Noise Analysis of Eigenlevel Sequences in Quantum Chaotic Systems

Eugene Kanzieper

Department of Applied Mathematics @ Holon Institute of Technology

joint work with

Roman Riser @ H.I.T. ➢ University of Haifa (Israel)
Vladimir Osipov @ UCLA Irvine (USA)

Centro di Ricerca Matematica Ennio De Giorgi, Pisa, Italy – May 18, 2018

PRL 118, 204101 (2017) PHYSICAL REVIEW LETTERS week ending 19 MAY 2017

Power Spectrum of Long Eigenlevel Sequences in Quantum Chaotic Systems

Roman Riser,[1] Vladimir Al. Osipov,[2] and Eugene Kanzieper[1]
[1]*Department of Applied Mathematics, H.I.T.–Holon Institute of Technology, Holon 5810201, Israel*
[2]*Division of Chemical Physics, Lund University, Getingevägen 60, Lund 22241, Sweden*
(Received 23 March 2017; published 16 May 2017)

Editors' Suggestion

Eugene Kanzieper

Department of Applied Mathematics @ Holon Institute of Technology

joint work with

Roman Riser @ H.I.T. ➢ University of Haifa (Israel)
Vladimir Osipov @ UCLA Irvine (USA)

Centro di Ricerca Matematica Ennio De Giorgi, Pisa, Italy – May 18, 2018

Outline

- Introduction
  - Two universality classes in quantum chaology (billiards)
  - Level spacing distributions: "Poisson" vs Wigner-Dyson, random matrices (RMT) and Painlevé V
- Power spectrum analysis of quantum chaotic spectra
  - Definition and numerics
  - Theory – do we need a revision?
  - Non-perturbative theory and RMT Painlevé VI solution
  - Universal law for the power spectrum: Painlevé V through asymptotic analysis
- Power spectrum for zeros of the Riemann zeta function
  (work in progress)
- Conclusions

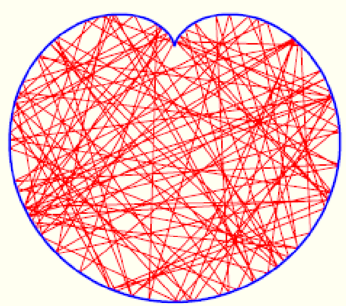
Introduction: Two universality classes

Quantum Chaology

statistical properties of quantum systems whose underlying classical dynamics is either regular or chaotic

Billiards: classical geodesics vs quantum behavior

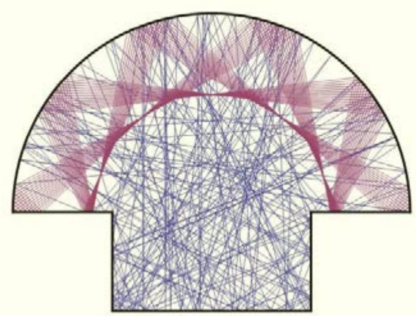
Cardioid



chaotic geodesics

© Bäcker 2007

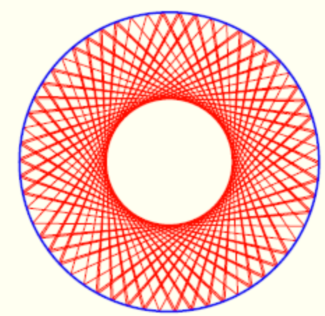
Mushroom



mixed phase space

© Dettmann & Georgiou 2011

Circular



regular geodesics

© Bäcker 2007

Introduction: Two universality classes

$$
\begin{cases}
-(\hbar^2/2m)\Delta\psi_n(\boldsymbol{r}) = E_n\psi_n(\boldsymbol{r}), & \boldsymbol{r} \in \mathcal{B}; \\
\psi_n(\boldsymbol{r}) = 0, & \boldsymbol{r} \in \partial\mathcal{B}.
\end{cases}
$$

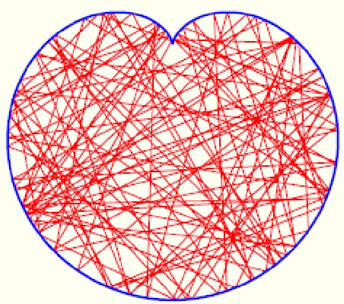
spectral

$$\{E_n\}$$

statistics

Billiards: classical geodesics vs quantum behavior

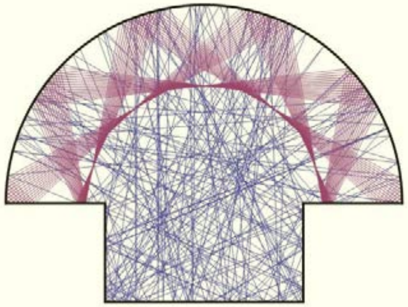
Cardioid



chaotic geodesics

© Bäcker 2007

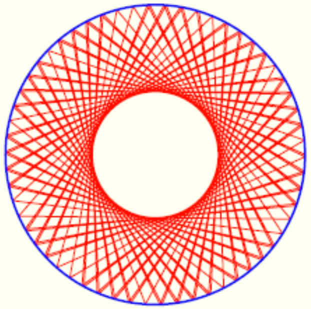
Mushroom



mixed phase space

© Dettmann & Georgiou 2011

Circular



regular geodesics

© Bäcker 2007

Introduction: Two universality classes

level spacing distribution

$$P(s,N) = \frac{1}{N}\sum_{j=1}^{N} \delta(s - E_{j+1} + E_j)$$

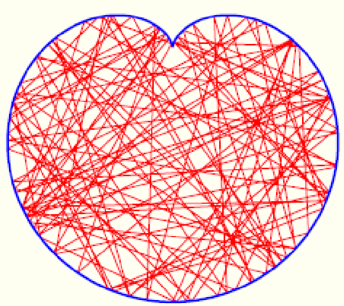
$P(s)$:

$$\lim_{N\to\infty} \int_0^\infty P(s,N)h(s)ds = \int_0^\infty P(s)h(s)ds$$

spectral $\{E_n\}$ statistics

Billiards: classical geodesics vs quantum behavior

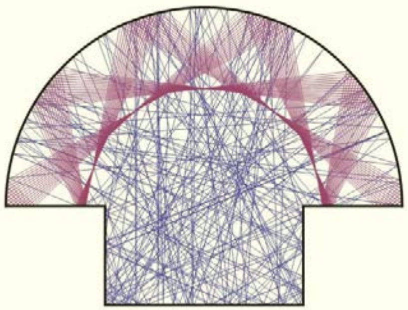
Cardioid



chaotic geodesics

© Bäcker 2007

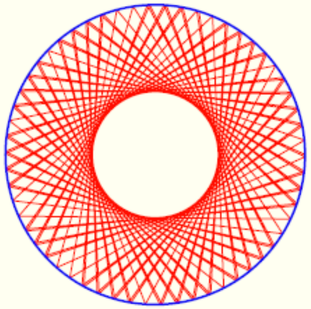
Mushroom



mixed phase space

© Dettmann & Georgiou 2011

Circular



regular geodesics

© Bäcker 2007

Introduction: Two universality classes

spectral

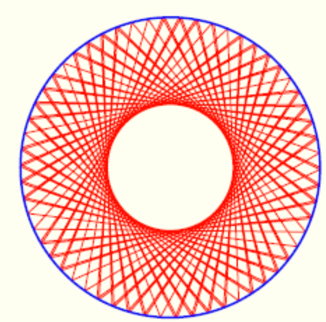
$\{E_n\}$

statistics

- **Billiards: classical geodesics vs quantum behavior**
- **Conjecture by Berry and Tabor (1977)**
  - *Eigenlevels of* ***generic*** *quantum systems with completely* ***integrable classical dynamics*** *exhibit statistics of a* ***Poisson*** *point process.*
  - ***Level spacings*** $s_n = E_{n+1} - E_n$ *fluctuate as the* ***waiting time*** *between consecutive events*

level spacing distribution

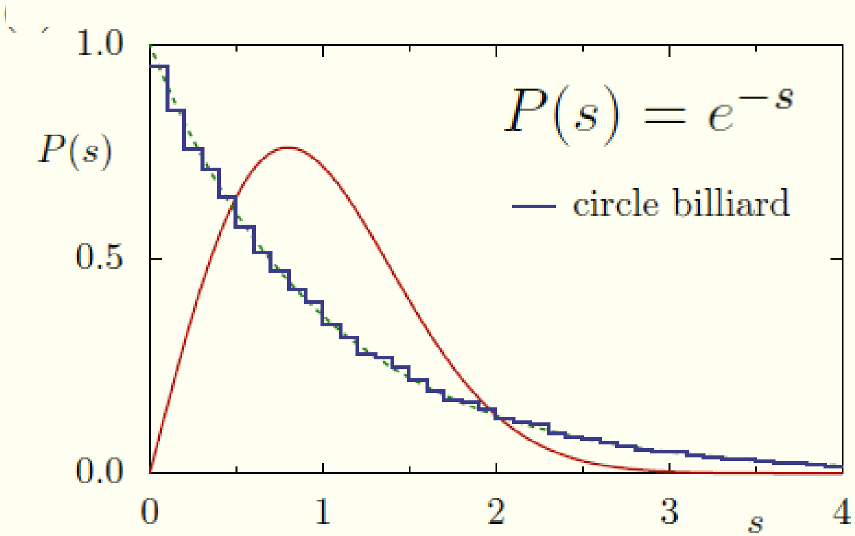
Circular



regular geodesics

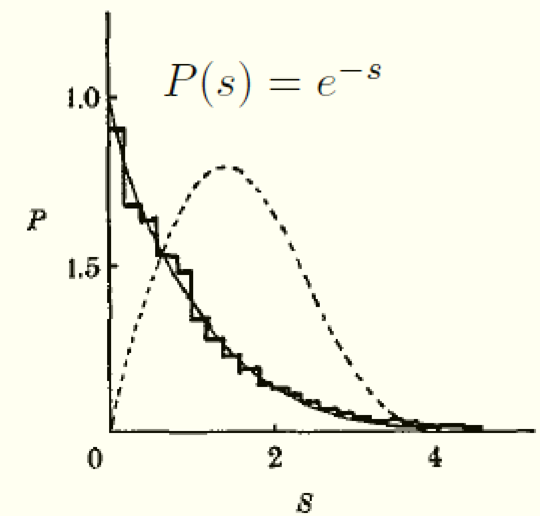
Introduction: Two universality classes

Circular billiard



Bäcker 2007

Rectangular billiard

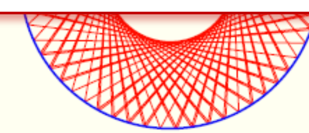


Berry 1987

level
distrib

point process.

- ***Level spacings*** $s_n = E_{n+1} - E_n$ *fluctuate as the* ***waiting time*** *between consecutive events*



regular geodesics

© Bäcker 2007

Introduction: Two universality classes

spectral

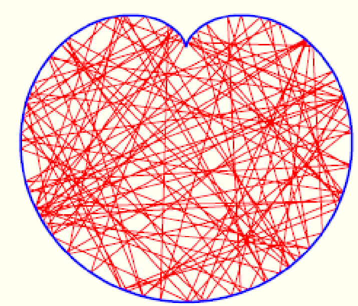
$\{E_n\}$

statistics

- **Billiards: classical geodesics vs quantum behavior**
- **Bohigas-Giannoni-Schmit Conjecture**
  - *Eigenlevels of* ***generic*** *quantum systems with completely* ***chaotic classical dynamics*** *exhibit the* ***RMT statistics*** *(for the proper symmetry index).*
  - ***Level spacings****:* ***Painleve V*** *(exact) or can be approximated by the* ***Wigner surmise***

level spacing distribution

Cardioid



chaotic geodesics

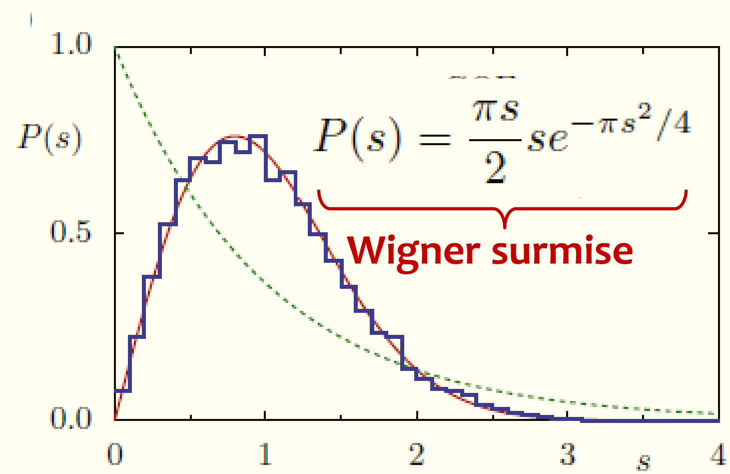
© Bäcker 2007

Introduction: Two universality classes

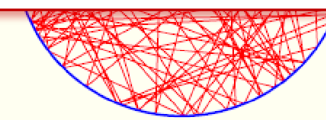
- **Billiards:** classical geo
- **Bohigas-Giannoni-Sch**
  - *Eigenlevels of* ***ge*** *systems with cor* ***classical dynami*** ***RMT statistics*** *(for the proper symmetry index).*
  - ***Level spacings:*** ***Painleve V*** *(exact) or can be approximated by the* ***Wigner surmise***

level spacing distribution

Cardioid billiard



Bäcker 2007



chaotic geodesics

© Bäcker 2007

Level spacing distributions (summary)

$$\begin{cases} -(\hbar^2/2m)\Delta\psi_n(\boldsymbol{r}) = E_n\psi_n(\boldsymbol{r}), & \boldsymbol{r}\in\mathcal{B}; \\ \psi_n(\boldsymbol{r}) = 0, & \boldsymbol{r}\in\partial\mathcal{B}. \end{cases}$$

spectral $\{E_n\}$ statistics

- Regular classical dynamics $P_{\text{reg}}(s) = \exp(-s)$ "Poisson"
- Chaotic classical dynamics (RMT – Wigner-Dyson statistics)

$$P_{\text{chaos}}(s) = \frac{d^2}{ds^2}\boxed{\exp\left(\int_0^{2\pi s}\frac{\sigma_0(t)}{t}\,dt\right)}$$

Painlevé V

Fredholm determinant

+ BCs

$$(t\sigma_\nu'')^2 + (t\sigma_\nu' - \sigma_\nu)(t\sigma_\nu' - \sigma_\nu + 4(\sigma_\nu')^2) - 4\nu^2(\sigma_\nu')^2 = 0$$

$$\boxed{E_0(s) = \sum_{p=0}^{\infty}\frac{(-1)^p}{p!}\int_0^s dx_1\cdots\int_0^s dx_p\;R_p(x_1,\dots,x_p)}$$

correlation functions of all orders

Gap formation probability out of the entire set $\{E_j(s)\}$

Outline

- Introduction
  - Two universality classes in quantum chaology (billiards)
  - Level spacing distributions: “Poisson” vs Wigner-Dyson, random matrices (RMT) and Painlevé V
- **Power spectrum analysis of quantum chaotic spectra**
  - **Definition and numerics**
  - Theory – do we need a revision?
  - Non-perturbative theory and RMT Painlevé VI solution
  - Universal law for the power spectrum: Painlevé V through asymptotic analysis
- Power spectrum for zeros of the Riemann zeta function
  (work in progress)
- Conclusions

Power spectrum analysis of quantum spectra

VOLUME 89, NUMBER 24 PHYSICAL REVIEW LETTERS 9 DECEMBER 2002

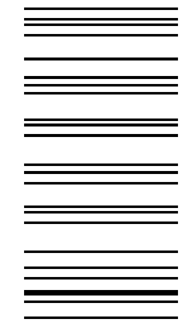
Quantum Chaos and $1/f$ Noise

A. Relaño, J. M. G. Gómez, R. A. Molina, and J. Retamosa
Departamento de Física Atómica, Molecular y Nuclear, Universidad Complutense de Madrid, E-28040 Madrid, Spain

E. Faleiro
Departamento de Física Aplicada, E.U.I.T. Industrial, Universidad Politécnica de Madrid, E-28012 Madrid, Spain
(Received 25 February 2002; published 22 November 2002)

It is shown that the energy spectrum fluctuations of quantum systems can be formally considered as a discrete time series. The power spectrum behavior of such a signal for different systems suggests the following conjecture: The energy spectra of chaotic quantum systems are characterized by $1/f$ noise.

RMT



eigenvalues

- **unfolded ordered eigenlevels** $\{\epsilon_1, \epsilon_2, \ldots, \epsilon_{N-1}, \epsilon_N\}$ $\quad \langle \epsilon_{k+1} - \epsilon_k \rangle = 1$
  mean spacing

- **deviations from the mean**
  (discrete "time" random process)
  $\{\delta\epsilon_1, \delta\epsilon_2, \ldots, \delta\epsilon_N\} \quad \delta\epsilon_\ell = \epsilon_\ell - \langle \epsilon_\ell \rangle$

- **discrete Fourier transforms**
  $k = 1, 2, \ldots, N/2$
  $$a_k = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \delta\epsilon_j \, e^{ij\omega_k}, \quad \omega_k = \frac{2\pi k}{N}$$

- **power spectrum**
  NEW statistics
  $$S_N(\omega_k) = \langle |a_k|^2 \rangle = \frac{1}{N} \sum_{j,\ell=1}^{N} \langle \delta\epsilon_j \delta\epsilon_\ell \rangle e^{i\omega_k (j-\ell)}$$

Power spectrum analysis of quantum spectra

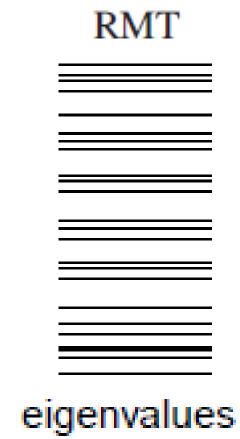
VOLUME 89, NUMBER 24 PHYSICAL REVIEW LETTERS 9 DECEMBER 2002

Quantum Chaos and $1/f$ Noise

A. Relaño, J. M. G. Gómez, R. A. Molina, and J. Retamosa
Departamento de Física Atómica, Molecular y Nuclear, Universidad Complutense de Madrid, E-28040 Madrid, Spain

E. Faleiro
Departamento de Física Aplicada, E.U.I.T. Industrial, Universidad Politécnica de Madrid, E-28012 Madrid, Spain
(Received 25 February 2002; published 22 November 2002)

It is shown that the energy spectrum fluctuations of quantum systems can be formally considered as a discrete time series. The power spectrum behavior of such a signal for different systems suggests the following conjecture: The energy spectra of chaotic quantum systems are characterized by $1/f$ noise.



- **deviations from the mean**
  (discrete "time" random process)

$$\{\delta\epsilon_1, \delta\epsilon_2, \ldots, \delta\epsilon_N\} \quad \delta\epsilon_\ell = \epsilon_\ell - \langle\epsilon_\ell\rangle$$

$$\boxed{\omega_k \sim 1 \quad (k \sim O(N))}: \quad \text{finite } |j-\ell| \text{ contribute}$$ nearby eigenlevels

$$\boxed{\omega_k \ll 1 \quad (k \ll N)}: \quad \text{large } |j-\ell| \text{ contribute}$$ distant eigenlevels

$$\omega_k = \frac{2\pi k}{N}$$

- **power spectrum**
  NEW statistics

$$S_N(\omega_k) = \langle |a_k|^2 \rangle = \frac{1}{N} \sum_{j,\ell=1}^{N} \langle \delta\epsilon_j \delta\epsilon_\ell \rangle e^{i\omega_k (j-\ell)}$$

Power spectrum analysis of quantum spectra

VOLUME 89, NUMBER 24 PH

A. Relaño

Departamento de Física Atómica, Mo

Departamento de Física Aplicada, I

(Receive

It is shown that the energy s
discrete time series. The pow
following conjecture: The ene

Random matrix simulations

$$S(\omega_k) \sim \frac{1}{\omega_k^{\alpha}}$$?

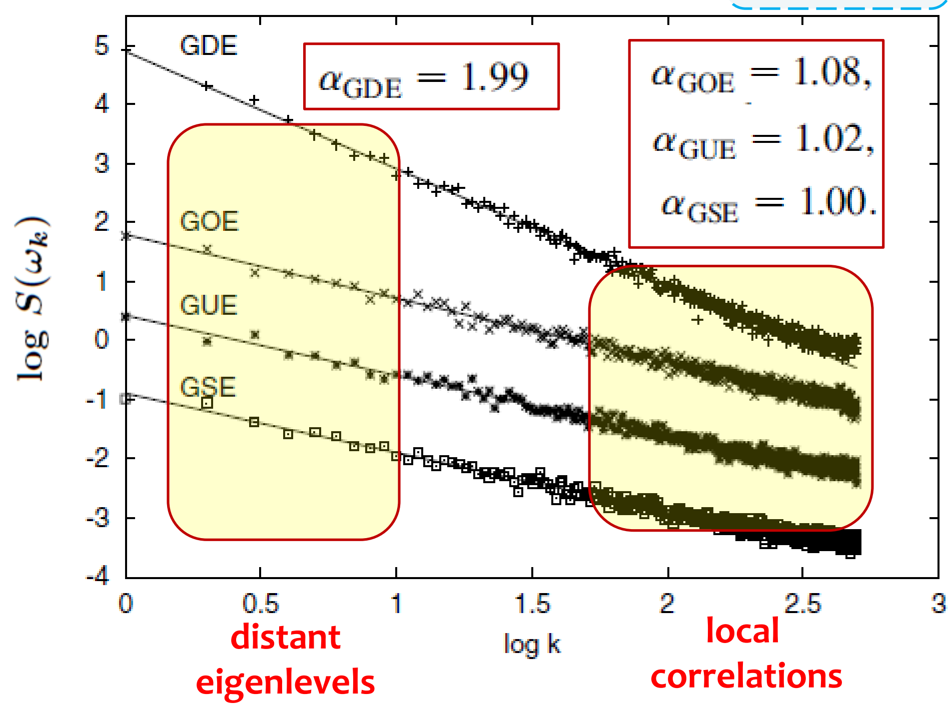


FIG. 3. Power spectrum of the δ_n function for GDE (Poisson) energy levels, compared to GOE, GUE, and GSE. The plots are displaced to avoid overlapping.

- **deviations from**

  (discrete "time" ran

$\omega_k \sim 1 \quad (k \sim O(N$

$\omega_k \ll 1 \quad (k \ll N$

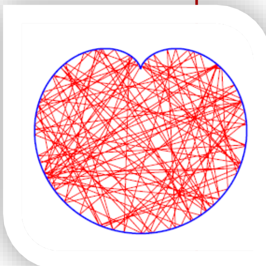
$$\omega_k = \frac{2\pi k}{N}$$

- power spectrum

  NEW statistics

$$S_N(\omega_k) = \langle |a_k|^2 \rangle = \frac{1}{N} \sum_{j,\ell=1}^{N} \langle \delta\epsilon_j \delta\epsilon_\ell \rangle e^{i\omega_k (j-\ell)}$$

Shell-model spectra in 24Mg and 34Na



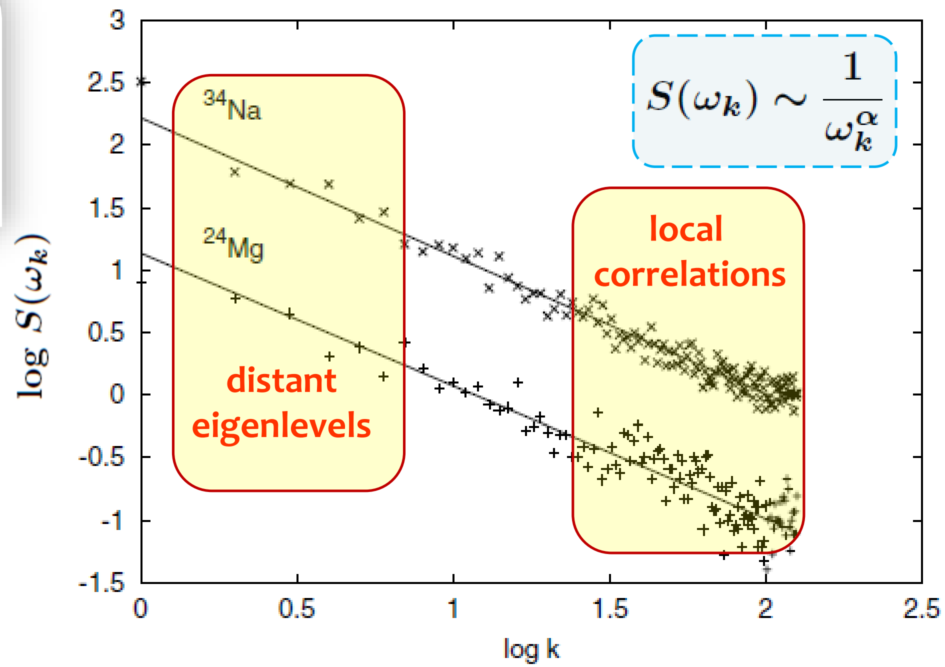
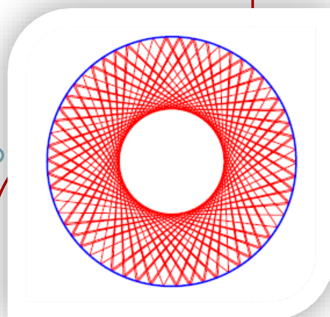


FIG. 1. Average power spectrum of the δ_n function for ^{24}Mg and ^{34}Na, using 25 sets of 256 levels from the high level density region. The plots are displaced to avoid overlapping.

Figure 1 shows the results for a typical stable *sd* shell nucleus, ^{24}Mg, with matrix dimensionalities up to about 2000; and for a very exotic nucleus, ^{34}Na, with dimensions up to about 5000, in the *sd* proton and *pf* neutron shells. Clearly, the power spectrum of δ_n follows closely a

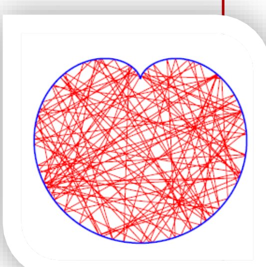
A least squares fit to the data of Fig. 1 gives $\alpha = 1.11 \pm 0.03$ for ^{34}Na, and $\alpha = 1.06 \pm 0.05$ for ^{24}Mg. These re-

Rectangular billiard



$$S_N^{(\mathrm{reg})}(\omega_k) \sim \frac{1}{\omega_k^2}$$

$$S_N^{(\mathrm{chaos})}(\omega_k) \sim \frac{1}{\omega_k}$$



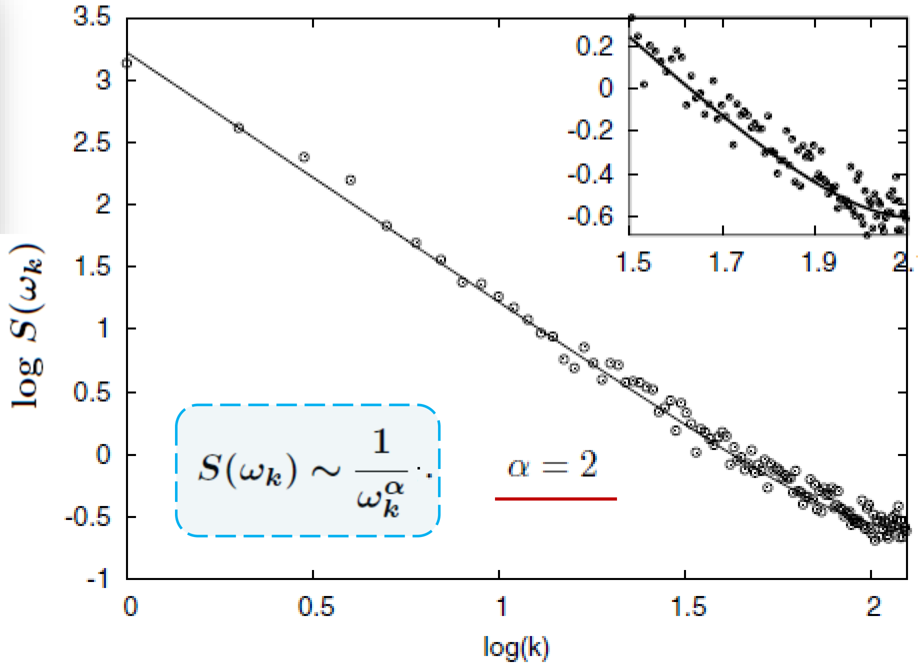


FIG. 3. Numerical average power spectrum of δ_q for a rectangular billiard, calculated using 25 sets of 256 consecutive levels, compared to the parameter free theoretical values (solid line) for integrable systems.

a rectangular billiard with sides of length $a = \sqrt{\lambda}$ and $b = 1/\sqrt{\lambda}$, with $\lambda = (\sqrt{5}+1)/2$; this geometry gives rise to an irrational ratio $a/b = \lambda$, and thus there are no degeneracies in the spectrum. We have calculated the

Power spectrum analysis of quantum spectra

PRL 93, 244101 (2004) PHYSICAL REVIEW LETTERS week ending 10 DECEMBER 2004

Theoretical Derivation of $1/f$ Noise in Quantum Chaos

E. Faleiro,[1] J. M. G. Gómez,[2] R. A. Molina,[3] L. Muñoz,[2] A. Relaño,[2] and J. Retamosa[2]

[1]*Departamento de Física Aplicada, E. U. I. T. Industrial, Universidad Politécnica de Madrid, E-28012 Madrid, Spain*

[2]*Departamento de Física Atómica, Molecular y Nuclear, Universidad Complutense de Madrid, E-28040 Madrid, Spain*

[3]*Max-Planck-Institut für Komplexer Systeme, D-01187 Dresden, Germany*

(Received 24 February 2004; published 6 December 2004)

It was recently conjectured that $1/f$ noise is a fundamental characteristic of spectral fluctuations in chaotic quantum systems. This conjecture is based on the power spectrum behavior of the excitation energy fluctuations, which is different for chaotic and integrable systems. Using random matrix theory, we derive theoretical expressions that explain without free parameters the universal behavior of the excitation energy fluctuations power spectrum. The theory gives excellent agreement with numerical calculations and reproduces to a good approximation the $1/f$ ($1/f^2$) power law characteristic of chaotic (integrable) systems. Moreover, the theoretical results are valid for semiclassical systems as well.

- **assumption: power spectrum contributed by two-point correlations**

spectral form-factor

$$\boxed{K_\beta(\tau) = \frac{1}{N} \iint d\epsilon d\epsilon' e^{2i\pi\tau(\epsilon-\epsilon')} \langle \delta\varrho_N(\epsilon)\delta\varrho_N(\epsilon')\rangle}$$

$$\boxed{K_\beta(\tau) = \frac{1}{N}\left(\left\langle \sum_{\ell=1}^{N}\sum_{m=1}^{N} e^{2i\pi\tau(\epsilon_\ell-\epsilon_m)}\right\rangle - \left\langle \sum_{\ell=1}^{N} e^{2i\pi\tau\epsilon_\ell}\right\rangle\left\langle \sum_{m=1}^{N} e^{-2i\pi\tau\epsilon_m}\right\rangle\right)}$$

Power spectrum analysis of quantum spectra

PRL 93, 244101 (2004) PHYSICAL REVIEW LETTERS week ending 10 DECEMBER 2004

Theoretical Derivation of 1/f Noise in Quantum Chaos

E. Faleiro,[1] J. M. G. Gómez,[2] R. A. Molina,[3] L. Muñoz,[2] A. Relaño,[2] and J. Retamosa[2]

$$S_N(\omega_k) = \mathfrak{F}\left[K_\beta\left(\frac{\omega_k}{2\pi}\right)\right]$$

- assumption: power spectrum contributed by two-point correlations

spectral form-factor

$$K_\beta(\tau) = \frac{1}{N}\iint d\epsilon d\epsilon' e^{2i\pi\tau(\epsilon-\epsilon')}\langle\delta\varrho_N(\epsilon)\delta\varrho_N(\epsilon')\rangle$$

$$K_\beta(\tau) = \frac{1}{N}\left(\left\langle\sum_{\ell=1}^{N}\sum_{m=1}^{N} e^{2i\pi\tau(\epsilon_\ell-\epsilon_m)}\right\rangle - \left\langle\sum_{\ell=1}^{N} e^{2i\pi\tau\epsilon_\ell}\right\rangle\left\langle\sum_{m=1}^{N} e^{-2i\pi\tau\epsilon_m}\right\rangle\right)$$

Power spectrum analysis of quantum spectra

PRL 93, 244101 (2004) PHYSICAL REVIEW LETTERS week ending 10 DECEMBER 2004

Theoretical Derivation of 1/f Noise in Quantum Chaos

E. Faleiro,[1] J. M. G. Gómez,[2] R. A. Molina,[3] L. Muñoz,[2] A. Relaño,[2] and J. Retamosa[2]

$$\boxed{S_N(\omega_k \ll 1) \simeq \frac{1}{\omega_k^2} K_\beta\left(\frac{\omega_k}{2\pi}\right)} \longrightarrow \simeq \frac{1}{\omega_k^2}$$

instead of $\simeq \frac{2}{\omega_k^2}$

- regular classical dynamics ⟶ $K_\beta^{(\mathrm{reg})}(\tau) = 1$

higher order correlations !?

- chaotic classical dynamics ⟶ $K_\beta^{(\mathrm{chaos})}(\tau) = \frac{2\tau}{\beta} \xrightarrow{\text{status?}} S_N(\omega_k) \simeq \frac{1}{\pi\beta\omega_k}$

➢ assumption: power spectrum contributed by two-point correlations

spectral form-factor

$$\boxed{K_\beta(\tau) = \frac{1}{N} \iint d\epsilon d\epsilon' e^{2i\pi\tau(\epsilon-\epsilon')} \langle \delta\varrho_N(\epsilon)\delta\varrho_N(\epsilon')\rangle}$$

$$\boxed{K_\beta(\tau) = \frac{1}{N}\left(\left\langle \sum_{\ell=1}^{N}\sum_{m=1}^{N} e^{2i\pi\tau(\epsilon_\ell-\epsilon_m)}\right\rangle - \left\langle \sum_{\ell=1}^{N} e^{2i\pi\tau\epsilon_\ell}\right\rangle\left\langle \sum_{m=1}^{N} e^{-2i\pi\tau\epsilon_m}\right\rangle\right)}$$

Power spectrum analysis of quantum spectra

- power spectrum

$$S_N(\omega_k) = \langle |a_k|^2 \rangle = \frac{1}{N} \sum_{j,\ell=1}^{N} \langle \delta\epsilon_j \delta\epsilon_\ell \rangle e^{i\omega_k (j-\ell)}$$

$$S_N(\omega_k) = \frac{1}{N\omega_k^2} \iint d\epsilon d\epsilon' e^{i\omega_k(\epsilon-\epsilon')} \left\langle \delta\varrho_N(\epsilon)\delta\varrho_N(\epsilon')\, \mathcal{V}_{\delta\varrho_N}[\omega_k;\epsilon,\epsilon'] \right\rangle$$

$$\mathcal{V}_{\delta\varrho_N}[\omega_k;\epsilon,\epsilon'] = \exp\left(i\omega_k \int_{\epsilon'}^{\epsilon} d\lambda\, \delta\varrho_N(\lambda) \right)$$

Exact !!

- assumption: power spectrum contributed by two-point correlations

Form-factor approximation:

$$\boxed{S_N(\omega_k \ll 1) \simeq \frac{1}{N\omega_k^2} \iint d\epsilon d\epsilon' e^{i\omega_k(\epsilon-\epsilon')} \langle \delta\varrho_N(\epsilon)\delta\varrho_N(\epsilon') \rangle}$$

Outline

- Introduction
  - Two universality classes in quantum chaology (billiards)
  - Level spacing distributions: "Poisson" vs Wigner-Dyson, random matrices (RMT) and Painlevé V
- **Power spectrum analysis of quantum chaotic spectra**
  - Definition and numerics
  - **Theory – do we need a revision?** YES !
  - Non-perturbative theory and RMT Painlevé VI solution
  - Universal law for the power spectrum: Painlevé V through asymptotic analysis
- Power spectrum for zeros of the Riemann zeta function (work in progress)
- Conclusions

Outline

- Introduction
  - Two universality classes in quantum chaology (billiards)
  - Level spacing distributions: "Poisson" vs Wigner-Dyson, random matrices (RMT) and Painlevé V
- **Power spectrum analysis of quantum chaotic spectra**
  - Definition and numerics
  - Theory – do we need a revision?
  - **Non-perturbative theory** and RMT Painlevé VI solution
  - Universal law for the power spectrum: Painlevé V through asymptotic analysis
- Power spectrum for zeros of the Riemann zeta function
  (work in progress)
- Conclusions

Non-perturbative theory for spectra with stationary spacings

$$S_M(\omega) = \frac{1}{M}\sum_{\ell=1}^{M}\sum_{m=1}^{M}\langle\delta\varepsilon_\ell\delta\varepsilon_m\rangle\, e^{i\omega(\ell-m)} = \frac{1}{M}\sum_{\ell=1}^{M}\sum_{m=1}^{M}\langle\delta\varepsilon_\ell\delta\varepsilon_m\rangle\, z^{\ell-m}$$

$$\boxed{z = e^{i\omega}}$$

(1) **Definition.** Consider an ordered sequence of (unfolded) eigenlevels $\{\varepsilon_1 \le \cdots \le \varepsilon_M\}$. Let $\{s_1, \cdots, s_M\}$ be a sequence of spacings between consecutive eigenlevels such that $s_k = \varepsilon_k - \varepsilon_{k-1}$ with $\varepsilon_0 = 0$. The **sequence of level spacings** is said to be **stationary** if (i) the average

$$\langle s_k\rangle = 1$$

is independent of k and (ii) the covariance matrix of *spacings* is of the Toeplitz type:

$$\mathrm{cov}(s_j, s_k) = I_{|j-k|} - 1.$$

(2) **Lemma.** A sequence of spacings is stationary if and only if

$$\langle(\varepsilon_\ell - \varepsilon_m)^q\rangle = \langle\varepsilon_{\ell-m}^q\rangle$$

for both $q = 1$ and $q = 2$, and $1 \le m < \ell \le M$.

(3) **Remark.**

$$\langle\delta\varepsilon_\ell\delta\varepsilon_m\rangle = \frac{1}{2}\left(\langle\delta\varepsilon_\ell^2\rangle + \langle\delta\varepsilon_m^2\rangle - \langle\delta\varepsilon_{|\ell-m|}^2\rangle\right),$$

where $\delta\varepsilon_\ell = \varepsilon_\ell - \ell$.

Non-perturbative theory for spectra with stationary spacings

$$S_M(\omega) = \frac{1}{M}\sum_{\ell=1}^{M}\sum_{m=1}^{M}\langle\delta\varepsilon_\ell\delta\varepsilon_m\rangle\, e^{i\omega(\ell-m)} = \frac{1}{M}\sum_{\ell=1}^{M}\sum_{m=1}^{M}\langle\delta\varepsilon_\ell\delta\varepsilon_m\rangle\, z^{\ell-m}$$

$$z = e^{i\omega}$$

$$S_M(\omega) = \frac{1}{M}\mathrm{Re}\left(z\frac{\partial}{\partial z} - M - \frac{1-z^{-M}}{1-z}\right)\sum_{\ell=1}^{M}\mathrm{var}[\varepsilon_\ell]\, z^\ell$$

GF of variances

(3) **Remark.**

$$\langle\delta\varepsilon_\ell\delta\varepsilon_m\rangle = \frac{1}{2}\left(\langle\delta\varepsilon_\ell^2\rangle + \langle\delta\varepsilon_m^2\rangle - \langle\delta\varepsilon_{|\ell-m|}^2\rangle\right),$$

where $\delta\varepsilon_\ell = \varepsilon_\ell - \ell$.

Non-perturbative theory for spectra with stationary spacings

$$S_M(\omega) = \frac{1}{M}\sum_{\ell=1}^{M}\sum_{m=1}^{M}\langle \delta\varepsilon_\ell \delta\varepsilon_m\rangle\, e^{i\omega(\ell-m)} = \frac{1}{M}\sum_{\ell=1}^{M}\sum_{m=1}^{M}\langle \delta\varepsilon_\ell \delta\varepsilon_m\rangle\, z^{\ell-m}$$

$$z = e^{i\omega}$$

$$S_M(\omega) = \frac{1}{M}\mathrm{Re}\left(z\frac{\partial}{\partial z} - M - \frac{1-z^{-M}}{1-z}\right)\sum_{\ell=1}^{M}\mathrm{var}[\varepsilon_\ell]\, z^\ell$$

GF of variances

$$\mathrm{var}[\varepsilon_\ell] = \int_0^\infty d\epsilon\, \epsilon^2 p_\ell(\epsilon) - \ell^2$$

probability density of ℓ-th ordered eigenvalue

Lemma. Let $\{\varepsilon_1 \le \cdots \le \varepsilon_M\}$ be an ordered sequence of eigenlevels supported on the half axis $(0,\infty)$, and let $E_M(\ell;\varepsilon)$ be the probability to find exactly ℓ eigenvalues below the energy ε. The following relation holds:

$$\frac{d}{d\varepsilon}E_M(\ell;\varepsilon) = p_\ell(\varepsilon) - p_{\ell+1}(\varepsilon).$$

Here, $p_\ell(\varepsilon)$ is the probability density of ℓ-th ordered eigenlevel such that $p_0(\varepsilon) = p_{M+1}(\varepsilon) = 0$ for $\varepsilon > 0$. Equivalently,

$$p_\ell(\varepsilon) = -\sum_{j=0}^{\ell-1}\frac{d}{d\varepsilon}E_M(j;\varepsilon).$$

$$\Phi_M(\varepsilon; 1-z) = \sum_{\ell=0}^{M} z^\ell E_M(\ell;\varepsilon)$$

Non-perturbative theory for spectra with stationary spacings

$$S_M(\omega) = \frac{1}{M}\sum_{\ell=1}^{M}\sum_{m=1}^{M}\langle\delta\varepsilon_\ell\delta\varepsilon_m\rangle\, e^{i\omega(\ell-m)} = \frac{1}{M}\sum_{\ell=1}^{M}\sum_{m=1}^{M}\langle\delta\varepsilon_\ell\delta\varepsilon_m\rangle\, z^{\ell-m}$$

$$z = e^{i\omega}$$

$$S_M(\omega) = \frac{1}{M}\mathrm{Re}\left(z\frac{\partial}{\partial z} - M - \frac{1-z^{-M}}{1-z}\right)\sum_{\ell=1}^{M}\mathrm{var}[\varepsilon_\ell]\, z^\ell$$

GF of variances

Main result (stationarity of level spacings, no RMT argument involved)

$$S_M(\omega_k) = \frac{2}{M}\mathrm{Re}\left(z_k\frac{\partial}{\partial z_k} - M\right)\frac{z_k}{1-z_k}\int_0^\infty d\epsilon\,\epsilon\,[\Phi_M(\epsilon;1-z_k)-1] - \frac{M}{|1-z_k|^2}$$

$$z_k = e^{i\omega_k}$$

$$\omega_k = \frac{2\pi k}{M}$$

GF of the probabilities

$\{E_j(s)\}$

$$\Phi_M(\varepsilon;1-z) = \sum_{\ell=0}^{M} z^\ell E_M(\ell;\varepsilon)$$

Non-perturbative theory for spectra with stationary spacings

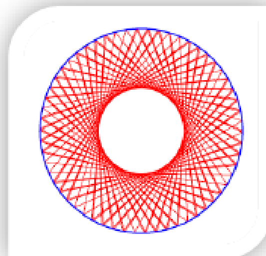
$S_M(\omega) =$

Main result (s

$S_M(\omega_k) = $

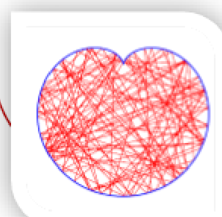
$z_k = e^{i\omega_k}$

$\omega_k = \frac{2\pi k}{M}$



$$S_N^{(\mathrm{reg})}(\omega_k) \sim \frac{1}{\omega_k^2}$$

$$S_N^{(\mathrm{chaos})}(\omega_k) \sim \frac{1}{\omega_k}$$



Shell-model spectra in $_{24}$Mg and $_{34}$Na

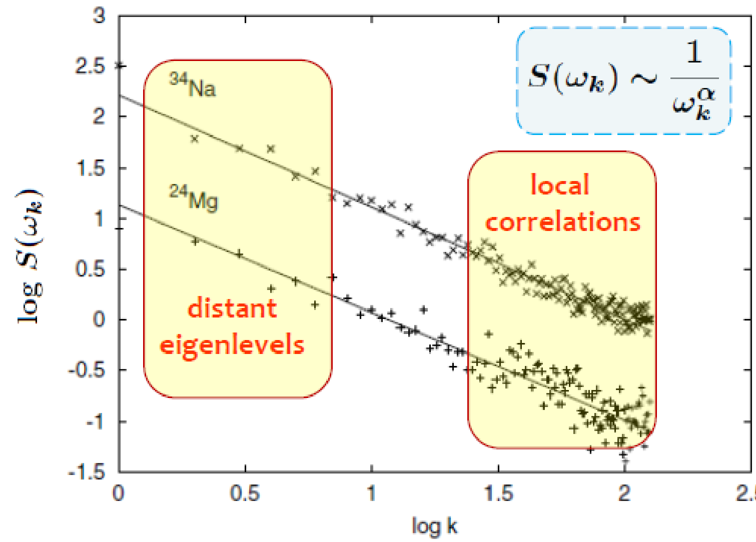


FIG. 1. Average power spectrum of the δ_n function for ^{24}Mg and ^{34}Na, using 25 sets of 256 levels from the high level density region. The plots are displaced to avoid overlapping.

Figure 1 shows the results for a typical stable *sd* shell nucleus, ^{24}Mg, with matrix dimensionalities up to about 2000; and for a very exotic nucleus, ^{34}Na, with dimensions up to about 5000, in the *sd* proton and *pf* neutron shells. Clearly, the power spectrum of δ_n follows closely a

A least squares fit to the data of Fig. 1 gives $\alpha = 1.11 \pm 0.03$ for ^{34}Na, and $\alpha = 1.06 \pm 0.05$ for ^{24}Mg. These re-

ω

es

$|^2$

$\ell=0$

Outline

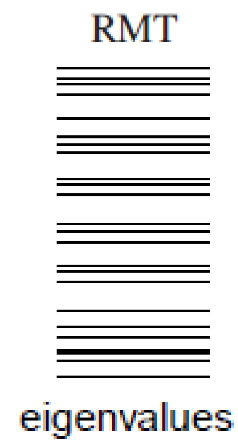
- Introduction
  - Two universality classes in quantum chaology (billiards)
  - Level spacing distributions: "Poisson" vs Wigner-Dyson, random matrices (RMT) and Painlevé V
- **Power spectrum analysis of quantum chaotic spectra**
  - Definition and numerics
  - Theory – do we need a revision?
  - **Non-perturbative theory and RMT Painlevé VI solution**
  - Universal law for the power spectrum: Painlevé V through asymptotic analysis
- Power spectrum for zeros of the Riemann zeta function
  (work in progress)
- Conclusions

Quantum chaotic systems & RMT Painlevé VI solution

BGS: Eigenlevels of **generic** quantum systems with completely **chaotic classical dynamics** exhibit the **RMT statistics – "(T)CUE"**

- **Sequence of CUE level spacings is not stationary!**
  (even though the **mean density** is a **constant**)

$$P_M(\boldsymbol{\theta}) \propto \prod_{\ell<m} \left|e^{i\theta_\ell} - e^{i\theta_m}\right|^2$$



Lemma. A sequence of spacings is stationary if and only if

$$\langle(\varepsilon_\ell - \varepsilon_m)^q\rangle = \langle\varepsilon^q_{\ell-m}\rangle$$

for both $q = 1$ and $q = 2$, and $1 \le m < \ell \le M$.

- **Sequence of Tuned CUE level spacings is stationary!**
  (even though the **mean density** is **not a constant**)

$$P_M(\boldsymbol{\theta}) \propto \prod_{\ell=1}^{M} \left|1 - e^{i\theta_\ell}\right|^2 \prod_{\ell<m}^{M} \left|e^{i\theta_\ell} - e^{i\theta_m}\right|^2$$

Quantum chaotic systems & RMT Painlevé VI solution

BGS: Eigenlevels of **generic** quantum systems with completely **chaotic classical dynamics** exhibit the **RMT statistics – "(T)CUE"**

Lemma 1. For $q = 0, 1, \ldots$ and $\ell = 1, 2, \ldots, M$ it holds that

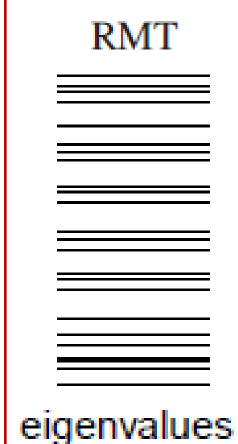
$$\langle \theta_\ell^q \rangle = \langle (2\pi - \theta_{M-\ell+1})^q \rangle .$$

Lemma 2. For $q = 0, 1, \ldots$ and $1 \le \ell < m \le M$ it holds that

$$\langle (\theta_m - \theta_\ell)^q \rangle = \langle \theta_{m-\ell}^q \rangle .$$

Lemma 3. It holds that

$$\langle \theta_\ell \rangle = \Delta_M \ell, \qquad \Delta_M = \frac{2\pi}{M+1}.$$



- **Sequence of Tuned CUE level spacings is stationary!** (even though the **mean density** is **not a constant**)

$$P_M(\boldsymbol{\theta}) \propto \prod_{\ell=1}^{M} \left|1 - e^{i\theta_\ell}\right|^2 \prod_{\ell<m}^{M} \left|e^{i\theta_\ell} - e^{i\theta_m}\right|^2$$

Quantum chaotic systems & RMT Painlevé VI solution

BGS: Eigenlevels of **generic** quantum systems with completely **chaotic classical dynamics** exhibit the **RMT statistics – "(T)CUE"**

- **Exact RMT solution for the power spectrum in the Tuned CUE**

$$S_M(\omega_k) = \frac{M+1}{4\pi^2}\mathrm{Re}\left[\left(z_k\frac{\partial}{\partial z_k} - (M+1)\right)\frac{z_k}{1-z_k}\int_0^{2\pi} d\phi\,\phi\,\Phi_M(\phi;1-z_k)\right]$$

$$\underline{\Phi_M(\phi;1-z)} = \sum_{\ell=0}^{M} z^\ell E_M(\ell;\phi) = \prod_{\ell=1}^{M}\left(\int_0^{2\pi}\frac{d\theta_\ell}{2\pi} - (1-z)\int_0^{\phi}\frac{d\theta_\ell}{2\pi}\right)P_M(\boldsymbol{\theta})$$

Representations: **Painlevé VI** (Forrester & Witte); **Fredholm determinant**; **Toeplitz determinant**

- **Sequence of Tuned CUE level spacings is stationary!**
(even though the **mean density** is **not a constant**)

$$P_M(\boldsymbol{\theta}) \propto \prod_{\ell=1}^{M}\left|1-e^{i\theta_\ell}\right|^2 \prod_{\ell<m}^{M}\left|e^{i\theta_\ell}-e^{i\theta_m}\right|^2$$

Quantum chaotic systems & RMT Painlevé VI solution

BGS: Eigenlevels of **generic** quantum systems with completely **chaotic classical dynamics** exhibit the **RMT statistics – "(T)CUE"**

- **Exact RMT solution for the power spectrum in the Tuned CUE**

$$S_M(\omega_k) = \frac{M+1}{4\pi^2}\mathrm{Re}\left[\left(z_k\frac{\partial}{\partial z_k} - (M+1)\right)\frac{z_k}{1-z_k}\int_0^{2\pi} d\phi\,\phi\,\Phi_M(\phi;1-z_k)\right]$$

$$\underline{\Phi_M(\phi;1-z)} = \sum_{\ell=0}^{M} z^\ell E_M(\ell;\phi) = \prod_{\ell=1}^{M}\left(\int_0^{2\pi}\frac{d\theta_\ell}{2\pi} - (1-z)\int_0^{\phi}\frac{d\theta_\ell}{2\pi}\right)P_M(\boldsymbol{\theta})$$

- **Painlevé VI representation** (Forrester & Witte)**:**

$$\Phi_M(\phi;\zeta) = \exp\left(-\int_{\cot(\phi/2)}^{\infty}\frac{dt}{1+t^2}\left(\tilde\sigma_M(t;\zeta)+t\right)\right)$$

$$\left((1+t^2)\,\tilde\sigma_M''\right)^2 + 4\tilde\sigma_M'(\tilde\sigma_M - t\tilde\sigma_M')^2 + 4(\tilde\sigma_M'+1)^2\left(\tilde\sigma_M' + (M+1)^2\right) = 0$$

BCs at infinity: $\tilde\sigma_M(t;\zeta) = -t + \frac{M(M+1)(M+2)}{3\pi t^2}\zeta + \mathcal{O}(t^{-4}),\ t\to\infty$

Quantum chaotic systems & RMT Painlevé VI solution

BGS: Eigenlevels of **generic** quantum systems with completely **chaotic classical dynamics** exhibit the **RMT statistics – “(T)CUE”**

- **Exact RMT solution for the power spectrum in the Tuned CUE**

$$S_M(\omega_k) = \frac{M+1}{4\pi^2}\mathrm{Re}\left[\left(z_k\frac{\partial}{\partial z_k} - (M+1)\right)\frac{z_k}{1-z_k}\int_0^{2\pi} d\phi\,\phi\,\Phi_M(\phi;1-z_k)\right]$$

$$\underline{\Phi_M(\phi;1-z)} = \sum_{\ell=0}^{M} z^\ell E_M(\ell;\phi) = \prod_{\ell=1}^{M}\left(\int_0^{2\pi}\frac{d\theta_\ell}{2\pi} - (1-z)\int_0^{\phi}\frac{d\theta_\ell}{2\pi}\right)P_M(\boldsymbol{\theta})$$

- **Fredholm determinant representation:**

$$\Phi_M(\phi;\zeta) = \det\left(\mathbb{1} - \zeta\tilde{\kappa}_M^{(0,\phi)}\right)$$

scalar kernel for the Tuned CUE
(Szegö – Askey OPs)

$$\left(\tilde{\kappa}_M^{(0,\phi)} f\right)(\theta) = \int_0^{\phi}\frac{d\theta'}{2\pi}\,\kappa_M(\theta,\theta')\,f(\theta')$$

$$P_M(\boldsymbol{\theta}) \propto \prod_{\ell=1}^{M}\left|1-e^{i\theta_\ell}\right|^2\prod_{\ell<m}^{M}\left|e^{i\theta_\ell}-e^{i\theta_m}\right|^2$$

Quantum chaotic systems & RMT Painlevé VI solution

BGS: Eigenlevels of **generic** quantum systems with completely **chaotic classical dynamics** exhibit the **RMT statistics – "(T)CUE"**

- **Exact RMT solution for the power spectrum in the Tuned CUE**

$$S_M(\omega_k) = \frac{M+1}{4\pi^2}\mathrm{Re}\left[\left(z_k\frac{\partial}{\partial z_k} - (M+1)\right)\frac{z_k}{1-z_k}\int_0^{2\pi} d\phi\, \phi\, \Phi_M(\phi; 1-z_k)\right]$$

$$\underline{\Phi_M(\phi; 1-z)} = \sum_{\ell=0}^{M} z^\ell E_M(\ell;\phi) = \prod_{\ell=1}^{M}\left(\int_0^{2\pi}\frac{d\theta_\ell}{2\pi} - (1-z)\int_0^{\phi}\frac{d\theta_\ell}{2\pi}\right) P_M(\boldsymbol{\theta})$$

- **Toeplitz determinant representation:**

$$\Phi_M(\phi;\zeta) = \prod_{\ell=1}^{M}\left(\int_0^{2\pi}\frac{d\theta_\ell}{2\pi} - \zeta\int_0^{\phi}\frac{d\theta_\ell}{2\pi}\right)\left|1-e^{i\theta_\ell}\right|^2\left|\Delta_M(e^{i\boldsymbol{\theta}})\right|^2$$

$$= \det_{\ell,m=1,\dots,M}\left[\left(\int_0^{2\pi}\frac{d\theta}{2\pi} - \zeta\int_0^{\phi}\frac{d\theta}{2\pi}\right)\left|1-e^{i\theta}\right|^2 e^{i\theta(\ell-m)}\right]$$

(Fisher-Hartwig symbol with a power-type singularity & two jump discontinuities)

Outline

- Introduction
  - Two universality classes in quantum chaology (billiards)
  - Level spacing distributions: "Poisson" vs Wigner-Dyson, random matrices (RMT) and Painlevé V
- **Power spectrum analysis of quantum chaotic spectra**
  - Definition and numerics
  - Theory – do we need a revision?
  - Non-perturbative theory and RMT Painlevé VI solution
  - **Universal law for the power spectrum: Painlevé V through asymptotic analysis**
- Power spectrum for zeros of the Riemann zeta function (work in progress)
- Conclusions

Universal Painlevé V law for the power spectrum

How to perform the limit of infinite-dimensional matrices?

Asymptotic analysis of Toeplitz determinants (uniform asymptotics!)

- **Exact RMT solution for the power spectrum in the Tuned CU[illegible]**

$$S_M(\omega_k) = \frac{M+1}{4\pi^2}\mathrm{Re}\left[\left(z_k\frac{\partial}{\partial z_k} - (M+1)\right)\frac{z_k}{1-[illegible]}\int^{2\pi}[illegible]\right]$$

$$\Phi_M(\phi; 1-z) = \sum_{\ell=0}^{M} z^\ell E_M(\ell;\phi) [illegible] \prod^{M} [illegible] z)\int_0^{\phi}\frac{d\theta_\ell}{2\pi}\right) P_M(\boldsymbol{\theta})$$

Toeplitz determinants with merging singularities

T. Claeys* and I. Krasovsky†

December 15, 2015

- **Toeplitz determinant represen[illegible]ion:**

$$\Phi_M(\phi;\zeta) = \prod_{\ell=1}^{M}\left(\int_0^{2\pi}\frac{d\theta_\ell}{2\pi} - \zeta\int_0^{\phi}\frac{d\theta_\ell}{2\pi}\right)\left|1-e^{i\theta_\ell}\right|^2\left|\Delta_M(e^{i\boldsymbol{\theta}})\right|^2$$

$$= \det_{\ell,m=1,\dots,M}\left[\left(\int_0^{2\pi}\frac{d\theta}{2\pi} - \zeta\int_0^{\phi}\frac{d\theta}{2\pi}\right)\left|1-e^{i\theta}\right|^2 e^{i\theta(\ell-m)}\right]$$

(Fisher-Hartwig symbol with a power-type singularity & two jump discontinuities)

Universal Painlevé V law for the power spectrum

$$S_\infty(\omega_k) = \mathcal{A}(\varpi_k)\Big\{ \mathrm{Im} \int_0^\infty \frac{d\lambda}{2\pi}\, \lambda^{1-2\varpi_k^2}\, e^{i\varpi_k \lambda}$$

$$\times \left[\exp\left(- \int_\lambda^\infty \frac{dt}{t} \left(\sigma(t;\varpi_k) - i\varpi_k t + 2\varpi_k^2 \right) \right) - 1 \right] + \mathcal{B}(\varpi_k) \Big\}$$

$$\mathcal{A}(\varpi_k) = \frac{1}{2\pi} \frac{\prod_{j=1}^2 G(j+\varpi_k) G(j-\varpi_k)}{\sin(\pi \varpi_k)},$$

$$\mathcal{B}(\varpi_k) = \frac{1}{2\pi} \sin(\pi \varpi_k^2)\, \varpi_k^{2\varpi_k^2 - 2}\, \Gamma(2 - 2\varpi_k^2),$$

$$\varpi_k = \frac{\omega_k}{2\pi}$$

$\sigma(t;\varpi_k) = \sigma_1(t)$ **solution to Painlevé V at** $\nu = 1$

$$(t\sigma_\nu'')^2 + (t\sigma_\nu' - \sigma_\nu)(t\sigma_\nu' - \sigma_\nu + 4(\sigma_\nu')^2) - 4\nu^2(\sigma_\nu')^2 = 0$$

BCs at infinity: $\sigma_1(t) \sim i\varpi_k t - 2\varpi_k^2$ as $t \to \infty$.

Universal Painlevé V law for the power spectrum

$$S_\infty(\omega_k) = \mathcal{A}(\varpi_k) \Big\{ \mathrm{Im} \int_0^\infty \frac{d\lambda}{2\pi} \lambda^{1-2\varpi_k^2} e^{i\varpi_k \lambda} \times \left[\exp\left(-\int_\lambda^\infty \frac{dt}{t} \left(\sigma(t;\varpi_k) - i\varpi_k t + 2\varpi_k^2 \right) \right) - 1 \right] + \mathcal{B}(\varpi_k) \Big\}$$

- **Accounts for correlation functions of all orders + parameter-free**

- **Nothing in common with the form-factor approximation claimed by various groups since 2004**

$$S_\infty(\omega_k) = \frac{1}{\omega_k^2}\left[K_\beta\left(\frac{\omega_k}{2\pi}\right) - 1\right] + \frac{1}{(2\pi-\omega_k)^2}\left[K_\beta\left(1-\frac{\omega_k}{2\pi}\right) - 1\right] + \frac{1}{4\sin^2(\omega_k/2)}$$

- **Only the leading-order term coincides in the domain $\omega_k \ll 1$**

$$S_\infty(\omega_k \ll 1) = \frac{1}{4\pi^2\varpi_k} + \frac{1}{2\pi^2}\varpi_k \log\varpi_k + \frac{\varpi_k}{12} + \mathcal{O}\left(\varpi_k^2 \log\varpi_k\right)$$

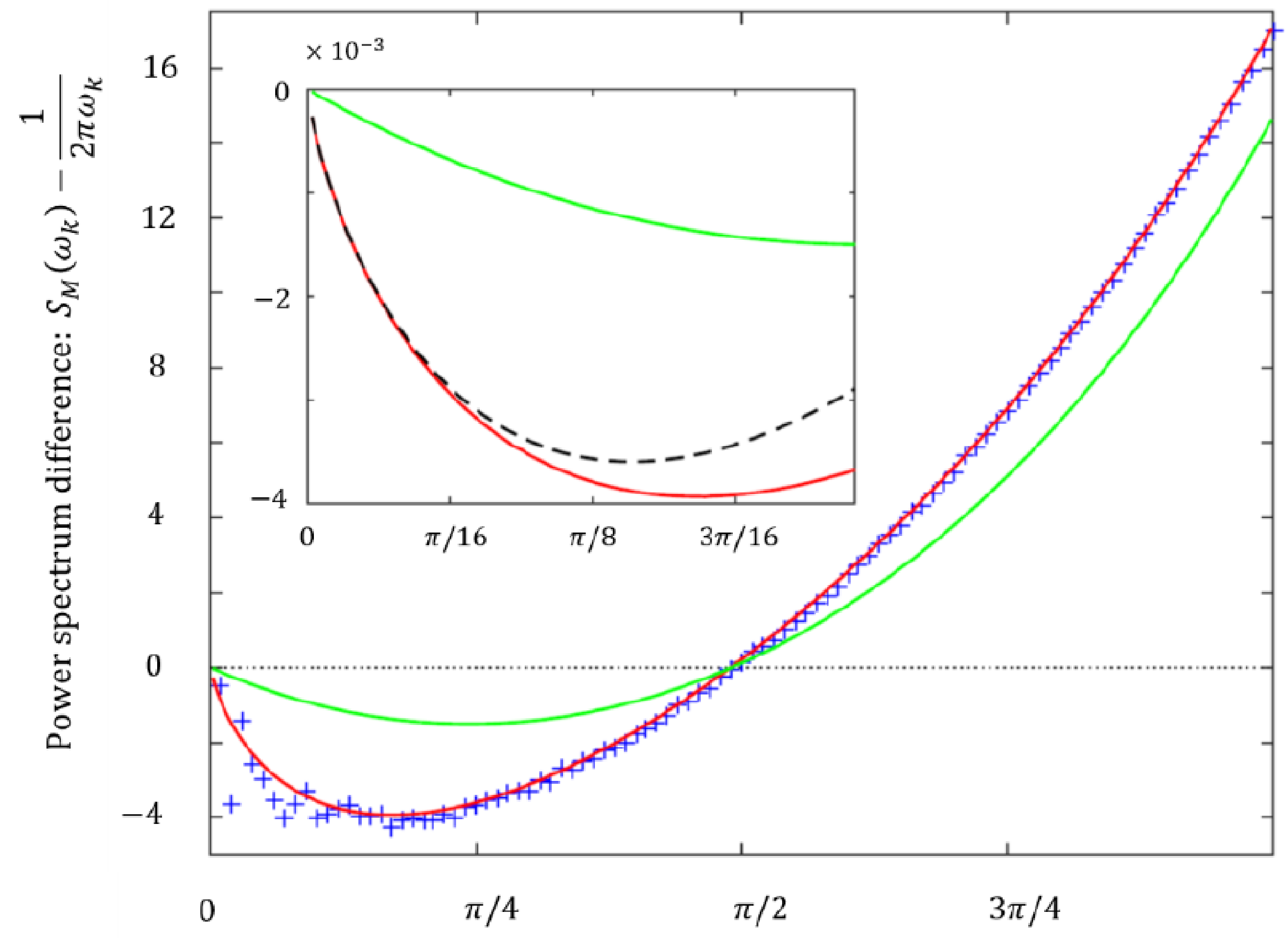
Universal Painlevé V law: Numerical tests (1)

Blue crosses: sequences of 200 unfolded eigenvalues in the CUE averaged over 4×10^6 realizations

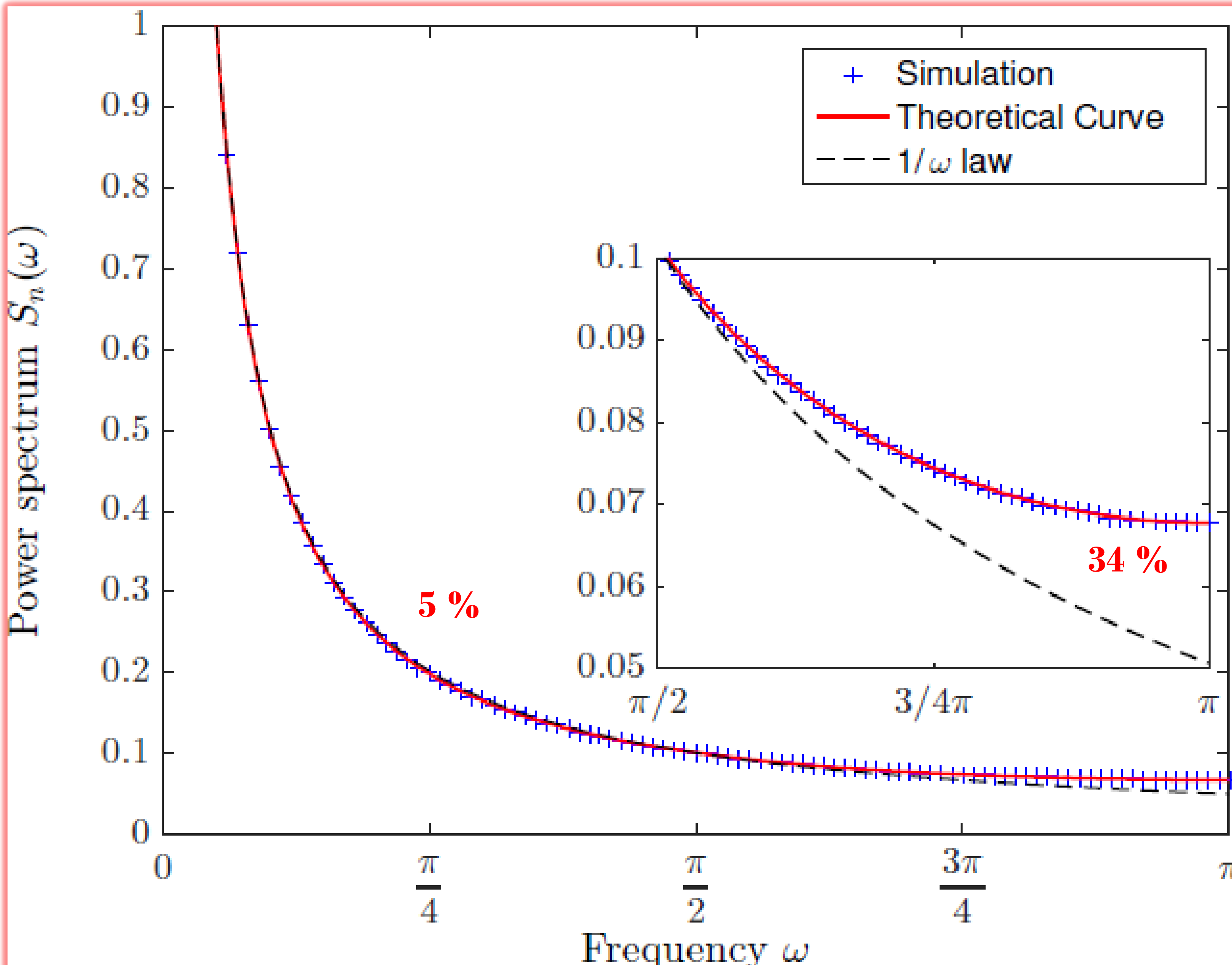
Red solid line: the *exact* theory (PV)

Green solid line: heuristic form-factor model

Dashed line (inset): small-ω_k expansion of the exact solution



Universal Painlevé V law: Numerical tests (2)



Outline

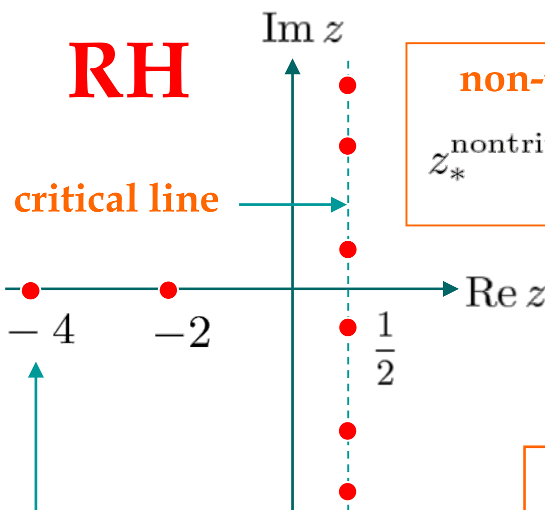
- Introduction
  - Two universality classes in quantum chaology (billiards)
  - Level spacing distributions: “Poisson” vs Wigner-Dyson, random matrices (RMT) and Painlevé V
- Power spectrum analysis of quantum chaotic spectra
  - Definition and numerics
  - Theory – do we need a revision?
  - Non-perturbative RMT theory and Painlevé VI solution
  - Universal law for the power spectrum: Painlevé V through asymptotic analysis
- **Power spectrum for zeros of the Riemann zeta function**
  (work in progress)
- Conclusions

Power spectrum for zeros of the Riemann zeta function

Riemann Zeta Function

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} = \prod_{\text{all primes } p} \left(1 - \frac{1}{p^z}\right)^{-1}$$

RH



non-trivial (complex) zeros

$$z_*^{\mathrm{nontriv}} = \frac{1}{2} \pm i\gamma_n, \ n = 1, 2, \cdots$$

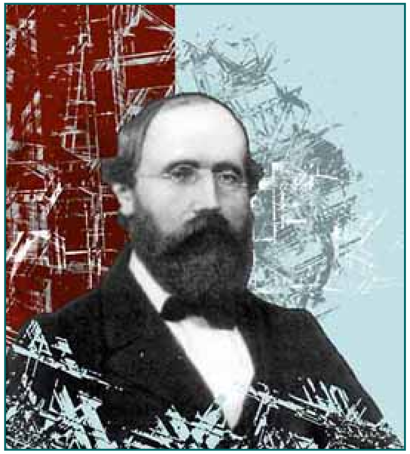
trivial zeros

$$z_*^{\mathrm{triv}} = -2n, \ n = 1, 2, \cdots$$

Distribution of spacing between consecutive zeros ?

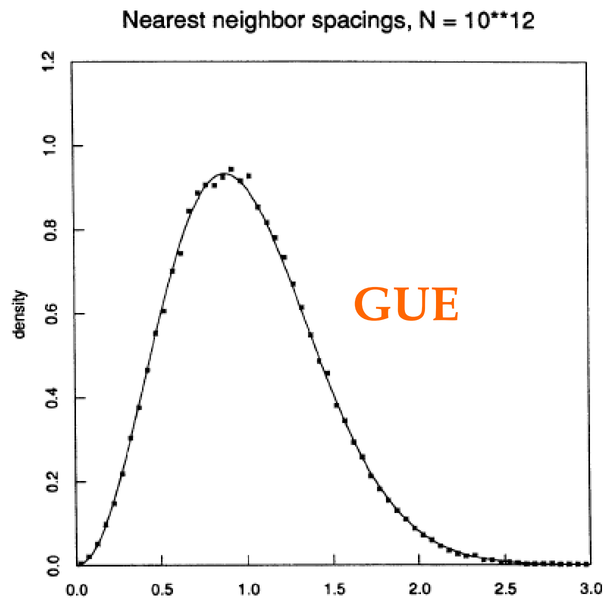
$$\{\mathrm{Im}\, z_*^{\mathrm{nontriv}}\}_n = \gamma_n$$

Leonhard Euler (~1730)



Bernhard Riemann (1859)

Power spectrum for zeros of the Riemann zeta function



MATHEMATICS OF COMPUTATION
VOLUME 48, NUMBER 177
JANUARY 1987, PAGES 273–308

Probability density of the normalized spacings δ_n. Solid line: GUE prediction. Scatter plot: empirical data based on zeros γ_n, $10^{12} + 1 \leqslant n \leqslant 10^{12} + 10^5$.

On the Distribution of Spacings Between Zeros of the Zeta Function

By A. M. Odlyzko

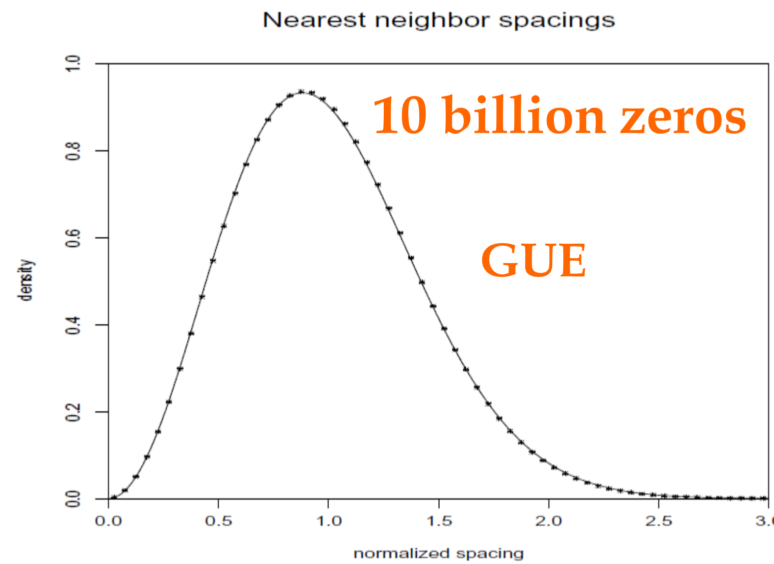


FIGURE 1. Probability density of the normalized spacings δ_n. Solid line: Gue prediction. Scatterplot: empirical data based on a billion zeros near zero # $1.3 \cdot 10^{16}$.

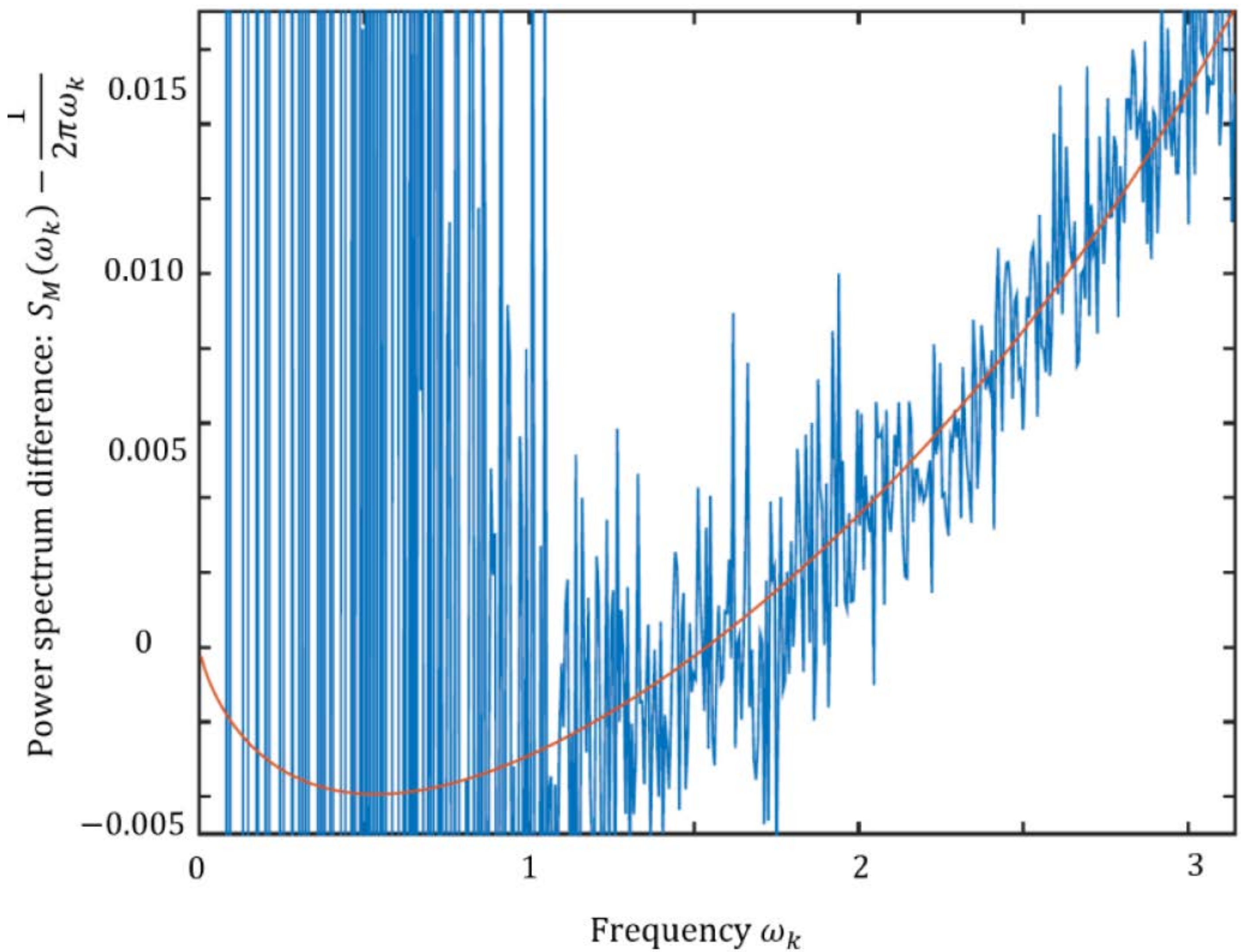
The 10^{22}-nd zero of the Riemann zeta function

A. M. Odlyzko

ABSTRACT. Recent and ongoing computations of zeros of the Riemann zeta function are described. They include the computation of 10 billion zeros near zero number 10^{22}. These computations verify the Riemann Hypothesis for those zeros, and provide evidence for additional conjectures that relate these zeros to eigenvalues of random matrices.

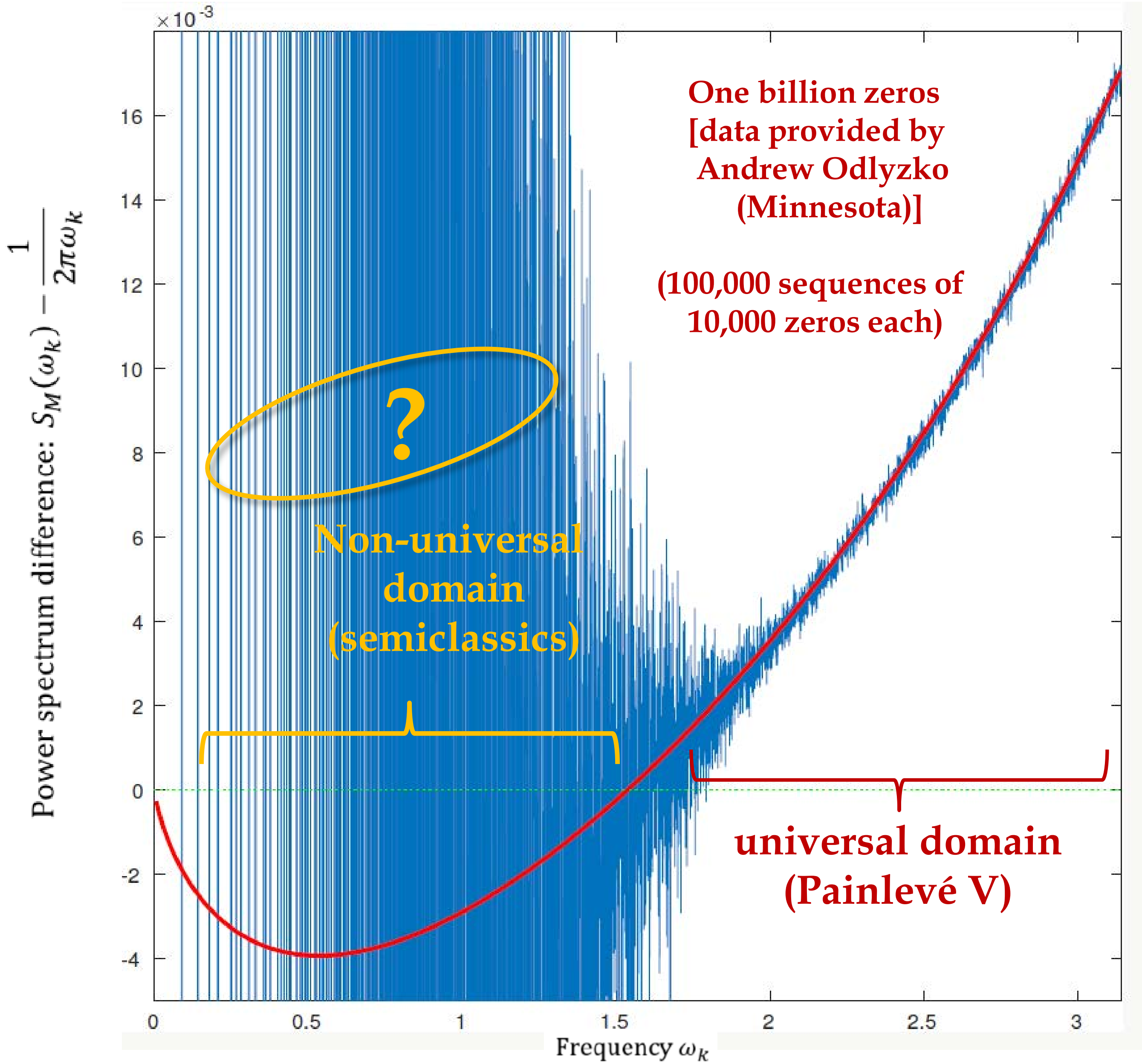
Spacing distribution (PV)

Power spectrum for zeros of the Riemann zeta function



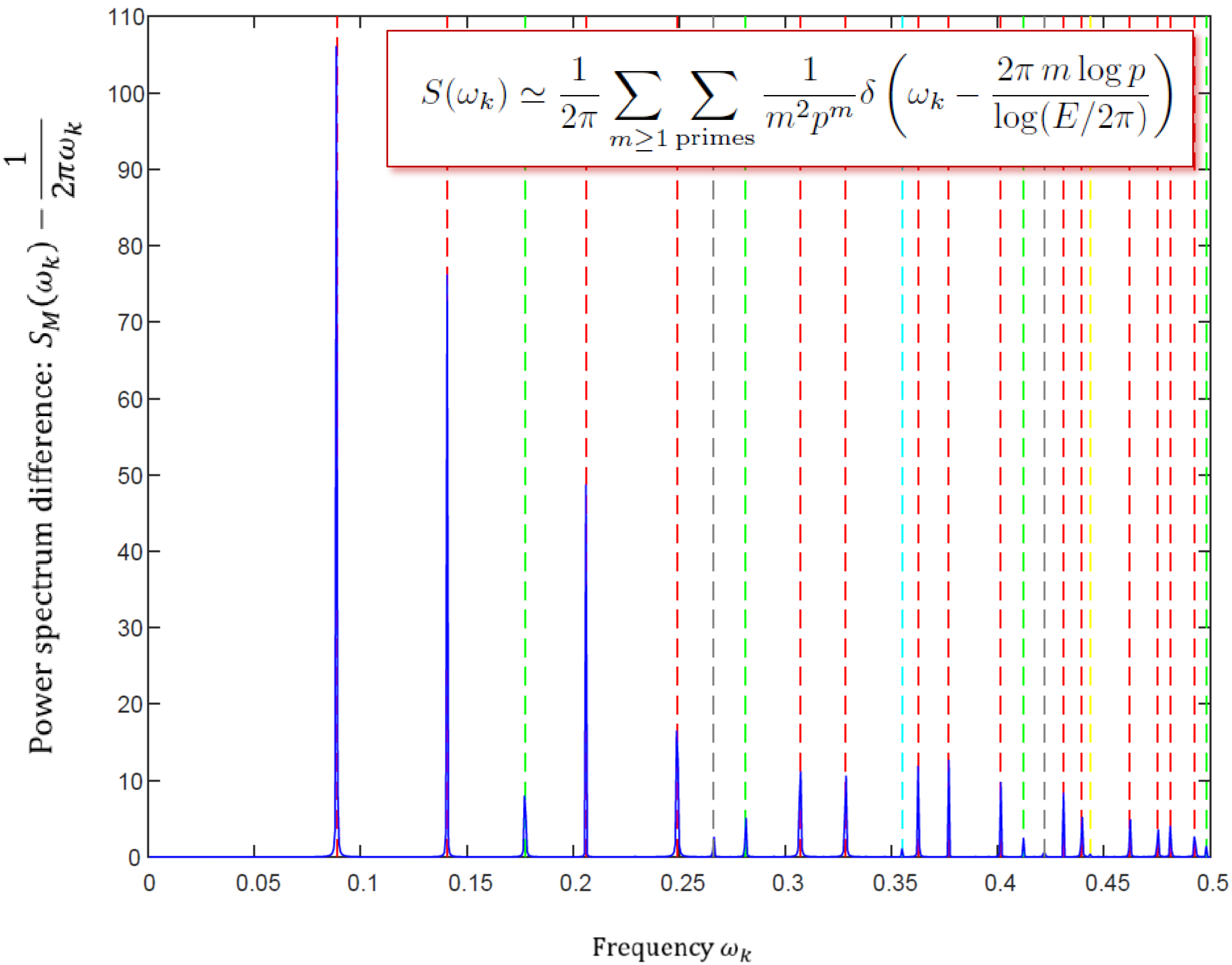
One million zeros (1,000 sequences of 1,000 zeros each)
data provided by Andrew Odlyzko (Minnesota)

Power spectrum for zeros of the Riemann zeta function



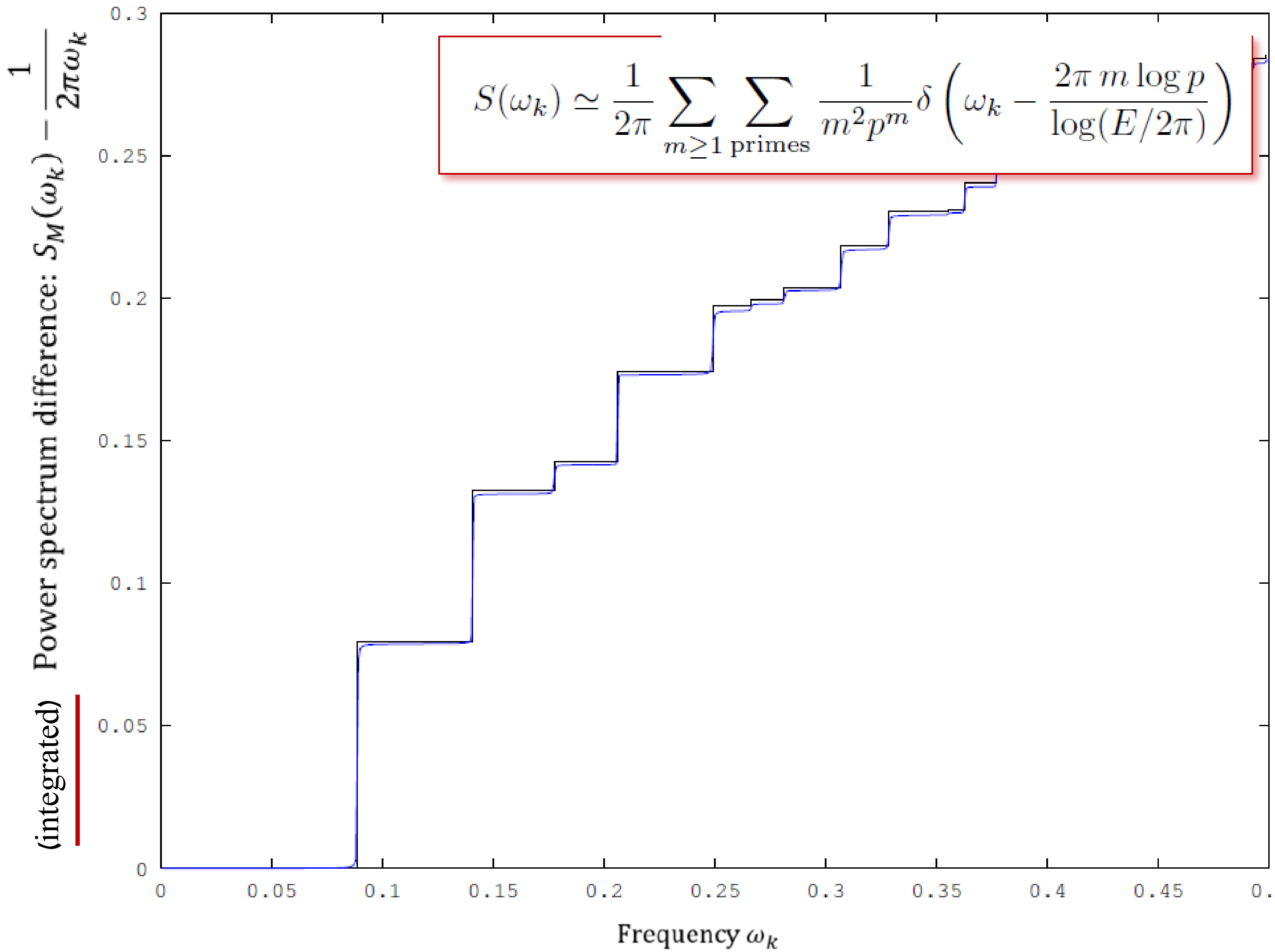
Power spectrum for zeros of the Riemann zeta function

Non-universal domain: Positions (semiclassical theory)



Power spectrum for zeros of the Riemann zeta function

Non-universal domain: Heights (semiclassical theory)



$$S(\omega_k) \simeq \frac{1}{2\pi} \sum_{m \geq 1} \sum_{\text{primes}} \frac{1}{m^2 p^m} \delta\left(\omega_k - \frac{2\pi\, m \log p}{\log(E/2\pi)}\right)$$

Conclusions

- A general framework for the power spectrum of eigenlevel sequences with stationary spacings
- Exact finite-N RMT solution in terms of Painlevé VI (correlation functions of all orders are accounted for) – $(\beta = 2)$
- Asymptotic (large-N) analysis brings a universal parameter-free prediction the power spectrum in terms of Painlevé V transcendent
- Remarkable agreement with the CUE numerics & the Riemann zeta (in the universal domain)
- Deviations from the $\frac{1}{\omega_k}$ law can reach 34% at frequencies close to the Nyquist frequency
- Open questions:

  - fully chaotic systems belonging to other Dyson's symmetry classes
  - problem of missing eigenlevels (work in progress)
  - quantum systems with mixed classical dynamics
  - econophysics (power spectrum of empirical correlation matrices)
  - ...

Thank you