Orbit determination for the Fibonacci dynamical system and generalizations: asymptotic behavior

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The classical orbit determination problem (Gauss 1809) becomes difficult, both conceptually and numerically, when applied to a chaotic orbit, if the time span of the observations being used in a least squares fit is much longer than the Lyapounov time. Nevertheless, such problems practically occur, both in Solar System Dynamics (e.g., in long term impact monitoring for NEO) and in Astrodynamics (e.g., in long satellite tours around the giant planets). Spoto and Milani (2016), and Serra, Spoto and Milani (2018) have tried to tackle such a complex problem by starting from a model problem, the discrete dynamical system defined by the standard map (discretization of a nonlinear pendulum): they have provided numerical evidence that the asymptotic behavior (as the observed time span goes to infinity) changes radically whenever dynamical parameters are included in the fit parameters, besides the initial conditions, and pointed out the relationship between these results and a finite time version of the shadowing lemma.

In this work we give for the first time analytical, rigorous proofs of theorems on some other model problems, starting from the Fibonacci map on the two-dimensional torus. For the non-homogeneous Fibonacci map

$$x_{k+1} = x_k + x_{k-1} + \mu,$$

where the variable x is considered as an angle, if the observations of both coordinates (x_k, x_{k-1}) are available for $-N \leq k \leq N$ (with standard deviation σ), we prove that the fit limited to the two initial conditions (x_1, x_0) has a post-fit standard deviation $STD(x_0) = STD(x_1) \simeq \sigma\, c\, \alpha^{-N}$ where α is the golden ratio (the largest Lyapounov multiplier) and $c = 1$. If also the dynamical parameter μ is included in the fit, then $STD(x_0) = STD(x_1) = STD(\mu) \simeq \sigma\, d/\sqrt{N}$ where $d = 1/2$. In short, the uncertainty of the solution for μ decreases as $1/\sqrt{N}$, like in the standard Gaussian statistics (e.g., in the central limit theorem). This result is enough to prove that the conjecture proposed by Wisdom (Icarus 1987) is not true, at least not in this case.

Then, we have considered a more general dynamical system on the 2-torus of the form

$$X_{k+1} = AX_k + \mu F$$

where A is a 2×2 matrix with integer coefficients and with an integer inverse, thus $\det A = \pm 1$ (belonging to the group $GL(2, \mathbb{Z})$). The matrix A is hyperbolic, with irrational eigenvalues, in all but a finite number of distinct cases. In the hyperbolic cases we have studied the same problem and proven, under the additional hypothesis that the matrix A is symmetric, that the same results apply for the asymptotic behavior, only with different values of the constants c, d, depending on A and on the vector F. These examples are an infinite, enumerable set of dynamical systems on the two-dimensional torus which are distinct (neither conjugate nor subgroups in $GL(2, \mathbb{Z})$). All these examples have the common property of being Anosov diffeomorphisms on a compact manifold, uniformly chaotic and ergodic (Arnold and Sinai 1962), and structurally stable (Moser 1969).

We propose the conjecture that the same asymptotic behavior is shared by the orbit determination on all Anosov diffeomorphisms. The case of the dynamical systems, such as the standard map and the three-body problem, in which there are chaotic orbits (on invariant hyperbolic sets) and ordered orbits (on invariant KAM tori), plus other less known behaviors, is necessarily more difficult: indeed the numerical evidence points to a less uniform behavior. Still some of the ideas developed in our research on model problems can already be applied to difficult practical problems.