f(x) r_i^{(a)}(x)\right)_{x_a}^{(\kappa_i^{(a)}-1)}}{(\kappa_i^{(a)}-1)!} \right] \in \mathbb{C}^{p \times \kappa_i^{(a)}}$$

$$\mathcal{J}_f(x_a) := \left[\mathcal{J}_f^{(1)}(x_a), \dots, \mathcal{J}_f^{(s_a)}(x_a) \right] \in \mathbb{C}^{p \times \alpha_a},$$

$$\mathcal{J}_f := \left[\mathcal{J}_f(x_1), \dots, \mathcal{J}_f(x_q) \right] \in \mathbb{C}^{p \times Np}.$$

...Come back to the matrix Geronimus–Uvarov transformation

The most general matrix Geronimus–Uvarov transformation is given by

$$\hat{u}_{x,y} := W_C(x)u_{x,y}(W_G(y))^{-1} + v_{x,y},$$

$$v_{x,y} := \sum_{a=1}^{q_G}\sum_{j=1}^{s_{G,a}}\sum_{m=0}^{\kappa_{G,j}^{(a)}-1} \frac{(-1)^m}{m!} W_C(x)\left(\xi_{j,m}^{[a]}\right)_x \otimes \delta^{(m)}(y - x_{G,a}) l_{G,j}^{(a)}(y),$$

expressed in terms of the spectrum $\sigma(W_G(y)) = \{x_{G,a}\}_{a=1}^{q_G}$, number of Jordan blocks $s_{G,a}$, partial multiplicities $\kappa_{G,j}^{(a)}$, and corresponding adapted left root polynomials $l_{G,j}^{(a)}(y)$ of the matrix polynomial $W_G(y)$ and for vectors of generalized functions $\left(\xi_{j,m}^{[a]}\right)_x \in \left(\mathbb{C}^p[x]\right)'$.

A explicit form for our sesquilinear forms is the following

$$
\begin{aligned}
\langle P(x), Q(y)\rangle_{\hat{u}} = {}& \left\langle P(x) W_C(x), Q(y)(W_G(y))^{-\top} \right\rangle_u \\
&+ \sum_{a=1}^{q_G} \sum_{j=1}^{s_{G,a}} \sum_{m=0}^{\kappa_{G,j}^{(a)}-1} \left\langle P(x) W_C(x), \left(\xi_{j,m}^{[a]}\right)_x \right\rangle \frac{1}{m!} \left(l_{G,j}^{(a)}(y) \left(Q(y)\right)^{\top} \right)_{x_{G,a}}^{(m)}.
\end{aligned}
$$

Given a set of generalized functions $(\xi_{i,m}^{[a]})_x$, we introduce the matrices

$$
\begin{aligned}
\left\langle \check{P}_n^{[1]}(x), (\xi_i^{[a]})_x \right\rangle &:= \left[\left\langle \check{P}_n^{[1]}(x), \left(\xi_{i,0}^{[a]}\right)_x \right\rangle, \dots, \left\langle \check{P}_n^{[1]}(x), \left(\xi_{i,\kappa_i^{(a)}-1}^{[a]}\right)_x \right\rangle \right] \in \mathbb{C}^{p \times \kappa_i^{(a)}}, \\
\left\langle \check{P}_n^{[1]}(x), (\xi^{[a]})_x \right\rangle &:= \left[\left\langle \check{P}_n^{[1]}(x), \left(\xi_1^{[a]}\right)_x \right\rangle, \dots, \left\langle \check{P}_n^{[1]}(x), \left(\xi_{s_a}^{[a]}\right)_x \right\rangle \right] \in \mathbb{C}^{p \times \alpha_a}, \\
\left\langle \check{P}_n^{[1]}(x), (\xi)_x \right\rangle &:= \left[\left\langle \check{P}_n^{[1]}(x), \left(\xi^{[1]}\right)_x \right\rangle, \dots, \left\langle \check{P}_n^{[1]}(x), \left(\xi_{s_a}^{[q]}\right)_x \right\rangle \right] \in \mathbb{C}^{p \times Np}.
\end{aligned}
$$

Spectral Christoffel–Geronimus–Uvarov formulas

For monic Geronimus–Uvarov perturbations, with masses, when $n \geq N_G$, we have the following last quasideterminant expressions for the perturbed biorthogonal matrix polynomials

$$\hat{P}^{[1]}_n(x) W_C(x) = \Theta_* \begin{bmatrix} \mathcal{J}_{C,P^{[1]}_{n-N_G}} & \mathcal{J}_{G,C^{[1]}_{n-N_G}} & -\left\langle P^{[1]}_{n-N_G}(x),(\xi)_x \right\rangle \mathcal{W}_G & P^{[1]}_{n-N_G}(x) \\ \vdots & & \vdots & \vdots \\ \mathcal{J}_{C,P^{[1]}_{n+N_C}} & \mathcal{J}_{G,C^{[1]}_{n+N_C}} & -\left\langle P^{[1]}_{n+N_C}(x),(\xi)_x \right\rangle \mathcal{W}_G & P^{[1]}_{n+N_C}(x) \end{bmatrix},$$

$$\left(\hat{P}^{[2]}_n(y)\right)^\top = -\Theta_* \begin{bmatrix} \mathcal{J}_{C,P^{[1]}_{n-N_G}} & \mathcal{J}_{G,C^{[1]}_{n-N_G}} - \left\langle P^{[1]}_{n-N_G}(x),(\xi)_x \right\rangle \mathcal{W}_G & H_{n-N_G} \\ \mathcal{J}_{C,P^{[1]}_{n-N_G+1}} & \mathcal{J}_{G,C^{[1]}_{n-N_G+1}} - \left\langle P^{[1]}_{n-N_G+1}(x),(\xi)_x \right\rangle \mathcal{W}_G & 0_p \\ \vdots & \vdots & \\ \mathcal{J}_{C,P^{[1]}_{n+N_C-1}} & \mathcal{J}_{G,C^{[1]}_{n+N_C-1}} \left\langle P^{[1]}_{n+N_C-1}(x),(\xi)_x \right\rangle \mathcal{W}_G & 0_p \\ W_G(y)\mathcal{J}_{C,K_{n-1}}(y) & W_G(y)\left(\mathcal{J}_{G,K^{(pc)}_{n-1}}(y) - \left\langle K_{n-1}(x,y),(\xi)_x \right\rangle \mathcal{W}_G\right) + \mathcal{J}_{G,\mathcal{V}}(y) & 0_p \end{bmatrix}.$$

Where $\mathcal{W}_G \in \mathbb{C}^{Np^2 \times Np}$ is a numerical matrix known.

Outline

Mixed spectral/nonspectral method

Now, we consider the Geronimus–Uvarov transformation as a composition of a first Geronimus transformation, applying nonspectral techniques, and then a Christoffel transformation, for which we can apply spectral techniques. i.e,

$$\check{u}_{x,y} := u_{x,y}(W_G(y))^{-1} + v_{x,y}, \longmapsto \hat{u}_{x,y} = W_C(x)\check{u}_{x,y}.$$

with $v_{x,y}$ a matrix of bivariate generalized functions such that $v_{x,y}W_G(y) = 0_p$. In this situation, we still require the leading coefficient A_{C,N_C} to be nonsingular, but we free $W_G(x)$ of such a condition.

Definition

For a given perturbed matrix of functionals $\check{u}$ we define a semi-infinite block matrix of the form

$$
\begin{aligned}
R &:= \left\langle P^{[1]}(x), \chi(y) \right\rangle_{\check{u}} \\
&= \left\langle P^{[1]}(x), \chi(y) \right\rangle_{uW_G^{-1}} + \left\langle P^{[1]}(x), \chi(y) \right\rangle_{v_G}.
\end{aligned}
$$

We introduce the matrix polynomials $r^K_{n,l}(y) \in \mathbb{C}^{p\times p}[y]$, $l \in \{0, \dots, n-1\}$, given by

$$
\begin{aligned}
r^K_{n,l}(z) : &= \left\langle W_G(z) K_{n-1}(x,z), I_p y^l \right\rangle_{\check{u}} - I_p z^l \\
&= \left\langle W_G(z) K_{n-1}(x,z), I_p y^l \right\rangle_{uW_G^{-1}} + \left\langle W(z) K_{n-1}(x,z), I_p y^l \right\rangle_{v_G} - I_p z^l.
\end{aligned}
$$

Some Considerations...

For $n \geq N_G$, let us assume that the matrix

$$\Phi_n := \begin{bmatrix} \mathcal{J}_{C,P^{[1]}_{n-N_G}} & R_{n-N_G,0} & \cdots & R_{n-N_G,n-1} \\ \vdots & \vdots & & \vdots \\ \mathcal{J}_{C,P^{[1]}_{n+N_C-1}} & R_{n+N_C-1,0} & \cdots & R_{n+N_C-1,n-1} \end{bmatrix} \in \mathbb{C}^{(N_C+N_G)p\times(N_C+n)p}$$

is full rank, thus there exist $\Phi_n^{\square} \in \mathbb{C}^{(N_C+N_G)p\times(N_C+N_G)p}$ a nonsingular square submatrix of Φ_n. We will refer to them as poised submatrix. We also consider

$$\varphi_n := \left[\mathcal{J}_{C,P^{[1]}_{n+N_C}}, R_{n+N_C,0}, \dots, R_{n+N_C,n-1}\right] \in \mathbb{C}^{p\times(N_C+n)p}$$

and the *square* submatrix $\varphi_n^{\square} \in \mathbb{C}^{p\times(N_C+N_G)p}$ corresponding to the selection of columns to built the poised submatrix $\Phi_n^{\square}$. We also consider

$$\varphi_n^K(y) = \left[W_G(y)\mathcal{J}_{C,K_{n-1}}(y), R^K_{n,0}(y), \dots, R^K_{n,n-1}(y)\right] \in \mathbb{C}^{p\times(N_C+n)p}[y]$$

and $(\varphi_n^K(y))^{\square}$.

Matrix Christoffel–Geronimus–Uvarov formulas

Given a matrix Geronimus–Uvarov transformation the corresponding perturbed polynomials can be expressed, for $n \geq N_G$, as follows

$$\hat{P}_n^{[1]}(x) W_C(x) = \Theta_* \left[\begin{array}{c|c} \multirow{3}{*}{$\Phi_n^\square$} & P^{[1]}_{n-N_G}(x) \\ & \vdots \\ & P^{[1]}_{n+N_C-1}(x) \\ \hline \varphi_n^\square & P^{[1]}_{n+N_C}(x) \end{array}\right],$$

$$\left(\hat{P}_n^{[2]}(y)\right)^\top A_N = -\Theta_* \left[\begin{array}{c|c} \multirow{4}{*}{$\Phi_n^\square$} & H_{n-N_G} \\ & 0_p \\ & \vdots \\ & 0_p \\ \hline \left(\varphi_n^K(y)\right)^\square & 0_p \end{array}\right].$$

Christoffel transformations with singular leading coefficients

Recall that the classic adjoint satisfies

$$W(x)\,\mathrm{adj}(W(x)) = \mathrm{adj}(W(x))W(x) = \det(W(x))I_p,$$

We can study the Christoffel transformation of a Hankel matrix of generalized kernels

$$\hat{u}_{x,x} = u_{x,x}W(x),$$

as the following massless Geronimus–Uvarov transformation

$$\hat{u}_{x,x} = W_C(x)u_{x,x}\big(W_G(x)\big)^{-1}, \quad W_C(x) := I_p \det(W(x)), \quad W_G(x) := \mathrm{adj}(W(x)),$$

with $N_C = \deg(W_C(x)) \leq Np$ and $N_G = \deg(W_G(x)) \leq N(p-1)$.

Here,we can apply our results for Geronimus–Uvarov transformations, and use the mixed spectral-nonspectral Christoffel–Geronimus–Uvarov formula

More transformations

We could consider a more general situation with transformations of the following (non)symmetric form, where $W(x)$ has a nonsingular leading coefficient,

$$\hat{u} = W(x)uV(x) = W_C(x)u(W_G(x))^{-1},$$
$$W_C(x) = \det(V(x))W(x), \quad W_G(x) = \operatorname{adj}(V(x)),$$

or

$$\hat{u} = (W(x))^{-1}u(V(x))^{-1} = W_C(x)u(W_G(x))^{-1},$$
$$W_C(x) = \operatorname{adj}(W(x)), \quad W_G(x) = \det(W(x))V(x).$$

In this case, we also can apply the mixed spectral/nonspectral Christoffel–Geronimus–Uvarov formulas to find a representation of the new families of matrix bi-orthogonal polynomials.

Outline

Open problems

Let consider a sequence $\{P_n(x)\}_{n\geq 0}$ of monic matrix orthogonal polynomials with respect to a weight matrix $W(x)$ that is entrywise-smooth, Hermitian and positive definite with finite moments.Then

Proposition

There exist sequences of complex valued matrices $\{s_n\}_{n\geq 0}, \{t_n\}_{n\geq 0}$, where t_n is nonsingular for every $n \geq 0$ such that

$$xP_n(x) = P_{n+1}(x) + s_n P_n(x) + t_n P_{n-1}(x), n \geq 0.$$

The matrix weight belongs to the Nevai class if $s_n \to s$ and $t_n \to t$, where t is a nonsingular matrix.

Problem 1

Is the Nevai class preserved by Geronimus-Uvarov transformations?

In a similar way to the scalar case, let denote by $\{Q_n(x)\}_{n\geq 0}$ the sequence of monic matrix orthogonal polynomials with respect to a Geronimus-Uvarov transformation of the weight matrix $W(x)$ in the Nevai class.

Problem 2

What about the outer relative asymptotics $[P_n(x)]^{-1}Q_n(x)$?

Problem 3

What about the location of zeros of $Q_n(x)$ in terms of the zeros of $Q_n(x)$?

Outline

ODE and matrix orthogonal polynomials

Bochner Problem for Matrix differential operators

Determine all weight matrices $W(x)$ such that there exists a second order differential operator $\mathfrak{L}$ such that the corresponding sequence of monic matrix orthogonal are eigenfunctions, i.e.
$\mathfrak{L}(p)(x) = p(x)a_0(x) + p'(x)a_1(x) + p''(x)a_2(x) = \Lambda p(x)$.

Definition

Let $W(x)$ be a weight matrix and $\mathfrak{L}$ be a W-symmetric second order differential operator such that the sequence of matrix orthogonal polynomials (SMOP) with respect to W are eigenfunctions. Then $(W, \mathfrak{L})$ is said to be a Bochner pair.

The key problem is the determination of the algebra $D(W)$ of all matrix differential operators having the (SMOP) associated with W as eigenfunctions.

Theorem (A. J. Durán, A. Grünbaum, 2004)

Let $W(x)$ be a weight matrix. Then $D(W)$ contains a second order differential operator if and only if there exists $\mathfrak{L} = D^2 a_2 + D a_1 + a_0$, with $a_k(x)$ polynomials of degree at most k, $k = 0, 1, 2$ such that

- $a_2 W(x) = W(x) a_2^*$.
- $(a_2 W(x))'' - (a_1 W(x))' + a_0 W(x) = W(x) a_0^*$
- $a_2 W(x)$ and $(a_2 W(x))' - a_1 W(x)$ have vanishing limits at the endpoints of the support of $W(x)$.

From these conditions you have $2(a_2 W(x))' = a_1 W(x) + W(x) a_1^*$, a noncommutative Pearson equation.

The above result means that $(W, \mathfrak{L})$ is a solution to the Bochner problem for matrix differential operators.

Theorem (W. Riley Casper, M. Yakimov 2018)

Let $W(x)$ be a weight matrix and suppose that $D(W)$ contains a second order linear differential operator $\mathfrak{L}$ as above, where $a_2 W(x)$ is positive definite on the support of $W(x)$. Then the algebra $D(W)$ is full, in the sense that the rank is as large as possible, if and only if $W(x)$ is a noncommutative bispectral Darboux transformation of a direct sum of classical weights. Furthermore, in this case $W(x) = T(x) diag(f_1(x), \cdots f_N(x)) T^*(x)$, where $T(x)$ is a rational matrix and $f_k(x)$ is a classical weight.

In the case $N = 2$ we have the following classification result

Theorem (W. Riley Casper, M. Yakimov 2018)

Let $W(x)$ be a 2×2 weight matrix and suppose that $D(W)$ contains a second order linear differential operator $\mathfrak{L}$ as above, such that $a_2 W(x)$ is positive definite on the support of $W(x)$. Then the algebra $D(W)$ is noncommutative if an only if the weight $W(x)$ is a noncommutative bispectral Darboux transformation of $r(x)I$ for some classical weight $r(x)$.

Thus aside from various degenerate cases when $D(W)$ of polynomials of a single differential operator of order 2, this solves the Bochner problem for the 2×2 case.

Outline

Let $G = (m_{i,j})_{i,j=0}^{\infty}$, $m_{i,j} \in \mathbb{R}^{p\times p}$ be a Hankel semi-infinite block matrix associated to a generalized kernel $u_{x,y}$ having a Gaussian block factorization

$$M = (S_1)^{-1} H (S_2)^{-\top}.$$

Given a covector of times $t_i = \Big[t_{i,1}, t_{i,2}, \dots\Big]$, $i \in \{1,2\}$, $t_{i,k} \in \mathbb{C}$,, where we only have a finite number of entries different from zero, we consider the semi-infinite matrix $V_0^{t_i} := \exp\Big(\sum_{0<j\ll\infty} t_{i,j}\Lambda^j\Big)$.

Thus the perturbed Gram matrix is

$$G^t = V_0^{t_1} G \Big(V_0^{t_2}\Big)^{-\top}, \quad \text{with} \quad t = (t_1, t_2)$$

We also consider the polynomials

$$t_1(x) := \sum_{0<j\ll\infty} t_{1,j} x^j, \qquad t_2(y) := \sum_{0<j\ll\infty} t_{2,j} y^j$$

We can check that

$$
\begin{aligned}
G^t &= V_0^{t_1} G \left(V_0^{t_2} \right)^{-\top} \\
&= V_0^{t_1} \left\langle \chi(x), \chi(y) \right\rangle_u \left(V_0^{t_2} \right)^{-\top} \\
&= \left\langle e^{\sum_{j=1}^{\infty} t_{1,j} x^j} \chi(x), e^{-\sum_{j=1}^{\infty} t_{2,j} y^j} \chi(y) \right\rangle_u \\
&= \langle \chi(x), \chi(y) \rangle_{u^t},
\end{aligned}
$$

where the deformed bivariate matrix of generalized functions is given by

$$
u_{x,y}^t := e^{t_1(x) - t_2(y)} u_{x,y}.
$$

Observe that if the initial Hankel matrix of generalized kernels $u_{x,y}$, then so is $u_{x,y}^t$. Thus, the Hankel symmetry is preserved under these continuous transformations.

We will assume that $u^t_{x,y}$ is quasidefinite; i.e., the block Gauss–Borel factorization

$$G^t = (S^t_1)^{-1} H^t (S^t_2)^{-\top} \tag{1}$$

holds. Consequently, for the time-dependent matrix polynomials

$$P^{[1],t}(x) = S^t_1 \chi(x), \qquad P^{[2],t}(y) = S^t_2 \chi(y),$$

the biorthogonality conditions hold

$$\left\langle P^{[1],t}_n(x), P^{[2],t}_m(y) \right\rangle_{u^t} = \delta_{n,m} H^t_n.$$

We also will need the second kind functions

$$C^{[1],t}_n(z) = \left\langle P^{[1],t}_n(x), \frac{I_p}{z-y} \right\rangle_{u^t}, \qquad \left(C^{[2],t}_n(z)\right)^\top = \left\langle \frac{I_p}{z-x}, P^{[2],t}_n(y) \right\rangle_{u^t}.$$

and the Christoffel–Darboux kernel and its mixed versions

$$K^t_n(x,y) = \sum_{k=0}^{n} (P^{[2],t}_k(y))^\top (H^t_k)^{-1} P^{[1],t}_k(x), \quad K^{(pc),t}_n(x,y) = \sum_{k=0}^{n} \left(P^{[2],t}_k(y)\right)^\top (H^t_k)^{-1} C^{[1],t}_k(x).$$

Proposition (Non-Abelian 2D Toda lattice equations)

Using the notation $t_{1,1} = \eta$ *and* $t_{2,1} = \zeta$,

$$\frac{\partial}{\partial \zeta}\left(\frac{\partial H_k}{\partial \eta}(H_k)^{-1}\right) + H_{k+1}(H_k)^{-1} - H_k(H_{k-1})^{-1} = 0,$$

For the Hankel case we find a reduction to the non-Abelian 1D Toda lattice equation, where $\eta = \zeta$,

$$\frac{\partial}{\partial \eta}\left(\frac{\partial H_k}{\partial \eta}(H_k)^{-1}\right) + H_{k+1}(H_k)^{-1} - H_k(H_{k-1})^{-1} = 0.$$

Transformation theory

We now proceed to the study of how the previous Geronimus–Uvarov transformations affect the solutions'of the non-Abelian 2D Toda lattice.

Notice that since $[\Lambda, V_0^t] = 0$, given a Geronimus–Uvarov transformation

$$\hat{u}_{x,y} W_G(y) = W_C(x) u_{x,y}, \qquad \hat{G} W_G(\Lambda^\top) = W_C(\Lambda) G,$$

we will have the corresponding *t*-evolved equations

$$\hat{u}^t_{x,y} W_G(y) = W_C(x) u^t_{x,y}, \qquad \hat{G}^t W_G(\Lambda^\top) = W_C(\Lambda) G^t,$$

where

$$\hat{u}^t_{x,y} = \mathrm{e}^{t_1(x) - t_2(y)} \, \hat{u}_{x,y}, \qquad \hat{G}^t = V_0^{t_1} \hat{G} \left(V_0^{t_2} \right)^{-\top}.$$

Christoffel–Geronimus–Uvarov type formulas

$$\hat{P}_n^{[1],t}(x) = \Theta_* \begin{bmatrix} \mathcal{J}_{C,P^{[1],t}_{n-N_G}} & \mathcal{J}_{G,C^{[1],t}_{n-N_G}} - \left\langle P^{[1],t}_{n-N_G}(x), (\xi)_x \right\rangle \mathcal{W}_G & P^{[1],t}_{n-N_G}(x) \\ \vdots & \vdots & \vdots \\ \mathcal{J}_{C,P^{[1]}_{n+N_C}} & \mathcal{J}_{G,C^{[1],t}_{n+N_C}} - \left\langle P^{[1],t}_{n+N_C}(x), (\xi)_x \right\rangle \mathcal{W}_G & P^{[1],t}_{n+N_C}(x) \end{bmatrix}$$

$$\left(\hat{P}_n^{[2],t}(y)\right)^{\top}$$

$$= -\Theta_* \begin{bmatrix} \mathcal{J}_{C,P^{[1],t}_{n-N_G}} & \mathcal{J}_{G,C^{[1],t}_{n-N_G}} - \left\langle P^{[1],t}_{n-N_G}(x), (\xi)_x \right\rangle \mathcal{W}_G & H^t_{n-N_G} \\ \mathcal{J}_{C,P^{[1],t}_{n-N_G+1}} & \mathcal{J}_{G,C^{[1],t}_{n-N_G+1}} - \left\langle P^{[1],t}_{n-N_G+1}(x), (\xi)_x \right\rangle \mathcal{W}_G & 0_p \\ \vdots & \vdots & \\ \mathcal{J}_{C,P^{[1],t}_{n+N_C-1}} & \mathcal{J}_{G,C^{[1],t}_{n+N_C-1}} - \left\langle P^{[1],t}_{n+N_C-1}(x), (\xi)_x \right\rangle \mathcal{W}_G & 0_p \\ W_G(y)\mathcal{J}_{C,K^t_{n-1}}(y) & W_G(y)\left(\mathcal{J}_{G,K^{(pc),t}_{n-1}}(y) - \left\langle K^t_{n-1}(x,y), (\xi)_x \right\rangle \mathcal{W}_G\right) + \mathcal{J}_{G,\mathcal{V}}(y) & 0_p \end{bmatrix}.$$

In particular, $\hat{H}_n^t$ is a new solution of the non-Abelian 2D Toda lattice equation

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