A distributional approach to fractional Sobolev spaces and fractional variation: existence of blow-up

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work in collaboration with

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Some topics of Geometric Analysis and Geometric Measure Theory

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If $p \in [1, +\infty)$, $\alpha \in (0,1)$ and $\Omega \subset \mathbb{R}^n$, the fractional Sobolev space $W^{\alpha,p}(\Omega)$ is the space

$$W^{\alpha,p}(\Omega) := \left\{ u \in L^p(\Omega) : [u]_{W^{\alpha,p}(\Omega)} < +\infty \right\},$$

where

$$[u]_{W^{\alpha,p}(\Omega)} := \left(\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{n+p\alpha}} \, dx \, dy \right)^{\frac{1}{p}}.$$

A measurable set $E \subset \mathbb{R}^n$ has finite fractional perimeter if

$$P_\alpha(E) := [\chi_E]_{W^{\alpha,1}(\mathbb{R}^n)} = 2 \int_{\mathbb{R}^n \setminus E} \int_E \frac{1}{|x - y|^{n+\alpha}} \, dx \, dy < +\infty,$$

and we define the fractional perimeter in an open set Ω as

$$P_\alpha(E; \Omega) := \int_\Omega \int_\Omega \frac{|\chi_E(x) - \chi_E(y)|}{|x - y|^{n+\alpha}} \, dx \, dy + 2 \int_\Omega \int_{\mathbb{R}^n \setminus \Omega} \frac{|\chi_E(x) - \chi_E(y)|}{|x - y|^{n+\alpha}} \, dx \, dy.$$

Differently from the standard Sobolev space $W^{1,p}(\Omega)$, the space $W^{\alpha,p}(\Omega)$ does not seem to have a clear distributional nature.

$$\text{Leibniz (1695): } \frac{d^{\frac{1}{2}}x}{dx^{\frac{1}{2}}} = c\sqrt{x}, \text{ and Lacroix (1819): } c = \frac{2}{\sqrt{\pi}}.$$

"Il y a de l'apparence qu'on tirera un jour des consequences bien utiles de ces paradoxes, car il n'y a gueres de paradoxes sans utilité." (Leibniz to de l'Hôpital, 30 September 1695)
"This is an apparent paradox from which, one day, useful consequences will be drawn, since there are no paradoxes without utility."

$$\text{Lacroix (1819): } \frac{d^\alpha x^m}{dx^\alpha} = \frac{\Gamma(m+1)}{\Gamma(m-\alpha+1)} x^{m-\alpha}, \ \frac{d^\alpha 1}{dx^\alpha} = \frac{1}{\Gamma(1-\alpha)} x^{-\alpha},$$

$$\text{Riemann-Liouville (1832-1847): } D^\alpha_a f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t \frac{f(\tau)}{(t-\tau)^\alpha} \, d\tau,$$

$$\text{Caputo (1967): } D^{C,\alpha}_a f(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{f'(\tau)}{(t-\tau)^\alpha} \, d\tau.$$

Other important contributions by Euler, Fourier, Grünwald, Letnikov, Weyl, Marchaud, Riesz, Horváth...

Following a recent work by Šilhavý, we define the *fractional gradient* of $u \in C_c^\infty(\mathbb{R}^n)$ as

$$\nabla^\alpha u(x) := \mu_{n,\alpha} \int_{\mathbb{R}^n} \frac{(u(y) - u(x))(y - x)}{|y - x|^{n+\alpha+1}} \, dy,$$

where

$$\mu_{n,\alpha} := 2^\alpha \pi^{-\frac{n}{2}} \frac{\Gamma\left(\frac{n+\alpha+1}{2}\right)}{\Gamma\left(\frac{1-\alpha}{2}\right)}.$$

Fractional gradient and divergence

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Analogously, we define the *fractional divergence* of $\varphi \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n)$ as

$$\mathrm{div}^\alpha \varphi(x) := \mu_{n,\alpha} \int_{\mathbb{R}^n} \frac{(\varphi(y) - \varphi(x)) \cdot (y - x)}{|y - x|^{n+\alpha+1}} \, dy.$$

The operators ∇^α and div^α are *dual*: for any $u \in C_c^\infty(\mathbb{R}^n)$ and $\varphi \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n)$

$$\int_{\mathbb{R}^n} u \, \mathrm{div}^\alpha \varphi \, dx = - \int_{\mathbb{R}^n} \varphi \cdot \nabla^\alpha u \, dx.$$

In addition, $\mathrm{div}^\alpha \nabla^\alpha = -(-\Delta)^\alpha$.

The operators ∇^α and div^α are $\mathcal{D}$-continuous, invariant by translations and rotations and α-homogeneous, in the sense that

$$\nabla^\alpha(f(\mathbf{R}^{-1}\cdot))(x) = \mathbf{R}(\nabla^\alpha f)(\mathbf{R}^{-1}x), \quad \mathrm{div}^\alpha(\mathbf{R}\varphi(\mathbf{R}^{-1}\cdot))(x) = (\mathrm{div}^\alpha\varphi)(\mathbf{R}^{-1}x),$$

and

$$(\nabla^\alpha f(\lambda\cdot))(x) = |\lambda|^\alpha \,\mathrm{sgn}\,(\lambda)(\nabla^\alpha f)(\lambda x),$$
$$(\mathrm{div}^\alpha\varphi(\lambda\cdot))(x) = |\lambda|^\alpha \,\mathrm{sgn}\,(\lambda)(\mathrm{div}^\alpha\varphi)(\lambda x),$$

for any $f \in C_c^\infty(\mathbb{R}^n)$, $\varphi \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$, $\mathbf{R} \in O(n)$, $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}^n$.

Theorem (Šilhavý (2018))

Up to multiplicative constants, ∇^α and div^α are the unique operators on $C_c^\infty(\mathbb{R}^n)$ and $C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$, respectively, for which the properties listed above hold.

- ∇^α extends to functions in $W^{\alpha,1}(\mathbb{R}^n)$, since

$$\|\nabla^\alpha f\|_{L^1(\mathbb{R}^n;\mathbb{R}^n)} \le \mu_{n,\alpha}[f]_{W^{\alpha,1}(\mathbb{R}^n)},$$

so that $\nabla^\alpha f(x)$ is well defined for $\mathscr{L}^n$-a.e. x. Analogously, div^α extends to $W^{\alpha,1}(\mathbb{R}^n;\mathbb{R}^n)$.

- For any $f, g \in C_c^\infty(\mathbb{R}^n)$ it holds

$$\nabla^\alpha(fg) = f\nabla^\alpha g + g\nabla^\alpha f + \nabla^\alpha_{\mathrm{NL}}(f,g),$$

where

$$\nabla^\alpha_{\mathrm{NL}}(f,g)(x) := \mu_{n,\alpha}\int_{\mathbb{R}^n} \frac{(f(y)-f(x))(g(y)-g(x))(y-x)}{|y-x|^{n+\alpha+1}}\,dy.$$

- For any $f \in C_c^\infty(\mathbb{R}^n)$ and $\varphi \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$ it holds

$$\mathrm{div}^\alpha(f\varphi) = f\,\mathrm{div}^\alpha\varphi + \varphi\cdot\nabla^\alpha f + \mathrm{div}^\alpha_{\mathrm{NL}}(f,\varphi),$$

where

$$\mathrm{div}^\alpha_{\mathrm{NL}}(f,\varphi)(x) := \mu_{n,\alpha}\int_{\mathbb{R}^n} \frac{(f(y)-f(x))(\varphi(y)-\varphi(x))\cdot(y-x)}{|y-x|^{n+\alpha+1}}\,dy.$$

- If $f \in \mathrm{Lip}_c(\mathbb{R}^n)$, then $\nabla^\alpha f \in L^1(\mathbb{R}^n;\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n;\mathbb{R}^n)$ and, for any $x \in \mathbb{R}^n$,

$$\nabla^\alpha f(x) = \nabla I_{1-\alpha} f(x) = I_{1-\alpha} \nabla f(x) = \frac{\mu_{n,\alpha}}{n+\alpha-1} \int_{\mathbb{R}^n} \frac{\nabla f(y)}{|y-x|^{n+\alpha-1}}\, dy,$$

where

$$I_\alpha f(x) := \frac{\Gamma\left(\frac{n-\alpha}{2}\right)}{2^\alpha \pi^{\frac{n}{2}} \Gamma\left(\frac{\alpha}{2}\right)} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy.$$

- If $\varphi \in \mathrm{Lip}_c(\mathbb{R}^n;\mathbb{R}^n)$, then $\mathrm{div}^\alpha \varphi \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and, for any $x \in \mathbb{R}^n$,

$$\mathrm{div}^\alpha \varphi(x) = \mathrm{div} I_{1-\alpha} \varphi(x) = I_{1-\alpha} \mathrm{div} \varphi(x) = \frac{\mu_{n,\alpha}}{n+\alpha-1} \int_{\mathbb{R}^n} \frac{\mathrm{div}\varphi(y)}{|y-x|^{n+\alpha-1}}\, dy.$$

We define

$$BV^\alpha(\mathbb{R}^n) := \left\{ u \in L^1(\mathbb{R}^n) : |D^\alpha u|(\mathbb{R}^n) < +\infty \right\},$$

where

$$|D^\alpha u|(\mathbb{R}^n) = \sup \left\{ \int_{\mathbb{R}^n} u \operatorname{div}^\alpha \varphi \, dx : \varphi \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n), \ \|\varphi\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^n)} \le 1 \right\}.$$

In perfect analogy with the classical framework,

1. $BV^\alpha(\mathbb{R}^n)$ is a Banach space and its norm is lower semicontinuous with respect to L^1-convergence;
2. given $f \in L^1(\mathbb{R}^n)$, then $f \in BV^\alpha(\mathbb{R}^n)$ if and only if there exists a finite vector valued Radon measure $D^\alpha f \in \mathcal{M}(\mathbb{R}^n; \mathbb{R}^n)$ such that
$$\int_{\mathbb{R}^n} f \operatorname{div}^\alpha \varphi \, dx = - \int_{\mathbb{R}^n} \varphi \cdot dD^\alpha f \quad \text{for any } \varphi \in \mathrm{Lip}_c(\mathbb{R}^n; \mathbb{R}^n);$$
3. any uniformly bounded sequence in $BV^\alpha(\mathbb{R}^n)$ admits limit points in $L^1(\mathbb{R}^n)$ with respect the L^1_{loc}-convergence.

- The inclusion $W^{\alpha,1}(\mathbb{R}^n) \subset BV^\alpha(\mathbb{R}^n)$ is continuous and, if $f \in W^{\alpha,1}(\mathbb{R}^n)$, then $D^\alpha f = \nabla^\alpha f \mathscr{L}^n$.
- We define the *distributional fractional Sobolev space*

$$S^{\alpha,p}(\mathbb{R}^n) := \{f \in L^p(\mathbb{R}^n) : \exists\, \nabla^\alpha_w f \in L^p(\mathbb{R}^n; \mathbb{R}^n)\}.$$

  If $f \in BV^\alpha(\mathbb{R}^n)$, then $f \in S^{\alpha,1}(\mathbb{R}^n)$ if and only if $|D^\alpha f| \ll \mathscr{L}^n$, in which case

$$D^\alpha f = \nabla^\alpha_w f\, \mathscr{L}^n,$$

  and this implies $BV^\alpha(\mathbb{R}^n) \setminus S^{\alpha,1}(\mathbb{R}^n) \neq \emptyset$.
- By an argument based on the Inverse Mapping Theorem, we get also $S^{\alpha,1}(\mathbb{R}^n) \setminus W^{\alpha,1}(\mathbb{R}^n) \neq \emptyset$ (since $W^{\alpha,1}(\mathbb{R}^n)$ is closed with respect to the pointwise convergence).
- If $f \in BV(\mathbb{R}^n)$, then

$$\nabla^\alpha f(x) = \frac{\mu_{n,\alpha}}{n+\alpha-1} \int_{\mathbb{R}^n} \frac{dDf(y)}{|y-x|^{n+\alpha-1}}$$

  for $\mathscr{L}^n$-a.e. $x \in \mathbb{R}^n$.

Definition

Let $\alpha \in (0,1)$ and let $E \subset \mathbb{R}^n$ be a measurable set. For any open set $\Omega \subset \mathbb{R}^n$, the *fractional Caccioppoli α-perimeter in Ω* is the *fractional variation* of χ_E in Ω, i.e.

$$|D^\alpha \chi_E|(\Omega) = \sup \left\{ \int_E \mathrm{div}^\alpha \varphi \, dx : \varphi \in C_c^\infty(\Omega; \mathbb{R}^n), \ \|\varphi\|_{L^\infty(\Omega;\mathbb{R}^n)} \le 1 \right\}.$$

We say that E is a set with *finite fractional Caccioppoli α-perimeter in Ω* if $|D^\alpha \chi_E|(\Omega) < +\infty$.

A measurable set $E \subset \mathbb{R}^n$ is a set with finite fractional Caccioppoli α-perimeter in Ω if and only if $D^\alpha \chi_E \in \mathcal{M}(\Omega; \mathbb{R}^n)$ and

$$\int_E \mathrm{div}^\alpha \varphi \, dx = - \int_\Omega \varphi \cdot dD^\alpha \chi_E$$

for all $\varphi \in C_c^\infty(\Omega; \mathbb{R}^n)$.

Definition

Let $\alpha \in (0,1)$ and let $\Omega \subset \mathbb{R}^n$ be an open set. If $E \subset \mathbb{R}^n$ is a set with finite fractional Caccioppoli α-perimeter in Ω, then we say that a point $x \in \Omega$ belongs to the *fractional reduced boundary* of E (inside Ω), and we write $x \in \mathscr{F}^\alpha E$, if

$$x \in \operatorname{supp}(D^\alpha \chi_E) \qquad \text{and} \qquad \exists \lim_{r \to 0} \frac{D^\alpha \chi_E(B_r(x))}{|D^\alpha \chi_E|(B_r(x))} \in \mathbb{S}^{n-1}.$$

We thus let

$$\nu_E^\alpha \colon \Omega \cap \mathscr{F}^\alpha E \to \mathbb{S}^{n-1}, \qquad \nu_E^\alpha(x) := \lim_{r \to 0} \frac{D^\alpha \chi_E(B_r(x))}{|D^\alpha \chi_E|(B_r(x))}, \quad x \in \Omega \cap \mathscr{F}^\alpha E,$$

be the (*measure-theoretic*) *inner unit fractional normal* to E (inside Ω).

Therefore, the following Gauss–Green formula holds for any $\varphi \in \mathrm{Lip}_c(\Omega; \mathbb{R}^n)$:

$$\int_E \operatorname{div}^\alpha \varphi \, dx = - \int_{\Omega \cap \mathscr{F}^\alpha E} \varphi \cdot \nu_E^\alpha \; d|D^\alpha \chi_E|.$$

Proposition

If $E \subset \mathbb{R}^n$ satisfies $P_\alpha(E;\Omega) < +\infty$, then E is a set with finite fractional Caccioppoli α-perimeter in Ω with

$$|D^\alpha \chi_E|(\Omega) \le \mu_{n,\alpha} P_\alpha(E;\Omega)$$

and $D^\alpha \chi_E = \nu_E^\alpha \, |D^\alpha \chi_E| = \nabla^\alpha \chi_E \, \mathscr{L}^n$. Moreover, if E is such that $|E| < +\infty$ and $\mathrm{P}(E) < +\infty$, then $\chi_E \in W^{\alpha,1}(\mathbb{R}^n)$ for any $\alpha \in (0,1)$, and

$$\nabla^\alpha \chi_E(x) = \frac{\mu_{n,\alpha}}{n+\alpha-1} \int_{\mathbb{R}^n} \frac{\nu_E(y)}{|y-x|^{n+\alpha-1}} \, d|D\chi_E|(y) \text{ for } \mathscr{L}^n\text{-a.e. } x \in \mathbb{R}^n.$$

Notice that we have

$$P_\alpha(E;\Omega) < +\infty \implies \mathscr{L}^n(\Omega \cap \mathscr{F}^\alpha E) > 0$$

including even the case $\chi_E \in BV(\mathbb{R}^n)$.

Thanks to the scaling properties, we get

$$D^\alpha \chi_{\frac{E-x}{r}} = \frac{1}{r^{n-\alpha}} (I_{x,r})_\# D^\alpha \chi_E,$$

where $I_{x,r}(y) = (y-x)/r$.

Theorem

Let $\alpha \in (0,1)$. There exist $A_{n,\alpha}, B_{n,\alpha} > 0$ with the following property. Let $E \subset \mathbb{R}^n$ be a set with locally finite fractional Caccioppoli α-perimeter in $\mathbb{R}^n$. For any $x \in \mathscr{F}^\alpha E$ there exists $r_x > 0$ such that

$$|D^\alpha \chi_E|(B_r(x)) \le A_{n,\alpha} r^{n-\alpha}, \quad |D^\alpha \chi_{E\cap B_r(x)}|(\mathbb{R}^n) \le B_{n,\alpha} r^{n-\alpha}$$

for all $r \in (0, r_x)$.

As an immediate consequence, we get

$$|D^\alpha \chi_E| \le 2^{n-\alpha} \frac{A_{n,\alpha}}{\omega_{n-\alpha}} \mathscr{H}^{n-\alpha} \llcorner \mathscr{F}^\alpha E \text{ and } \dim_{\mathscr{H}}(\mathscr{F}^\alpha E) \ge n-\alpha.$$

Let $\mathrm{Tan}(E, x)$ be the set of all *tangent sets of E at x*, i.e. the set of all limit points in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$-topology of the family $\left\{\frac{E-x}{r} : r > 0\right\}$ as $r \to 0$.

Theorem

Let $\alpha \in (0,1)$. Let E be a set with locally finite fractional Caccioppoli α-perimeter in $\mathbb{R}^n$. For any $x \in \mathscr{F}^\alpha E$, we have $\mathrm{Tan}(E, x) \neq \varnothing$.
In addition, if $F \in \mathrm{Tan}(E, x)$, then F is a set of locally finite fractional Caccioppoli α-perimeter in $\mathbb{R}^n$ such that $\nu_F^\alpha(y) = \nu_E^\alpha(x)$ for $|D^\alpha \chi_F|$-a.e. $y \in \mathscr{F}^\alpha F$.

If $\varphi \in C_c^2(\mathbb{R}^n;\mathbb{R}^n)$, then

$$\lim_{\alpha\to 1^-} \|\mathrm{div}^\alpha \varphi - \mathrm{div}\varphi\|_{L^1(\mathbb{R}^n)} = 0.$$

Hence, if E is a set of locally finite perimeter in $\mathbb{R}^n$, we have $D^\alpha \chi_E \rightharpoonup D\chi_E$.

Theorem (Γ-liminf inequality)

Let $\Omega \subset \mathbb{R}^n$ be an open set. If there exist measurable sets $(E_\alpha)_{\alpha\in(0,1)}$ and E satisfying $\chi_{E_\alpha} \to \chi_E$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ as $\alpha \to 1^-$, then

$$\mathrm{P}(E;\Omega) \le \liminf_{\alpha\to 1^-} |D^\alpha \chi_{E_\alpha}|(\Omega).$$

In addition, if $\mathrm{P}(E) < \infty$, we have $|D^\alpha \chi_E| \le I_{1-\alpha}(|D\chi_E|) \rightharpoonup |D\chi_E|$.

Theorem (Γ-limsup inequality)

Let E be a set of finite perimeter in $\mathbb{R}^n$ and Ω be an open bounded set such that $|D\chi_E|(\partial\Omega) = 0$. Then, we have

$$\limsup_{\alpha\to 1^-} |D^\alpha \chi_E|(\Omega) \le \mathrm{P}(E;\Omega).$$

Theorem ($\Gamma(L^1_{\mathrm{loc}})$ - lim of perimeters as $\alpha \to 1^-$)

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. For every measurable set $E \subset \mathbb{R}^n$, we have

$$\Gamma(L^1_{\mathrm{loc}})\text{-}\lim_{\alpha \to 1^-} |D^\alpha \chi_E|(\Omega) = P(E;\Omega).$$

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4. extend the Gauss–Green and integration by parts formulas to sets of finite fractional Caccioppoli α-perimeter;
5. give a good definition of BV^{α} functions on a general open set.

Thank you for your attention!