Entanglement Entropy and Quantum Field Theory

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Outline:

- Introduction to RG and CFT
- Entropy in QFT
- Path integral formula for the entropy
- Exact calculations with CFT in 1+1 dimensions
- Non critical $1+1$–dimensional systems

[P. Calabrese and J. Cardy, hep-th/0405152]

[JSTAT 0406: P002,2004]

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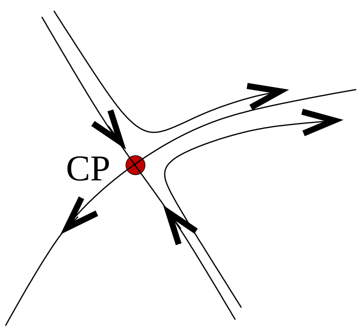
RG and CFT

Let us consider a model defined by an Hamiltonian $H(g)$. At $g = g_c$ it undergoes a phase transition

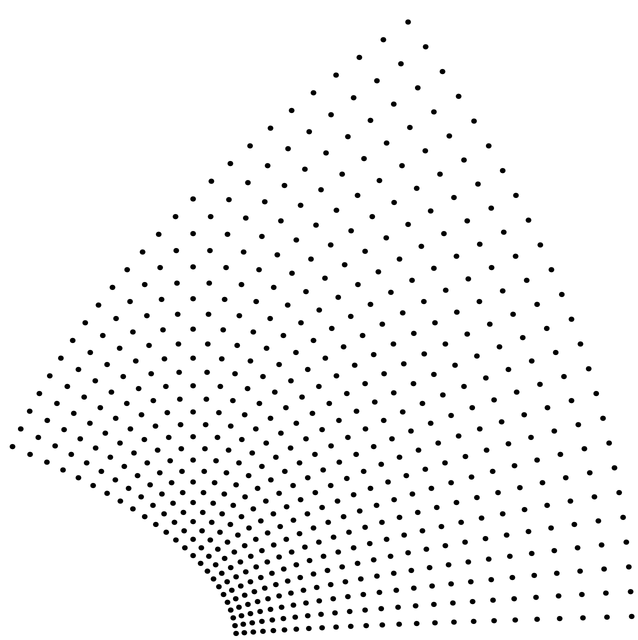
Universality idea: Physical properties of a system close to a phase transition do not depend on microscopic details, but only on global properties such as symmetries and dimensionality.

Explanation: Under RG transformations different Hamiltonians have the same fixed point that completely determines the long-distance (IR) behavior.



EG: A model defined on the lattice has an Hamiltonian invariant under lattice translation, but the corresponding fixed point Hamiltonian is generically invariant under arbitrary translations!

It is known that the critical point is invariant for scale transformations . . . but not only, in fact it is generically invariant under conformal transformations (roughly speaking, they are local combinations of translations, rotations and dilatations)



$$\langle \phi_1(r_1)\phi_2(r_2)\rangle =$$
$$b(r_1)^{x_1} b(r_2)^{x_2} \langle \phi_1(r_1')\phi_2(r_2')\rangle$$

What are the consequence of this further invariance?

In $d > 2$ no particular news

In $d = 2$ the conformal algebra is ∞ dimensional (analytic functions on the complex plane)
$\Rightarrow$ ∞ conservations laws $\Rightarrow$ Everything!!

EG under $z \to z' = w(z)$, the relation

$$\langle \phi_1(z_1,\bar{z}_1)\phi_2(z_2,\bar{z}_2)\rangle = w'(z_1)^{h_1}\overline{w'(z_1)}^{\bar{h}_1} w'(z_2)^{h_2}\overline{w'(z_2)}^{\bar{h}_2} \langle \phi_1(w_1)\phi_2(w_2)\rangle$$

relates the two-point function on the plane [$\langle \phi(z_1)\phi(z_2)\rangle = |z_1 - z_2|^{-4\Delta_\phi}$] to the one in other geometries (strip, torus . . .)

One of the most striking results:
For unitary minimal models the universality class is identified by the ubiquitous central charge

$$c = 1 - \frac{6}{m(m+1)} \qquad m = 2, 3, \ldots \infty$$

that determines all universal quantities [care with c=1]

Further c-theorem $\Rightarrow$

A function $C(\mu)$ exists such that it is a non-increasing function under RG flow, and at the FP it equals c $\Rightarrow$

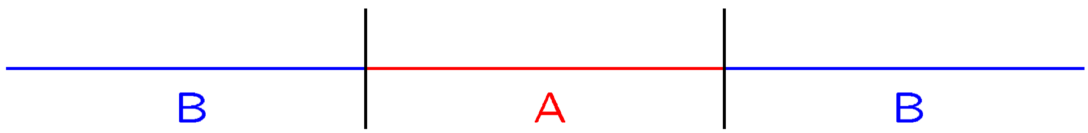
Irreversibility under RG flow

Entanglement Entropy and QFT

Quantum system in the ground state $|\Psi\rangle$

The density matrix is $\rho = |\Psi\rangle\langle\Psi|$ ($\text{Tr}\rho = 1$)

A measures a subset, B the remainder:



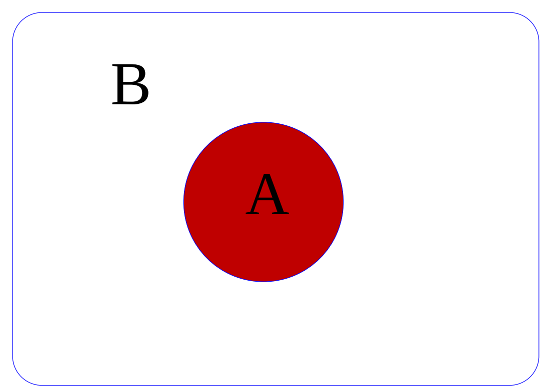
Reduced density matrix $\rho_A = \text{Tr}_B \rho$ ($\rho_B = \text{Tr}_A \rho$)

Entanglement Entropy $\equiv$ Von Neumann entropy of ρ_A:

$$S_A = -\text{Tr}\,\rho_A \ln \rho_A$$

[note $S_A = S_B$]

Historical review:



- Srednicki '93: Area Law
in a $d+1$ critical $T=0$ QFT
$S_A \propto \mathcal{A} \Rightarrow S \propto \mathcal{A}\Lambda^{d-1}$
and for $d=1$?
$S \propto \ln \Lambda \Rightarrow S \propto \ln \ell\Lambda$

Non extensive

- Holzhey, Larsen, Wilczek '94: In a 1+1D $T=0$ CFT

$$S_A = \frac{c}{3} \ln \frac{\ell}{a}$$

Entropy and path integral

Lattice QFT in 1+1 dimensions

$\{\hat{\phi}(x)\}$ a set of fundamental fields with eigenvalues $\{\phi(x)\}$ and eigenstates $\otimes_x|\{\phi(x)\}\rangle$

The density matrix at temperature β^{-1} is

$$\rho(\{\phi(x'')''\}|\{\phi(x')'\}) = Z^{-1}\langle\{\phi(x'')''\}|e^{-\beta\hat{H}}|\{\phi(x')'\}\rangle$$

$Z = \text{Tr}\, e^{-\beta\hat{H}}$ is the partition function.

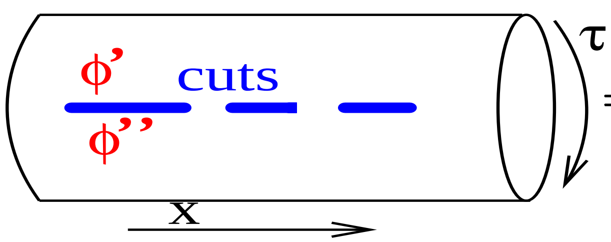
Euclidean path integral:

$$\rho = Z^{-1}\int[d\phi(x,\tau)]\prod_x \delta(\phi(x,0)-\phi(x')')\prod_x \delta(\phi(x,\beta)-\phi(x'')'')\, e^{-S_E}$$

$S_E = \int_0^\beta L_E d\tau$, with L_E the Euclidean Lagrangian

The trace has the effect of sewing together the edges along $\tau = 0$ and $\tau = \beta$ to form a cylinder of circumference β.

$A = (u_1, v_1), \ldots, (u_N, v_N)$: ρ_A sewing together only those points x which are not in A. This will have the effect of leaving open cuts, one for each interval (u_j, v_j), along the the line $\tau = 0$.

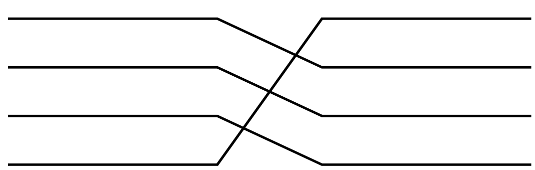
$\rho_A =$ 

$= \int_{x\in B}[d\phi(x,0)]\delta(\phi(x,\beta)-\phi(x,0))\rho$

"Replica trick"

$$S_A = -\text{Tr}\rho_A \log\rho_A = -\lim_{n\to 1}\frac{\partial}{\partial n}\text{Tr}\rho_A^n$$

$\text{Tr}\rho_A^n$ for integer n is the partition function on an $n-$sheeted Riemann surface


$= \text{“}\rho_A^{ij}\rho_A^{jk}\rho_A^{kl}\rho_A^{li}\text{”}$

It has a unique analytic continuation to $\text{Re}\, n > 1$ and that its first derivative at $n = 1$ gives the required entropy:

$$S_A = -\lim_{n\to 1}\frac{\partial}{\partial n}\frac{Z_n(A)}{Z^n}$$

[For $n \neq 1$ it is the Tsallis entropy]

Continuum limit: $a \to 0$ [Most of UV div cancel in the ratio]

Entropy and CFT

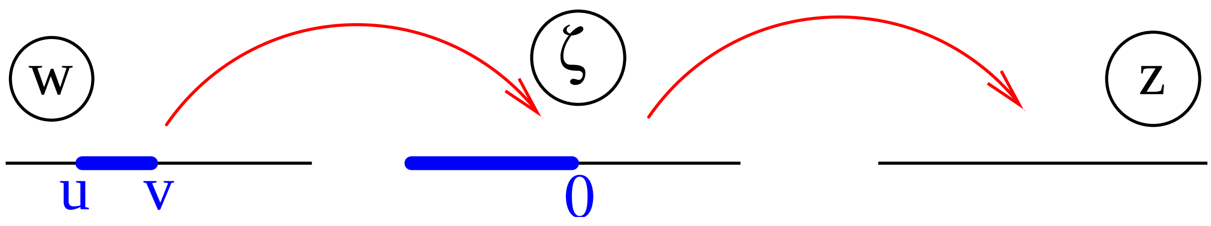
0. Single interval (u,v). We need $Z_n/Z^n = \langle 0|0\rangle_{\mathcal{R}_n}$. Thus we have to compute $\langle T(w)\rangle_{\mathcal{R}_n}$

Under a conformal transformation $w \to z$

$$T(w) = \left(\frac{dz}{dw}\right)^2 T(z) + \frac{c}{12}\frac{z'''z' - 3/2 z''^2}{z'^2}$$

Thus

$$w \to \zeta = \frac{w-u}{w-v};\ \zeta \to z = \zeta^{1/n} \Rightarrow w \to z = \left(\frac{w-u}{w-v}\right)^{1/n}$$



But $\langle T(z)\rangle_{\mathrm{C}} = 0 \Rightarrow$

$$\langle T(w)\rangle_{\mathcal{R}_n} = \frac{c(1-(1/n)^2)}{24}\frac{(v-u)^2}{(w-u)^2(w-v)^2}$$

To be compared with the Conformal Ward identities:

$$\frac{\langle T(w)\Phi_n(u)\Phi_{-n}(v)\rangle_{\mathrm{C}}}{\langle \Phi_n(u)\Phi_{-n}(v)\rangle_{\mathrm{C}}} = \frac{\Delta_\Phi (v-u)^2}{(w-u)^2(w-v)^2}$$

$$\Downarrow$$

Z_n/Z^n transforms under conformal transformations (acting identically on each sheet) as nth power of the two point function of a (fake) primary field on the complex plane with scaling dimension

$$\boxed{\Delta_\Phi = \overline{\Delta}_\Phi = \frac{c}{24}\left(1 - \frac{1}{n^2}\right)}$$

Recall that $\langle \phi(x)\phi(y)\rangle = |x-y|^{-4\Delta_\phi}$

$$\mathrm{Tr}\,\rho_A^n = \frac{Z_n}{Z^n} = c_n\left(\frac{v-u}{a}\right)^{-(c/6)(n-1/n)}$$

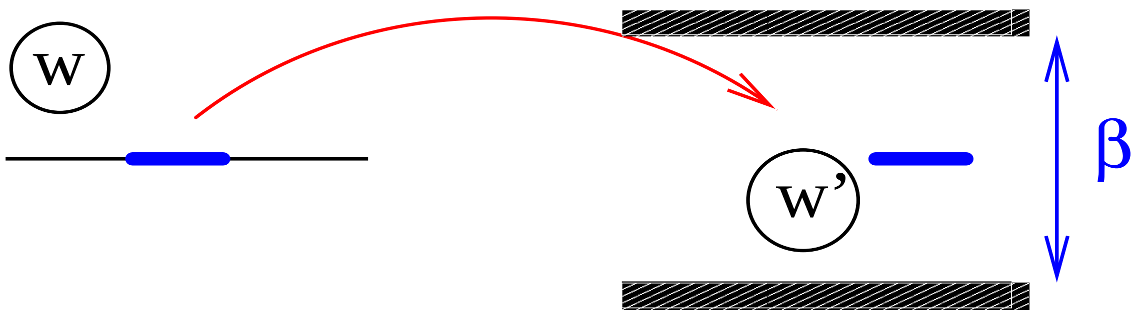
Finally with the replica trick $(v-u=\ell)$

$$S_A = \frac{c}{3}\ln\frac{\ell}{a} + c_1'$$

Generalizations

1. Finite temperature: map the plane into a cylinder

$$w \to w' = \frac{\beta}{2\pi} \ln w$$



$$S_A(\beta) \sim \frac{c}{3} \log \left(\frac{\beta}{\pi a} \sinh \frac{\pi \ell}{\beta} \right) + c_1' \,.$$

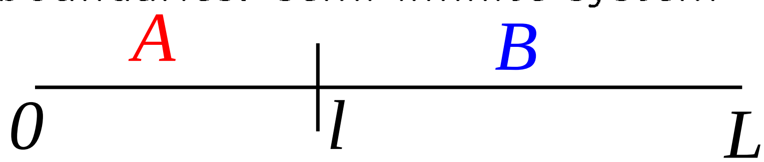
$$S_A \simeq \begin{cases} \dfrac{\pi c}{3} \dfrac{\ell}{\beta} \,, & \ell \gg \beta \qquad \text{classical extensive} \\ \\ \dfrac{c}{3} \log \dfrac{\ell}{a} \,, & \ell \ll \beta \qquad T = 0 \text{ non} - \text{extensive} \end{cases}$$

2. Finite size: orient the branch cut perpendicular to the axis $\beta \to L$ and $w \to iw$

$$S_A \sim \frac{c}{3} \log \left(\frac{L}{\pi a} \sin \frac{\pi \ell}{L} \right) + c_1'$$

It is symmetric under $\ell \to L - \ell$. It is maximal when $\ell = L/2$

3. Open boundaries: semi-infinite system



If $L = \infty$ and $T = 0$, it is uniformised by $z = \left(\frac{w - i\ell}{w + i\ell}\right)^{1/n}$

$$\mathrm{Tr}\,\rho_A^n \simeq \tilde{c}_n \left(\frac{2\ell}{a}\right)^{(c/12)(n-1/n)} \quad \Rightarrow S_A \simeq \frac{c}{6}\log\frac{2\ell}{a} + \tilde{c}_1'$$

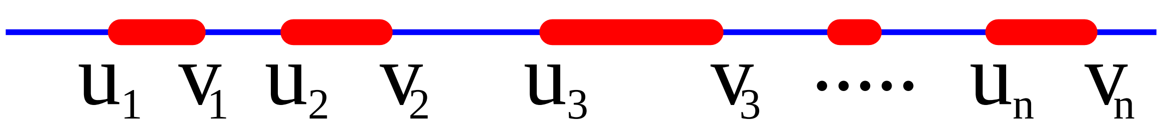
and at finite temperature β^{-1} and finite size

$$S_A(\beta) \simeq \frac{c}{6}\log\left(\frac{\beta}{\pi a}\sinh\frac{2\pi\ell}{\beta}\right) + \tilde{c}_1'$$

$$S_A(L) \simeq \frac{c}{6}\log\left(\frac{2L}{\pi a}\sin\frac{\pi\ell}{L}\right) + \tilde{c}_1'$$

Note: $\tilde{c}_1' - c_1' = g$ boundary entropy [Affleck, Ludwig]

4. General case:



Uniformised by $z = \prod_i (w - w_i)^{\alpha_i}$, with $\sum_i \alpha_i = 0$ ($w_k = u_i$ or v_j)

$$S_A = \frac{c}{3}\left(\sum_{j\le k}\log\frac{v_k - u_j}{a} - \sum_{j<k}\log\frac{u_k - u_j}{a} - \sum_{j<k}\log\frac{v_k - v_j}{a}\right) + Nc_1'$$

A similar expression holds in the case of a boundary, with half of the w_i corresponding to the image points

Entropy in non critical systems

Question: What about the entanglement entropy in the so-called critical domain, where $g \neq g_c$, but $|g - g_c| \ll 1$, i.e. the correlation length $\xi = |g - g_c|^{-\nu}$ is large but finite?

Following the line of the c-theorem proof, we showed

$$\boxed{S_A = \mathcal{A}\frac{c}{6}\log\frac{\xi}{a}}$$

where $\mathcal{A}$ is the number of boundary points between A and B (1D area).

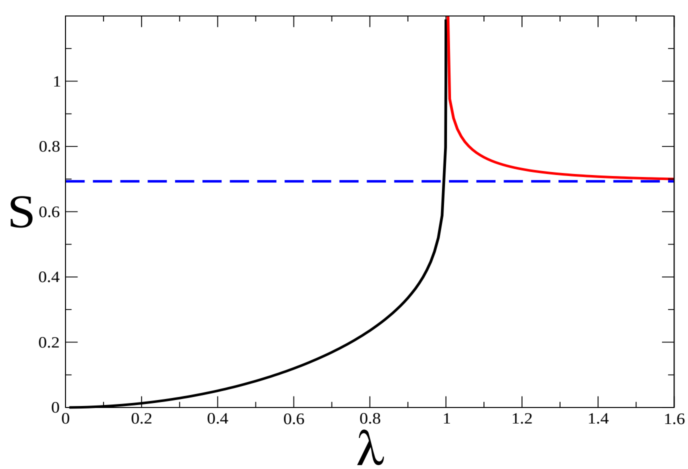
We checked this result in some cases with $\mathcal{A} = 1$

- Gaussian Massive FT
- Ising model in a transverse magnetic field

$$H_I = -\sum_{n=1}^{L-1}\sigma_n^x - \lambda\sum_{n=1}^{L-1}\sigma_n^z\sigma_{n+1}^z$$

by means of the corner transfer matrix ($\epsilon = \epsilon(\lambda)$)

$$S_A = \begin{cases} \epsilon\displaystyle\sum_{j=0}^{\infty}\frac{2j}{1+e^{2j\epsilon}} + \sum_{j=0}^{\infty}\log(1+e^{-2j\epsilon})\,, & \lambda > 1 \\ \epsilon\displaystyle\sum_{j=0}^{\infty}\frac{2j+1}{1+e^{(2j+1)\epsilon}} + \sum_{j=0}^{\infty}\log(1+e^{-(2j+1)\epsilon})\,, & \lambda < 1 \end{cases}$$



For $\lambda \to 1$

$$S \simeq \frac{1}{12}\log\xi + C_1$$

- XXZ model, similar results but $c = 1$
- In the finite slit geometry (i.e. $\mathcal{A} = 2$), it was exactly calculated by Its et al. and Peschel for the XY chain, finding agreement!