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Hypersurfaces with constant higher order mean curvature

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Pisa, SNS: November 25, 2019

THE OBJECTS

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- N will be the Euclidean space $\mathbb{R}^{n+1}$, the hyperbolic space $\mathbb{H}^{n+1}$ and the product space $\mathbb{H}^n \times \mathbb{R}$.

Where do the symmetric functions of the principal curvatures come from?

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- If M has constant H_{r+1}, then one can choose $\lambda = \binom{n}{r+1} H_{r+1}$ and M is a critical point for J_r.

Short digression about stability.

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- Notice that the linearized operator of $H_{r+1} = cost$ is

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- We are studying the properties of complete r-stable hypersurfaces.

Back to $H_r = constant$

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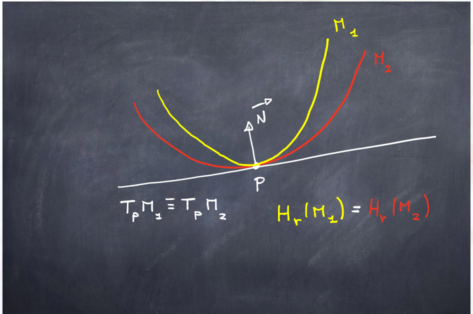

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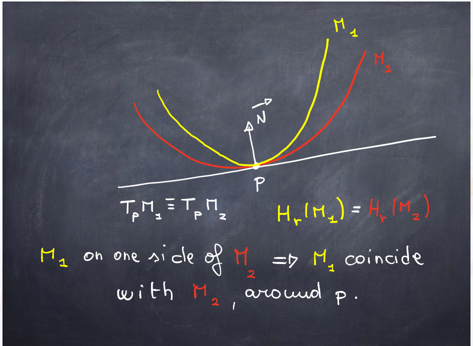

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THE CASE $H_r = \mathit{constant} > 0$

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- Ros-Korevaar used a technique based on Reilly formula and Minkowsky formula.

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Let C be a convex curve in a plane P and let M be a surface with constant H_1, embedded in $\mathbb{R}^3$ with boundary $\partial M = C$ and transverse to P along C. Then M inherits the symmetries of C. In particular M is a sphere if C is a circle.

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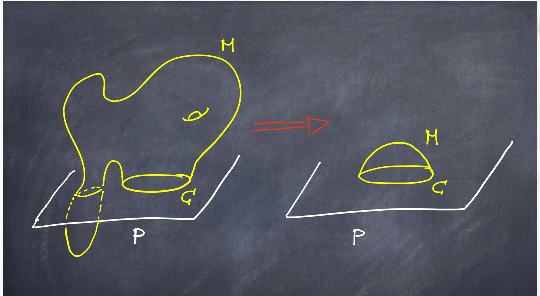

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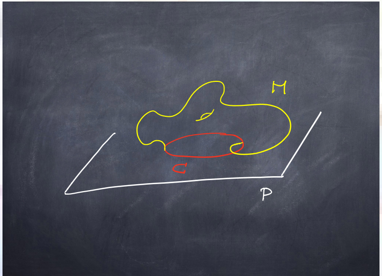
CONJECTURE (BRITO-MEEKS-ROSENBERG-SA EARP [IUMJ, 1991])

A compact constant mean curvature surface M bounded by a circle is a sphere provided either of the following conditions holds



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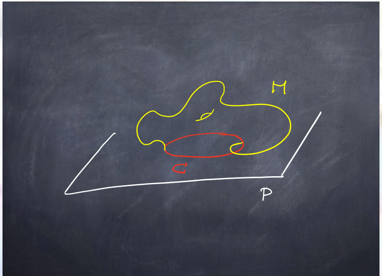

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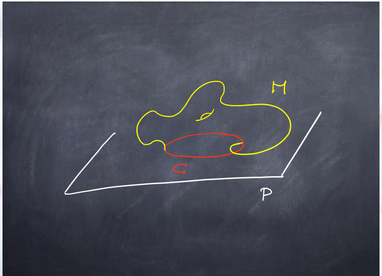
2. M is embedded.



$$\mathbf{H_r = c} \ \text{ in } \ \mathbb{H}^{n+1}$$

$H_r = \textit{constant} \geq 0$ IN $\mathbb{H}^{n+1}$ AND $\partial M \neq \emptyset$

THEOREM (N-ROSENBERG [AGAG, 1995])

Let C be a codimension two sphere in $\mathbb{H}^{n+1}$ and let M be a compact embedded hypersurface in $\mathbb{H}^{n+1}$ with boundary $\partial M = C$. Assume some H_r of M to be constant and $0 \leq H_r \leq 1$. Then M is a part of an equidistant sphere.

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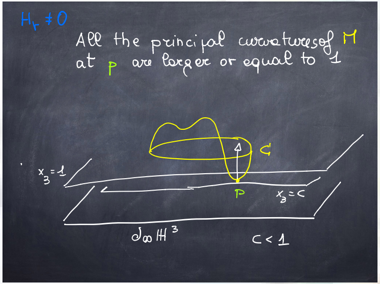
Notice that

- We do not need transversality along the boundary.
- We take $H_r \in [0, 1]$.
- Notice that $H_r = 0$ does not yields an elliptic PDE in general, hence we use purely geometric comparison arguments.

Proof. $H_r = constant > 0$

- We can assume that C is contained in a horosphere: then we can prove transversality at the boundary and that M is contained on one side of a horosphere.



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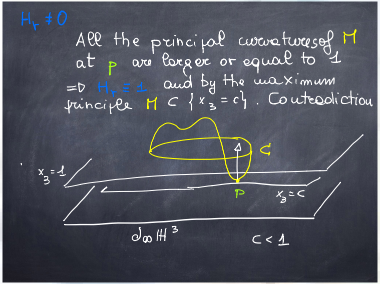

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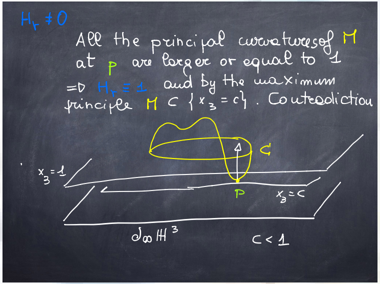
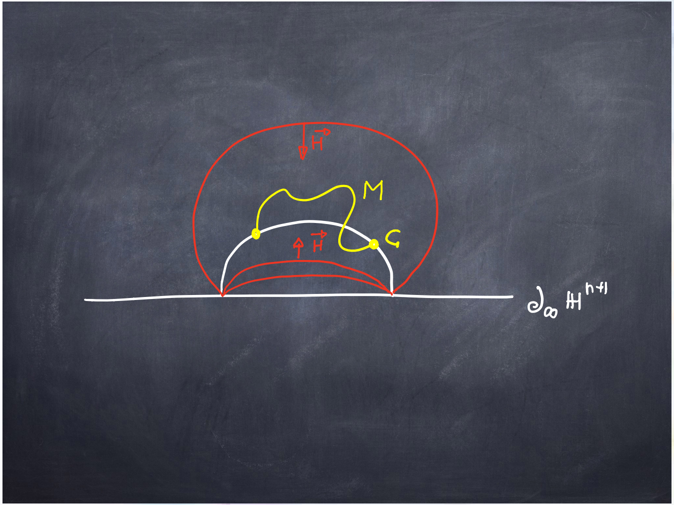


- One concludes using Alexandrov reflection method.

Proof. $H_r = 0$

- We prove, using curves on equidistant spheres, that M is the part of the geodesic hyperplane bounded by C.



$$H_r = 0$$

WHEN MAXIMUM PRINCIPLE HOLDS FOR $H_r = 0$

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Theorem

(Hounie-Leite [JDG, 1995]) *Let M, M' be hypersurfaces in $\mathbb{R}^{n+1}$ with $H_r = 0$ for some $1 \le r < n$ and let $p \in M \cap M'$ such that $T_pM \equiv T_pM'$, $\mathbf{N}(p) = \mathbf{N}'(p)$ and $\vec{\kappa}(p)$, $\vec{\kappa}'(p)$ belong to the same leaf. Then, if M remains on one side of M' around p and the rank of the Gauss map of either M or M' is $\ge r$, then M and M' must coincide in a neighbourhood of p.*

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- The rank of the Gauss map coincide with the number of κ_i's different from zero.

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- Hounie-Leite [JDG, 1995] established a maximum principle for hypersurfaces in $\mathbb{R}^{n+1}$ with $H_r = 0$. Due to the algebraic form of H_r, the set $\Gamma = \{x \in \mathbb{R}^n \mid H_r(x) = 0\}$ can be decomposed as union of r leaves. Denote by $\mathbf{N}$ an orientation and $\vec{\kappa} = (\kappa_1, \ldots \kappa_n)$, the principal curvatures of M.

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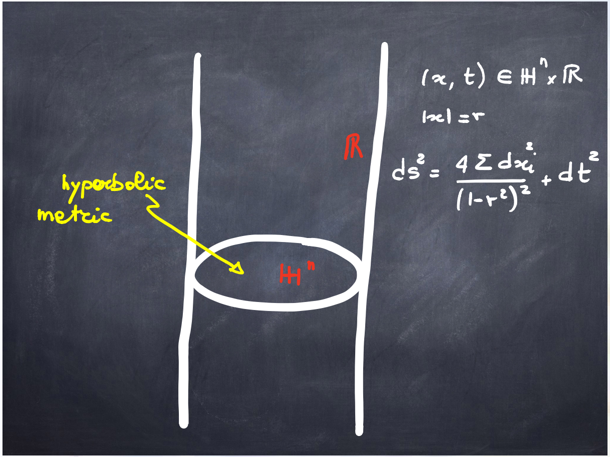
- Fontenele-Silva [AABS, 2002] extended it to any ambient manifold.

$\mathbb{H}^n \times \mathbb{R}$

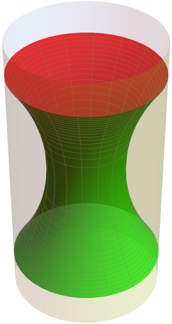


Two ends examples in $\mathbb{H}^2 \times \mathbb{R}$

TWO ENDS EXAMPLES IN $\mathbb{H}^2 \times \mathbb{R}$

Courtesy of Martin-Mazzeo-Rodriguez

Vertical Catenoid



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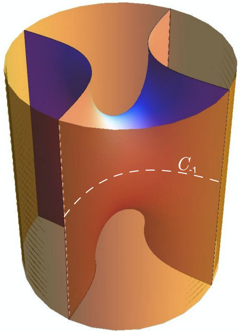

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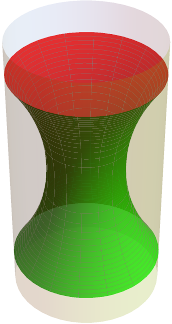

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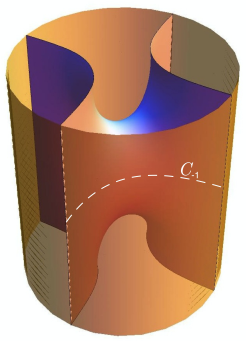

In $\mathbb{H}^n \times \mathbb{R}$, we revisit two classical results for minimal surfaces with finite number of ends (an end is a connected component of the complement of a compact set).

$H_r = 0$ in $\mathbb{H}^n \times \mathbb{R}$: Schoen type theorem.

Theorem Elbert-N-Santos [Manuscripta, 2016]

Let M be a complete, connected, embedded hypersurface in $\mathbb{H}^n \times \mathbb{R}$ with $H_r = 0$, $2 \leq r < n$ with $H_{r+1} \neq 0$, Assume that M has two ends, each a vertical graph whose asymptotic boundary is a copy of $\partial_\infty \mathbb{H}^n$. Then, M is isometric, by an ambient isometry to a rotational invariant hypersurface.

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- In Elbert-N-Santos [Manuscripta, 2016] r- minimal rotational hypersurfaces in $\mathbb{H}^n \times \mathbb{R}$ are classified.

$H_r = 0$ IN $\mathbb{H}^n \times \mathbb{R}$: OSSERMAN THEOREM.

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Theorem (Huber [CMH, 1957], Osserman [1969])
Let M be a complete oriented, immersed minimal surface in $\mathbb{R}^3$ with finite total curvature. Then M is conformally equivalent to a compact Riemann surface M with a finite number of points removed (the ends of M). Moreover, the Gauss map extends meromorphically to the punctures.

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- Then, we specialise to M with $H_r = 0$ for some $r = 1, \ldots, n$.

Finite Strong Total Curvature

Finite strong total curvature

Theorem (Elbert-N [Bull. London Math. Soc. 2019])
Let $\mathbf{x} : M \longrightarrow \mathbb{H}^n \times \mathbb{R}$, $n \geq 3$, be an orientable complete hypersurface with finite strong total curvature. Then:

1. *The immersion $\mathbf{x}$ is proper.*
2. *M is diffeomorphic to a compact manifold $\bar{M}$ minus a finite number of points $q_1, \dots, q_k$.*

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Let A be the shape operator of M. We define $|| \, |A| \, ||_{W^{1,q}_{-1}(M)}$ to be the *strong total curvature* of M and we say that M has *finite strong total curvature* if $|A| \in W^{1,q}_{-1}(M)$, for some $q > n$,

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DEFINITION.

Let A be the shape operator of M. We define $||\,|A|\,||_{W^{1,q}_{-1}(M)}$ to be the *strong total curvature* of M and we say that M has *finite strong total curvature* if $|A| \in W^{1,q}_{-1}(M)$, for some $q > n$, that is (ξ is the intrinsic distance from a fixed point of M)

$$||\,|A|\,||_{W^{1,q}_{-1}(M)} = \left(\int_M |A|^q \xi^{q-n}\, dM\right)^{1/q} + \left(\int_M |\nabla|A||^q \xi^{2q-n}\, dM\right)^{1/q} < \infty$$

Finite strong total curvature

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- Our proof is inspired in Do Carmo-Elbert [CAG, to appear]

Finite strong total curvature and $H_r = 0$

Theorem (Elbert-N [Bull. London Math. Soc. 2019])

Let $X : M \longrightarrow \mathbb{H}^n \times \mathbb{R}$, $n \geq 3$, be an orientable complete hypersurface with finite strong total curvature and $H_r = 0$. Let E be a punctured neighbourhood of one of the q_i' s and $N = (N_1, \ldots, N_{n+1})$ be a unit normal vector field on E. Let P be a hyperplane of $\mathbb{H}^n \times \mathbb{R}$ such that ∂E is contained in one of the half-spaces determined by P. Suppose that $\partial_\infty E \cap (\partial_\infty \mathbb{H}^n \times \mathbb{R}) \subset \partial_\infty P$. Then:

- *E is asymptotically close to P*
- *For any sequence of points $\{p_m\} \subset E$ converging to a point in $\partial_\infty E$, the sequence $\{N_{n+1}(p_m)\}$ converges uniformly to zero.*

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- When working with $n > 2$ one looses the technical support of the complex analysis and with $r > 1$ one weakens the technical support given by the theory of quasi-linear PDE. Then it seems reasonable to require a stronger assumption on the total curvature.


Thank You