Compensation Phenomena

in

Linear Elliptic Systems

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ETH Zürich

Recent Trends in Geometric Analysis and Applications

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Hence

$$-\Delta u^i = \nabla^{\perp} B^{ij} \cdot \nabla u_j \quad \text{where} \quad \nabla^{\perp} B^{ij} = u^i \nabla u^j - u^j \nabla u^i \in L^2(D^2)$$

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$$\begin{cases} -\Delta\psi = \nabla^\perp a \cdot \nabla b & \text{in } D^2 \\ \\ \partial_\nu \psi = 0 & \text{on } \partial D^2 \end{cases}$$

but $\psi \notin C^0$ and $\psi \notin W^{1,2}$ Da Lio-Pamurella 2017.

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$$\Omega := \nabla(P_T(u))P_N(u) - (\nabla(P_T(u))P_N(u))^t \in L^2(D^2, so(m) \otimes \mathbb{R}^2)$$

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but $u \notin C^0$.

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Then there exists $A, A^{-1} \in L^\infty \cap W^{1,2}(D^2, Gl_m(\mathbb{R}))$ such that

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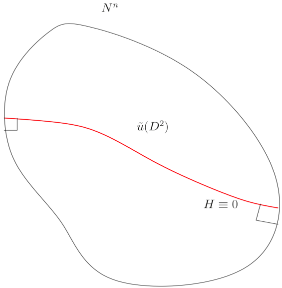

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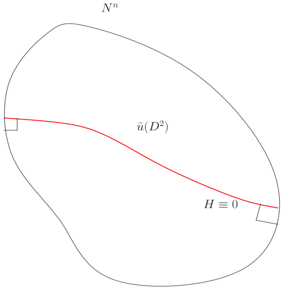

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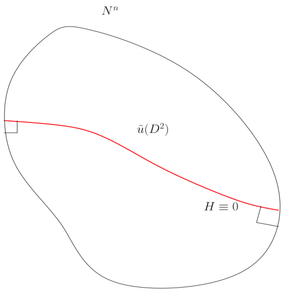

In other words u is 1/2−harmonic iff it's harmonic extension is a conformal param. of a free boundary disc. A similar caracterization exists for general free boundary surfaces Da Lio-Pigati 2017.

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$$\left\| \Delta^{1/4}(Q\, v) - Q\, \Delta^{1/4} v + \Delta^{1/4} Q\, v \right\|_{\dot{H}^{-1/2}} \leq C\ \|Q\|_{\dot{H}^{1/2}}\ \|v\|_{L^2}$$

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Question asked by Brezis

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Solutions to

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Weiertrass representation of Lagrangian Surfaces Hélein, Romon 2000.

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Classical arguments permit to deduce Morrey estimates of the form

$$\sup_{x_0,\rho>0}\,\rho^{-\delta}\,\|\mathfrak{F}\|_{L^{2,\infty}(B_\rho(x_0))} < +\infty$$

for some $\delta > 0$. This implies the result.