Covariant measurements that maximise the likelihood and optimal reference frame transmission

Massimiliano Federico Sacchi

INFM, Unita' di Pavia www.qubit.it

QUIT quantum information theory group

Quantum entanglement in physical and information sciences,
14-18 December 2004, Pisa

Joint work with
G. Chiribella, G.M. D'Ariano, and P. Perinotti

€

sponsors

Università degli Studi di Pavia

UNIONE EUROPEA

MINISTERO DELL'ISTRUZIONE DELL'UNIVERSITÀ E DELLA RICERCA

ISTITUTO NAZIONALE PER LA FISICA DELLA MATERIA

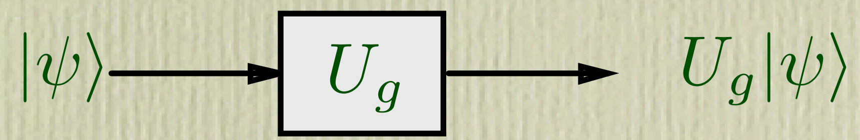
EUROPEAN SCIENCE FOUNDATION

DARPA

QUIPROCONE

Outline

Problem: a black box performs on the input $|\psi\rangle$ a random unitary transformation U_g from some group representation



We are asked to estimate the parameter g

- Optimal detection strategy for covariant estimation (rotation, displacement, phase, ...)
- Find the optimal state and the optimal measurement according to some figure of merit

- A review on phase estimation with any degenerate shift operator
- Covariant measurements for arbitrary group representations that maximise the likelihood
- Exploit all group representations (also equivalent)
- Efficient use of quantum resources
- A relevant application: absolute alignment of Cartesian reference frames
- Use N spins to encode the directions and exploit equivalent reps. to achieve sensitivity $\propto 1/N^2$

Mathematical description of quantum measurements

Measurement statistics - POVM

$$P_i \geq 0\,, \quad \sum_i P_i = I$$

Given quantum system in state ρ

$$p_i = \mathrm{Tr}[\rho P_i]$$

Covariant POVMs

$$P(\mathrm{d}\,g) = \mu(\mathrm{d}\,g) U_g \nu U_g^\dagger$$

Positivity and normalization conditions are

$$\nu \geq 0\,, \qquad \int \mu(\mathrm{d}g) U_g \nu U_g^\dagger = I$$

A review on phase estimation

Generalize Holevo's solution of the problem of estimating a phase shift to any degenerate shift operator H with discrete spectrum S

(D'Ariano, Macchiavello, Sacchi, PLA 1998)

$$\rho_\phi = e^{-i\phi H}\rho_0\, e^{i\phi H}$$

S=Z unbounded

S=N bounded from below

S=Z_q bounded

Find the optimal measurement and the optimal state for given cost function $C(\phi_*, \phi)$

Minimise the average cost

$$\bar{C} = \int_0^{2\pi} \mathrm{d}\phi \, p_0(\phi) \int_0^{2\pi} \mathrm{d}\phi_* C(\phi_*, \phi) \, p(\phi_*|\phi),$$

with conditional probability

$$p(\phi_*|\phi) \, \mathrm{d}\phi_* = \mathrm{Tr}[\mathrm{d}\mu(\phi_*) \, \mathrm{e}^{-\mathrm{i}\phi\hat{H}} \rho_0 \, \mathrm{e}^{\mathrm{i}\phi\hat{H}}]$$

$\mathrm{d}\mu(\phi_*)$ POVM

For $C(\phi_*, \phi) \equiv C(\phi_* - \phi)$ and uniform a priori

we can always find an optimal COVARIANT POVM

$$\mathrm{d}\mu(\phi_*) = \mathrm{e}^{-\mathrm{i}\hat{H}\phi_*} \xi \, \mathrm{e}^{\mathrm{i}\hat{H}\phi_*} \frac{\mathrm{d}\phi_*}{2\pi}$$

In fact, for any given POVM $\mathrm{d}\tilde{\mu}(\phi)$

$$\mathrm{d}\mu(\phi) = \frac{\mathrm{d}\phi}{2\pi} e^{-i\hat{H}\phi} \left(\int_0^{2\pi} e^{i\hat{H}\phi'} \mathrm{d}\tilde{\mu}(\phi') e^{-i\hat{H}\phi'} \right) e^{i\hat{H}\phi}$$

is covariant with the same average cost

Once the best POVM is found, one further optimises the input state

The solution

For pure input $|\psi_0\rangle$ the problem is restricted to $\mathcal{H}_{||}$ spanned by

$$|n\rangle \propto \Pi_n|\psi_0\rangle \neq 0$$

Π_n Projector on the degenerate eigenspace of H with eigenvalue n

Hence, choose POVM block-diagonal on $\mathcal{H} = \mathcal{H}_{||} \otimes \mathcal{H}_{\perp}$

$$\mathrm{d}\mu(\phi) = \mathrm{d}\mu_{||}(\phi) \oplus \mathrm{d}\mu_{\perp}(\phi)$$

The problem is reduced to the canonical phase estimation

$$|\psi_0\rangle \rightarrow \exp(\mathrm{i}H_{\parallel}\phi)|\psi_0\rangle$$

$$H_{\parallel} = \sum_{n\in S} n|n\rangle\langle n| \text{ and } |\psi_0\rangle = \sum_{n\in S} w_n|n\rangle$$

For Holevo's class of cost functions

$$C(\phi) = -\sum_{l=0}^{\infty} c_l \cos l\phi \qquad c_l \geqslant 0 \ \forall l \geqslant 1$$

the optimal POVM writes

$$\mathrm{d}\mu_{\parallel}(\phi) = \frac{\mathrm{d}\phi}{2\pi}|e(\phi)\rangle\langle e(\phi)| \qquad |e(\phi)\rangle = \sum_{n\in S} \mathrm{e}^{\mathrm{i}n\phi}|n\rangle$$

True also for phase-pure states: mixture of states in $\mathcal{H}_{\parallel}$ s.t.

$$\rho_{nm} \equiv \langle n|\hat{\rho}|m\rangle = |\rho_{nm}|\mathrm{e}^{i(\chi_n-\chi_m)}$$

Relevant cost functions

$C(\phi) = 4\sin^2(\phi/2)$ 2π-periodic "variance"

$C(\phi) = 1 - |\langle\psi_0|e^{i\hat{H}\phi}|\psi_0\rangle|^2$ fidelity

$C(\phi) = -\delta_{2\pi}(\phi)$ likelihood criterion

Example

Optimal POVM for phase-difference of two-mode field

$$\hat{H} = a^\dagger a - b^\dagger b \qquad S \equiv \mathbb{Z}$$

Eigenvectors

$$|d\rangle_\nu = |d+\nu\rangle|\nu\rangle,$$

$$d \in \mathbb{Z}, \quad \nu \in [\max(0,-d), +\infty)$$

Take input $|\psi_0\rangle = h_0|0\rangle|0\rangle + \sum_{n=1}^{+\infty}(h_n|n\rangle|0\rangle + h_{-n}|0\rangle|n\rangle)$, $\quad h_n \geqslant 0$

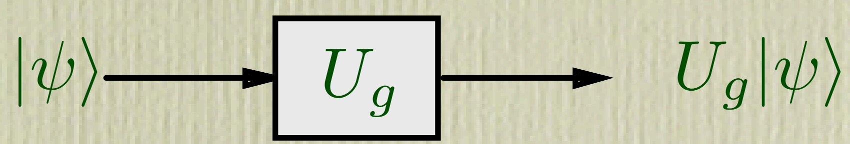
Optimal POVM with $|e(\phi)\rangle = |0\rangle|0\rangle + \sum_{d=1}^{+\infty}(\mathrm{e}^{\mathrm{i}\,d\phi}|d\rangle|0\rangle + \mathrm{e}^{-\mathrm{i}\,d\phi}|0\rangle|d\rangle)$

$$\langle e(\phi)|e(\phi')\rangle = \sum_{n=-\infty}^{+\infty} \mathrm{e}^{in(\phi-\phi')} = \delta_{2\pi}(\phi-\phi') \qquad \hat{\phi} = \int_{-\pi}^{+\pi} \mathrm{d}\phi |e(\phi)\rangle\langle e(\phi)|\phi$$

self-adjoint

The optimal state solves $h_{n+1} + h_{n-1} - \mu(\lambda + |n|)h_n = 0$

Quantum measurements that maximise the likelihood



Consider arbitrary group transformation
and look for the optimal strategy according to the
maximum likelihood principle
(Chiribella, D'Ariano, Perinotti, Sacchi, PRA 2004)

The group acts transitively on the states $|\Psi_x\rangle$

with $x \in \mathfrak{X} = \mathbf{G}/\mathbf{G}_\Psi$

$\mathbf{G}_\Psi$ stability group

$U_h|\Psi\rangle = e^{i\phi_h}|\Psi\rangle$

Equivalent to a problem of covariant state estimation

The optimal probability (density) $p(g|g^*)$ satisfies

$$p(hg|hg^*) = p(g|g^*) \qquad \forall h \in G$$

The optimal POVM is of the form: $U_g \Xi U_g^\dagger$

Take into account non-trivial stability group by replacing

$$\overline{\Xi} = \frac{\int_{G_\Psi} \mathrm{d}\, g\, U_g \Xi U_g^\dagger}{\int_{G_\Psi} \mathrm{d}\, g}$$

Maximum likelihood criterion

Maximise the probability

$$\mathcal{L}(g) = p(g|g) \qquad \forall g$$

of guessing the true value

Find positive Ξ that maximises $\mathrm{Tr}[\Xi\rho]$ with

$$\int_{\mathbf{G}} \mathrm{d}\, g\, U_g \Xi U_g^\dagger = I$$

Solution for pure state and arbitrary groups with square-summable (projective) reps., left-invariant measure and compact stability group

Observation for mixed states and arbitrary invariant cost function

The problem of the optimal measurement that maximises the average "score" of a function h

with h positive, summable, and satisfying $h(\hat{g}, g) = h(g^{-1}\hat{g}, e)$

is equivalent to the ML problem with mixed input state

$$\bar{s} = \int_{\mathbf{G}} \mathrm{d}\, g\, h(g,e)\, \mathrm{Tr}[\rho U_g \Xi U_g^\dagger] \quad = \left[\int_{\mathbf{G}} \mathrm{d}\, g\, h(g,e)\right] \mathcal{L}_{\mathscr{M}(\rho)}[\Xi]$$

with

$$\mathscr{M}(\rho) = \frac{\int_{\mathbf{G}} \mathrm{d}\, g\, h(g,e) U_g^\dagger \rho U_g}{\int_{\mathbf{G}} \mathrm{d}\, g\, h(g,e)}$$

Solution

Take an input state $|\psi\rangle \in H$

Decompose H into irreducible subspaces with its multiplicities

$$H = \oplus_\mu (H_\mu \otimes \mathbb{C}^{m_\mu})$$

such that $|\psi\rangle = \sum_\mu c_\mu |\psi_\mu\rangle$

and $\langle\psi_\mu| I^{(\mu)}_{m,m'} |\psi_\mu\rangle \propto \delta_{m,m'}$

$I^{(\mu)}_{m,m'} = I^{(\mu)} \otimes |m\rangle\langle m'|$ connects equivalent reps.

"The component of $|\psi\rangle$ on equivalent reps. are bi-orthogonal"

Each $|\psi_\mu\rangle$ is of the Schmidt form $|\psi_\mu\rangle = \sum_{i=1}^{r_\mu} \sqrt{\sigma_i^{(\mu)}} |i\rangle_\mu \otimes |i\rangle$

The optimal measurement

The likelihood is maximised by the POVM $M_g = U_g|\eta\rangle\langle\eta|U_g^\dagger$

with $$|\eta\rangle = \sum_\mu e^{i \arg c_\mu} \sqrt{d_\mu} |\phi_\mu\rangle \ , \qquad |\phi_\mu\rangle = \sum_{i=1}^{r_\mu} |i\rangle_\mu \otimes |i\rangle$$

For compact groups: $d_\mu = \dim H_\mu$

The value of the likelihood is given by $\left(\sum_{\mu,i} |c_\mu| \sqrt{d_\mu \sigma_i^{(\mu)}} \right)^2$

Ingredients of the proof

All inequalities leading to $\langle\psi|\Xi|\psi\rangle \leq \left(\sum_{\mu,i} |c_\mu| \sqrt{d_\mu \sigma_i^{(\mu)}}\right)^2$

are saturated

From Schur's lemma $\int dg\, U_g O U_g^\dagger = \sum_{\mu,i,j} \frac{1}{d_\mu} \mathrm{Tr}[O\, I_{j,i}^{(\mu)}]\, I_{i,j}^{(\mu)}$

and the condition $\langle\psi_\mu| I_{m,m'}^{(\mu)} |\psi_\mu\rangle \propto \delta_{m,m'}$

the completeness $\int dg\, U_g \Xi U_g^\dagger = I$ of the POVM is proved

The optimal state

We can optimise the likelihood $\left(\sum_{\mu,i} |c_\mu| \sqrt{d_\mu \sigma_i^{(\mu)}} \right)^2$

over the input states $|\psi\rangle = \sum_\mu \sum_{i=1}^{r_\mu} c_\mu \sqrt{\sigma_i^{(\mu)}} |i\rangle_\mu \otimes |i\rangle$

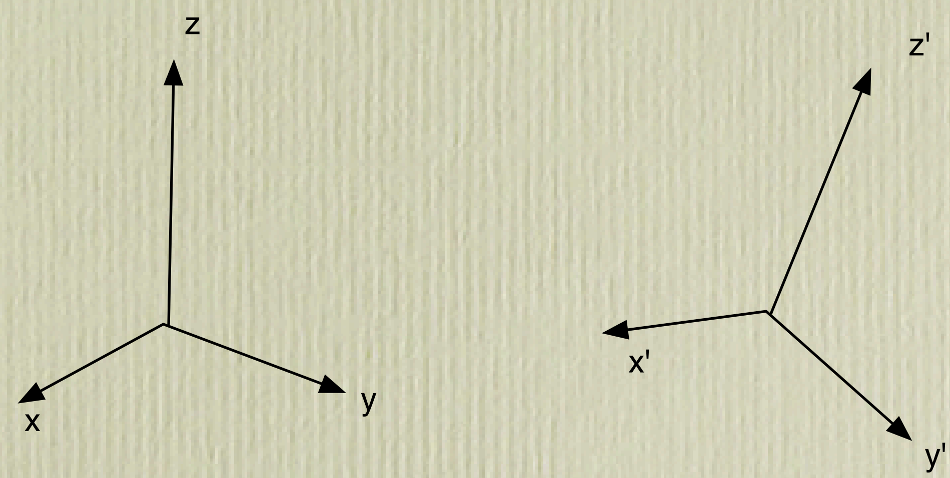
$|c_\mu| \sqrt{\sigma_i^{(\mu)}} = \sqrt{\frac{d_\mu}{d}}$ ⇒ the use of equivalent reps. improves the likelihood !

$r_\mu \leq \min(d_\mu, m_\mu)$ # useful equivalent reps. at most d_μ

Use maximal entanglement between representation and multiplicity spaces

Absolute alignment of reference frames

- Two distance parties (Alice and Bob) want to align their Cartesian reference frames



- Without "fixed stars" Alice has to send gyros to Bob

The protocol

- Idea: use spins that can carry information about the rotation g_* that connects the two frames

- Alice prepares N spins in $|A\rangle$ and sends them to Bob who receives $|A_{g^*}\rangle = U_{g*}^{\otimes N}|A\rangle$

- Bob performs a measurement to infer g_* and rotates his frame by the estimated rotation g

- The state and the measurement should be chosen to minimize the average transmission error

$$\langle e\rangle = \int dg_* \int dg\, p(g|g_*)\, e(g, g_*)$$

deviation

• It was argued that equivalent reps. are useless, and this led to false claim of optimality:

asymptotic sensitivity $\propto 1/N$

(Bagan, Baig, Munoz-Tapia, PRL 2001)

• Then, a non-covariant strategy was shown to do better, and this led to claim that covariant quantum measurements may not be optimal !!

(Peres & Scudo, JMO 2002)

BUT... equivalent irreducible representations cannot be neglected, and their use dramatically increases the efficiency of the protocol

The solution

(Chiribella, D'Ariano, Perinotti, Sacchi, PRL 2004)

Write the Clebsch-Gordan decomposition

$H^{\otimes N} \equiv \oplus_{j=0}^{J} H_j \otimes M_j \qquad J = N/2$

- Choose a state of the form

$$|A\rangle = A_J |JJ\rangle + \sum_{j=0}^{J-1} A_j |I_j\rangle\rangle$$

with

$$|I_j\rangle\rangle = \frac{1}{\sqrt{2j+1}} \sum_{m=-j}^{j} |jm\rangle \otimes |m\rangle$$

- Entanglement between representation and multiplicity space, but no shared entanglement between Alice and Bob

- Bob uses the POVM that maximises the likelihood

Comparison of the protocol exploiting equivalent representation with the optimal one without equivalent representations

Number of spins	$\langle e \rangle_{with}$	$\langle e \rangle_{without}$
N= 3	1.6114	1.8138
N= 5	0.9136	1.3292

Asymptotic behavior for large N:

$$\langle e \rangle_{with} \sim \frac{8\pi^2}{N^2} \qquad \langle e \rangle_{without} \sim \frac{8}{N}$$

Conclusion

- We gave the solution of the general problem of phase estimation for arbitrary shift generator and Holevo's cost functions with phase-pure input states
- We provided the optimal covariant measurement according to the ML principle for arbitrary groups, showing the relevance of equivalent representations
- We gave a covariant protocol for absolute alignment of reference frames using N spins that achieves sensitivity $\propto 1/N^2$

References

G. M. D'Ariano, C. Macchiavello, and M. F. Sacchi,
On the general problem of quantum phase estimation,
Phys. Lett. A 248, 103 (1998)

G. Chiribella, G. M. D'Ariano, P. Perinotti, and M. F. Sacchi,
Covariant quantum measurements that maximize the likelihood,
Phys. Rev. A 70, 062105 (2004)

Efficient use of quantum resources for the transmission of a reference frame,
Phys. Rev. Lett. 93, 180503 (2004)

Download at www.qubit.it