Entanglement Monotones for Multipartite Systems

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OUTLINE

- Entanglement in bipartite systems
  - Qubits (2 x 2)
  - Approaches to (d1 x d2)

- Mixed states

- Motivation from Spin chain analysis and CKW conjecture

- Approaches to multipartite Entanglement

- Genuine multipartite entanglement: antilinear formulation
  - Combs and Filters
  - Detected maximally entangled states

- Conclusions

Bipartite

Entanglement

Pure States, two Bits

†

†

Question : **?**

Why is it an interesting question?

- **product state: measuring** one site

  gains no information on the other

- **Can be created locally**

Qualitatively :

Indicators for Entanglement

†

$\Rightarrow$

ij i j

M

Look at

M †

Rank = 1 !

rank M_Ψ=2 $\Rightarrow$ entangled!

Measure Candidates

- $|\det M_\Psi|$
- $\tau_1 = \det \rho^{(1)} = |\det M_\Psi|^2$ (one)- tangle
- $\mathrm{Tr}\, \rho^{(1)} \log_2 \rho^{(1)}$ von Neumann entropy
- arbitrary monotonic function of

  $\rho^{(1)}$'s smallest eigenvalue

Example :

↔Schmidt decomposition

Bell states

det $_1$ †2 det M † p

measures portion of Bell state

Mixed states

- Decomposition of ρ **NOT unique**

∀ρ is product state *iff*

- **Minimization over all possible reps. necessary**

Criteria for separability

- Schmidt decomposition   (d_1 x d_2)

- positive partial transpose (2x2, 2x3)
  otherwise $\rightarrow$ bound entanglement

Peres PRL 1996
Horodecki, Horodecki PLA 1996

Entanglement measures

pure states

- v.Neumann entropy of reduced density matrix

- "Groverian" entanglement (Biham et al '02)

Biham et al. PRA 2002

Mixed states

- relative entropy of entanglement

Vedral & Plenio PRA 1998

- Negativity

Vidal&Werner PRA 2002

- Concurrence

Hill & Wootters PRL 1997
Wootters PRL 1998

Different Concepts

- distillable entanglement
- entanglement of formation
- entanglement cost

Bennett et al. PRA 1996

- entanglement of assistance

DiVincenzo et al.1998

Concurrence

Hill, Wootters PRL1997, Wootters PRL 1998

Ingredients:

R

EV's of R

Example :

Abouraddy et al. PRA 2001

↔Schmidt decomposition

Bell states

measures portion of Bell state

Approaches to (d1 x d2) Entanglement measures

- **Concurrence vectors**

  Audenaert et al. PRA 2001
  Li, Zhu *quant-ph/0308139*

- **Universal state inversion**

  Rungta et al. PRA 2001

$$C_{d1 \times d2} = \sqrt{2\left(1 - \mathrm{tr}\,\rho_A^2\right)}$$

Both measures are identical

Multipartite

Entanglement

Different Entanglement Classes

$GHZ^{\dagger}$

$W^{\dagger}$

1

W   4 det W $+\frac{8}{9}$; $C_{12}^2 + C_{13}^2 + \frac{4}{9}$

1

GHZ   4 det GHZ $+1$; $C_{12}^2 + C_{13}^2 + 0$

Residual entanglement measure (pure states)

3 1

1 $+4$ det 1 C_{12}^2 C_{13}^2

Coffman, Kundu, Wootters PRA 2000

Entanglement

in Spin chains

Anisotropic XY models

$$H\dagger \quad J \quad 1 \quad s_i^x s_i^x \;{}_1 \quad 1 \quad s_i^y s_i^y \;{}_1 \quad h s_i^z$$

$$i$$

- Exactly solved **(Jordan-Wigner + Bogoliubov)**
- Correlation functions accessible

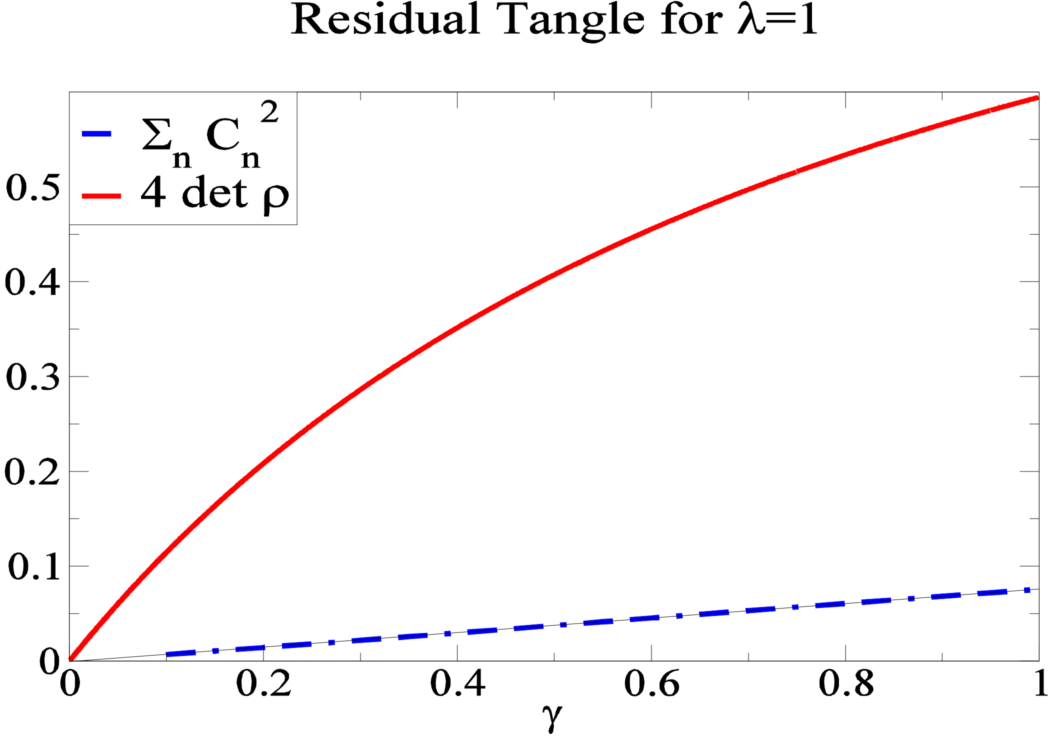
$$\dagger \; \frac{J}{h}$$

Lieb, Schulz, Matthis Ann. Phys.NY 16, 407 (1961)

Barouch, McCoy, Dresden PRA 2, 1075 (1970)

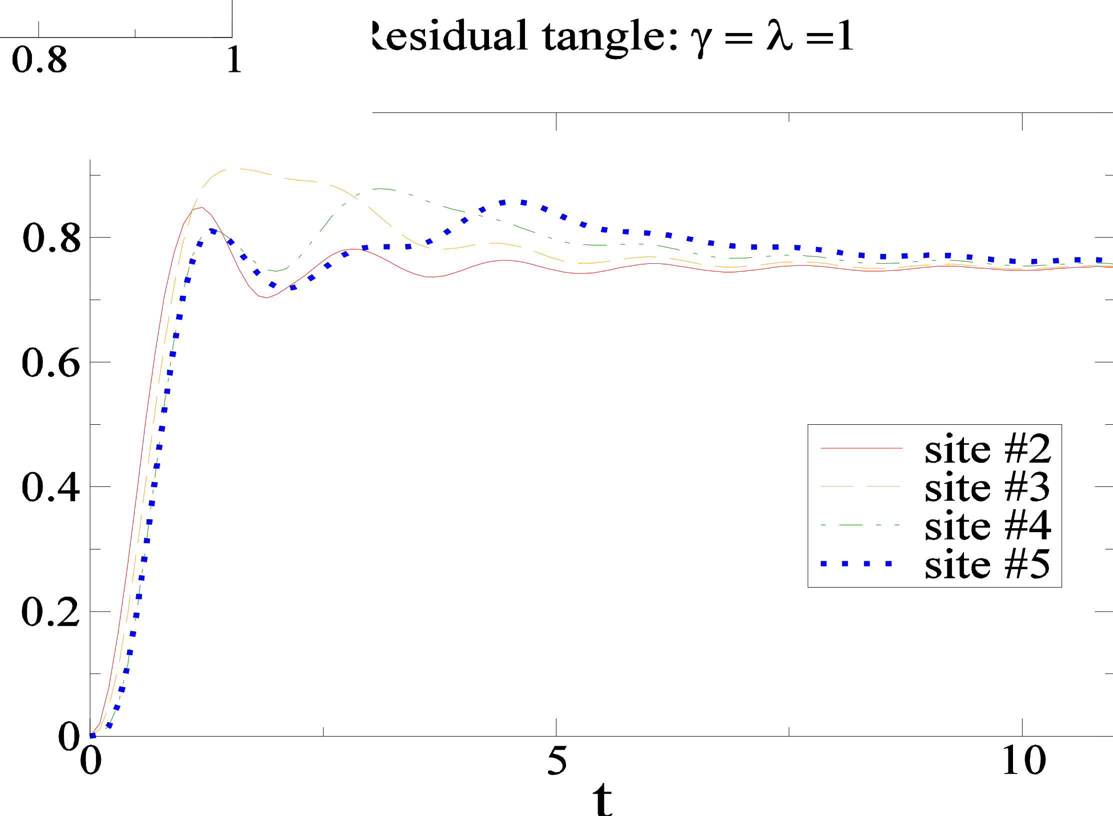
Barouch, McCoy PRA 3, 786 (1971)

Pfeuty Ann. Phys.NY 57, 79 (1970)



Coffman et al. PRA 2000

offman-Kundu-Wootters Conjecture



equilibrium

Osterloh et al. Nature 2002

dynamic

Amico et al. PRA 2004

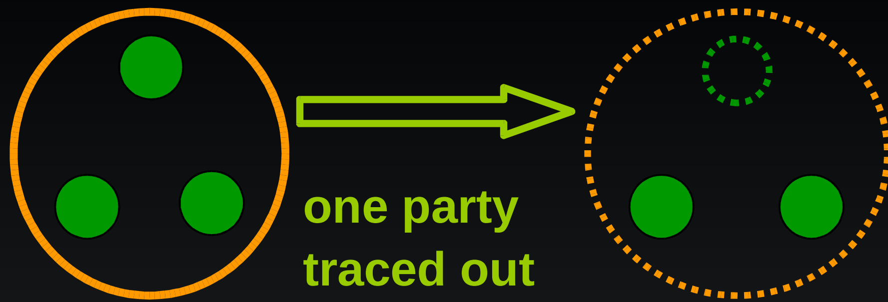
Approaches to Multipartite Entanglement measures

- N-Concurrence $\sigma_y^{\otimes N}$ Wong, Christensen PRA 2001
  - N even only
  - factorises
    $\rightarrow$ products of e.g. Bell states judged maximally entangled
- Concurrence vectors
  - detect one-tangle
- Q-measure = averaged tangle Meyer, Wallach quant-ph/0108104

Genuine Multipartite Entanglement Monotones on an antilinear footing

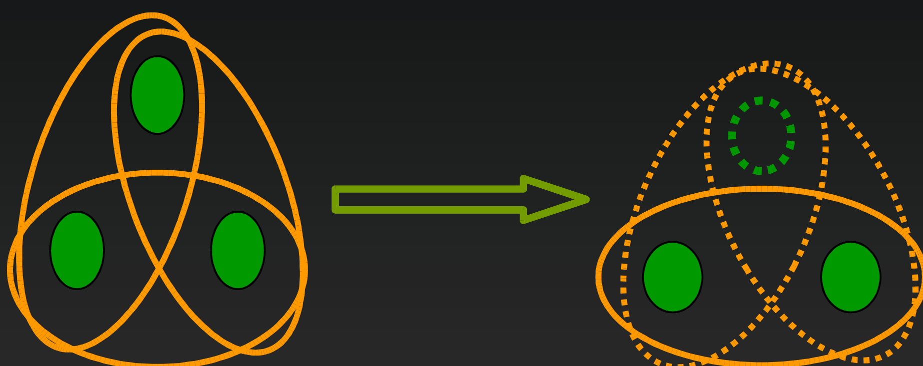
Osterloh, Siewert *quant-ph/0410102*

GHZ

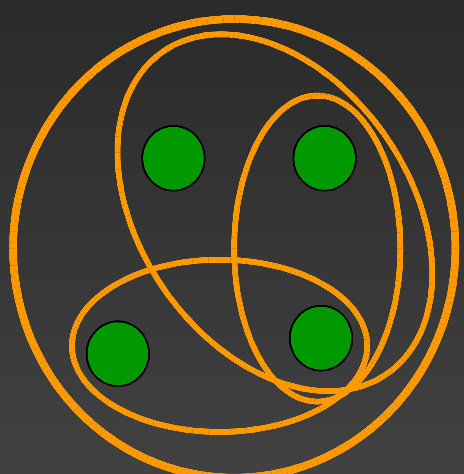


NO
entanglement
left !

W



entanglement
is still there



- τ_1 or its average take

  somewhat care of all types

- C or τ_2 cares for 2-party case

- τ_3 cares for 3-party case …

A purely n-tangled state ...

- contains no subtangle

- is not a tensor product of whatsoever kind

- rank[ρ_{red}] = 2 $\Rightarrow$ maximally mixed

Stochastic states: Verstraete, Dehaene, De Moor PRA 2003

- phase independent canonical form

GHZ states ➨

. . . Other states?

Central Idea

- **Antilinear local operator *filtering out* products in *combing* the local Hilbertspace**

find antilinear operators O_j such that

O_1 $\dagger 0$ H_{loc} *

O_2 $\dagger 0$ H_{loc}

Hairy ball theorem: * only possible for half integer spin

2nd condition defined on an embedding; not a Hilbert space

Building blocks: combs

2 $_0$ tr X† ^{3} X

†0

g † diag a, a, b, a

SL(2,C) invariant !

Convenient choice: a=1, b=0

Concurrence τ_2

- time reversal operator is a filter for two qubits

- Alternative filter expression found

➨Convex roof extension via R-Matrix works

Uhlmann PRA 2000

Three-tangle τ_3

- **Arbitrary filter construction $\equiv \tau_3$**

➘ **number of independent filters**
= number of different types of entanglement???

➘ **Convex roof extension via R-Matrix works …**
…in some cases

Four qubits

1

4 † , , , , , ,

2

4 † , , , , , , , ,

3

4 $\dagger\frac{1}{2}$, , , , 1,3 2,3

Three different states:

1 †GHZ† 0 15 $1,1,\frac{1}{2}$

2 † 1 2 4 8 2 15 0,0

3 † 1 2 12 15 0,0,1

Five qubits

1

5 †

2

5 †

...

1 †GHZ† 0 31 1, 1, 1, $\frac{1}{8}$

2 † 1 2 28 31 0, 0, 0, 1

3 † 1 2 4 24 2 31 0, 0, $\frac{2}{3}$, 0

4 † 1 2 4 8 16 3 31 $\frac{3}{32}$ 3, 0, 0, 0

Six qubits

- ➨ **straight forward extension to n-tangles**
- ➨ **candidates for n-tangled states suggested by the filters**

Spin S –2S+1 x 2S+1

- bilinear comb no longer unique
- S_y is a comb
- SL(2,C) invariant comb on hand
- SL(2S+1,C) invariant comb …?

Filters ...

- are entanglement monotones by construction
- consider only balanced part of a state
- straight forward *manufacturing*
- pave the way towards mixed states …
- …general subsystems

Summary and Outlook

- Simple ingredient: comb
- Unique comb $\rightarrow$ SL(2,C) invariance $\rightarrow$ filters are monotones

  Verstraete et al. PRA 2003
- Efficient way of generating entanglement monotones
- Monotones can be taylored according to the types of entanglement to be taken into consideration
- Well defined starting point for higher local dimension
- Connection with other approaches