The geometry of constant mean curvature surfaces in Euclidean space.

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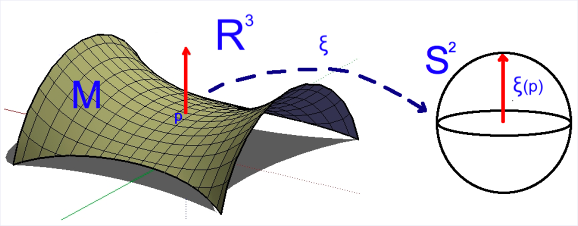
(Joint work with Meeks)



Let $\mathbf{M}$ be an oriented surface in $\mathbf{R^3}$, let ξ be the unit vector field normal to $\mathbf{M}$. Since $\langle \xi, \xi \rangle = 1$

$$d\xi_p \colon T_p\mathbf{M} \to T_{\xi(p)}\mathbf{S^2} \simeq T_p\mathbf{M}$$

Definition

The map $\mathbf{A} = -d\xi$ is the **shape operator** of $\mathbf{M}$.

Definition

$$\mathbf{A}_p\colon\ T_p\mathbf{M} \to T_p\mathbf{M}$$

- $\mathbf{A}_p$ is symmetric.
- The eigenvalues k_1, k_2 of $\mathbf{A}_p$ are the **principal curvatures** of $\mathbf{M}$ at p.
- $\mathbf{K_G} = \det(\mathbf{A}) = k_1 k_2$ is the **Gauss curvature**.
- $\mathbf{H} = \frac{1}{2}\mathrm{tr}(\mathbf{A}) = \frac{k_1+k_2}{2}$ is the **mean curvature**.
- $|\mathbf{A}| = \sqrt{k_1^2 + k_2^2}$ is the **norm of the shape operator** or **norm of the second fundamental form**.

Gauss equation

$$4\mathbf{H}^2 = |\mathbf{A}|^2 + 2\mathbf{K_G}$$

Question

What does the mean curvature say about $\mathbf{M}$?

First Variation Formula

Let $\mathbf{M}_t^{\phi}$ be a smooth normal variation of $\mathbf{M}$ fixing the boundary, i.e. let $\phi \in C_0^{\infty}(\mathbf{M})$ and

$$\mathbf{M}_t^{\phi} = \{p + t\phi(p)\xi(p) \mid p \in \mathbf{M}\}, \text{ then}$$

$$\frac{d}{dt}\text{Area}(\mathbf{M}_t^{\phi}) \Big|_{t=0} = -2\int_{\mathbf{M}} \mathbf{H}\phi$$

First Variation Formula

$$\frac{d}{dt}\text{Area}(\mathbf{M}_t^{\phi}) \Big|_{t=0} = -2\int_{\mathbf{M}} \mathbf{H}\phi, \quad \phi \in C_0^{\infty}(\mathbf{M})$$

Definition

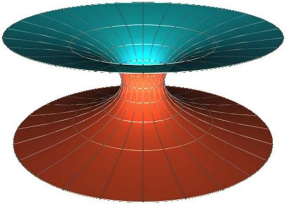
$\mathbf{M}$ is a **minimal surface** $\iff$ $\mathbf{M}$ is a critical point for the area functional $\iff$ $\mathbf{H} \equiv 0$.

Definition

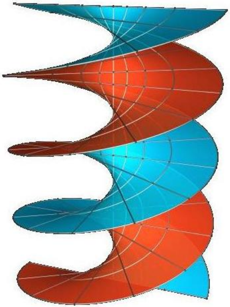
$\mathbf{M}$ is a **CMC surface** $\iff$ $\mathbf{M}$ is a critical point for the area functional under variations **preserving the volume**, $\int_{\mathbf{M}} \phi = 0$ $\iff$ $\mathbf{H} \equiv$ constant.

Catenoid

Image by Matthias Weber



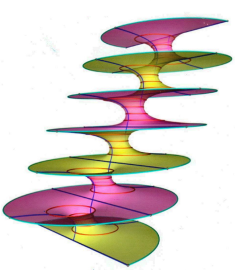
- In 1741, **Euler** discovered that when a catenary $x_1 = \cosh x_3$ is rotated around the x_3-axis, then one obtains a surface which minimizes area among surfaces of revolution after prescribing boundary values for the generating curves.
- In 1776, **Meusnier** verified that the catenoid is locally a solution of Lagrange's equation.
- Together with the plane, the catenoid is the only minimal surface of revolution (1860 **Bonnet**).
- The catenoid has genus zero. (The **genus** of a surface is the maximum number of pairwise disjoint simple closed curves which do not separate the surface)



- Proved to be minimal by **Meusnier** in 1776.
- Together with the plane, the helicoid is the only ruled minimal surface (1842, **Catalan**).
- The helicoid has genus zero.

Riemann minimal examples.

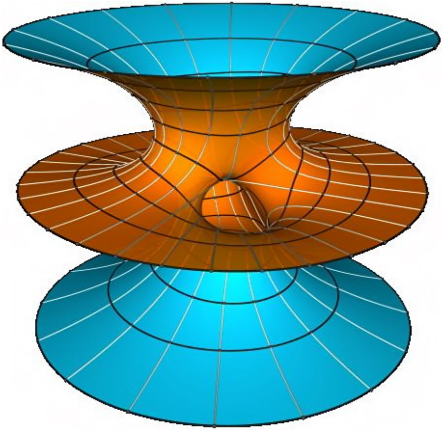
Image by Matthias Weber



- Discovered in 1860 by **Riemann**, these examples are invariant under reflection in the (x_1, x_3)-plane and by a translation T_λ.
- After appropriate scalings, they converge to catenoids as $t \to 0$ or to helicoids as $t \to \infty$.
- The Riemann minimal examples have the property that every horizontal plane intersects the surface in a circle or in a line.
- These surfaces have genus 0 and infinite topology. (A surface has **finite topology** if it is topologically equivalent to a compact surface with a finite subset of points removed)

Costa torus.

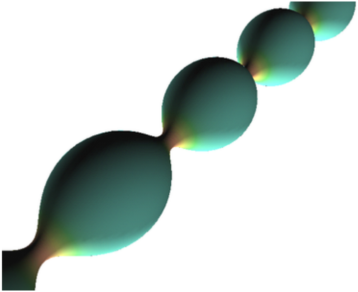
Image by Matthias Weber

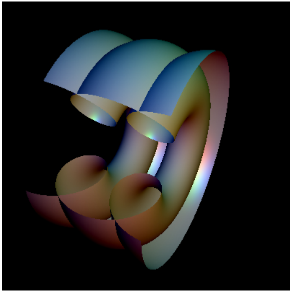


Key Properties:

- Discovered in 1982 by **Costa**.
- This is a thrice punctured torus with two catenoidal ends and one planar middle end.
- **Hoffman** and **Meeks** proved its global embeddedness.

Delaunay Surfaces





- In 1845, **Delaunay** discovered and classified the surfaces of revolution with constant mean curvature $\mathbf{H} = 1$
- He wrote down a 1-parameter family of surfaces now called **Delaunay surfaces** (unduloids and nodoids).
- Such family contains a chain of round spheres of radius 1 and the cylinder of radius $\frac{1}{2}$.

Example (Graph of a function)

$(x, y, u(x,y))$: $\xi = \frac{1}{\sqrt{1+|\nabla u|^2}}(-u_x, -u_y, 1)$

- $\mathbf{K}_G = \frac{\det(\mathrm{Hess}(u))}{(1+|\nabla u|^2)^2}$ Fully non-linear PDE
- $\mathbf{H} = \frac{1}{2}\mathrm{div}\frac{\nabla \mathbf{u}}{\sqrt{1+|\nabla \mathbf{u}|^2}} = \frac{1}{2}\frac{\Delta \mathbf{u}}{\sqrt{1+|\nabla u|^2}} + \frac{1}{2}\nabla u \cdot \nabla\left(\frac{1}{\sqrt{1+|\nabla u|^2}}\right)$
  Quasi-linear elliptic PDE
- $\frac{|\mathrm{Hess}(u)|}{(1+|\nabla u|^2)^{\frac{3}{2}}} \leq |\mathbf{A}| \leq 2\frac{|\mathrm{Hess}(u)|}{\sqrt{1+|\nabla u|^2}}$

PDE $\Longrightarrow$ minimal and CMC surfaces satisfy some nice properties.

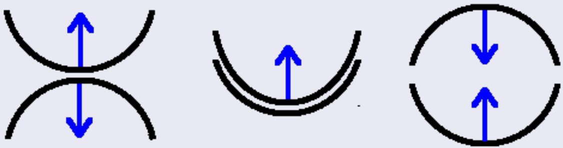
Mean Curvature Vector

$$\vec{\mathbf{H}} = \mathbf{H}\xi$$

Local properties of CMC surfaces, $\mathbf{H} \neq 0$:

- 1-sided maximum principle;
- 1-sided regular neighborhood.

What happens when CMC surfaces get close to each other?



Classification of complete, simply-connected surfaces embedded in $\mathbf{R}^3$ with constant mean curvature.

Compact case:

Let $\mathbf{M}$ be a **closed** (compact without boundary) CMC surface in $\mathbf{R}^3$:

- $\mathbf{M}$ cannot be minimal. ($\mathbf{K_G} \leq 0$)

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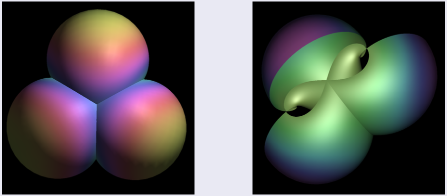
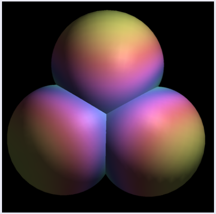
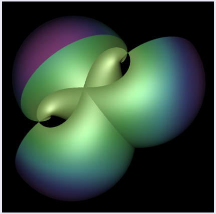


Wente Torus

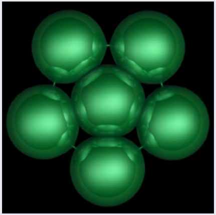
- In 1984, **Wente** constructed the first example of a **closed** (compact without boundary) CMC surface in $\mathbf{R}^3$ different from the round sphere.

- Existence of immersed CMC Tori (1984, **Wente**).





- Many examples of closed CMC surfaces (1994, **Kapouleas**; **Mazzeo-Pacard**, **Mazzeo-Pacard-Pollack**, et al.)



Classification of complete, simply-connected surfaces embedded in $\mathbf{R}^3$ with constant mean curvature.

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Theorem (2008, **Colding-Minicozzi** and **Meeks-Rosenberg**)

The plane and the helicoid are the only complete simply connected minimal surfaces **embedded** in $\mathbf{R}^3$.

Classification of complete, simply-connected surfaces embedded in $\mathbb{R}^3$ with NON-ZERO constant mean curvature.

Question

Is the round sphere the only complete simply connected surface **embedded** in $\mathbb{R}^3$ with nonzero constant mean curvature?

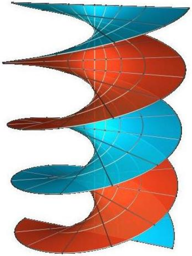
Answer (Meeks-T.)

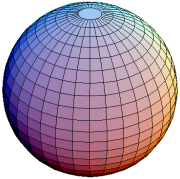
Yes.

Theorem (Meeks-T.)

Round spheres are the only complete simply connected surfaces **embedded** in $\mathbf{R}^3$ with nonzero constant mean curvature.

Let $\mathbf{M}$ be a complete, simply-connected CMC surface **embedded** in $\mathbf{R}^3$, then it is either **a plane, a helicoid or a round sphere**. (2008, **Colding-Minicozzi** and **Meeks-Rosenberg** for $\mathbf{H} = 0$)





Definition

A **1**-disk is a simply-connected surface (possibly with boundary) **embedded** in $\mathbf{R}^3$ with constant mean curvature **1**.

Definition

A $\mathbf{1}$-disk is a simply-connected surface (possibly with boundary) **embedded** in $\mathbf{R^3}$ with constant mean curvature $\mathbf{1}$.

Theorem (Radius Estimates for $\mathbf{H}$-Disks, Meeks-T.)

$\exists\ \mathbf{R_0} \geq \pi$ such that every $\mathbf{1}$-disk in $\mathbf{R^3}$ (with boundary) has radius $< \mathbf{R_0}$, *i.e.* $dist_{\mathbf{M}}(p, \partial\mathbf{M}) < \mathbf{R_0}$, *for any* $p \in \mathbf{M}$.

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In particular, if $\mathbf{M}$ is a complete $\mathbf{1}$-disk then
Radius Estimate $\implies$ $\mathbf{M}$ is compact

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In particular, if $\mathbf{M}$ is a complete $\mathbf{1}$-disk then
Radius Estimate $\implies$ $\mathbf{M}$ is compact $\implies$ $\mathbf{M}$ is an embedded sphere $\implies$ $\mathbf{M}$ is a round sphere.

Corollary (Meeks-T.)

A complete simply-connected $\mathbf{H}$-surface in $\mathbf{R^3}$ with $\mathbf{H} > 0$ is a round sphere.

The Radius Estimate is a non-trivial consequence of the following Intrinsic Curvature Estimate.

Theorem (Curvature Estimates for $\mathbf{H}$-Disks, Meeks-T.)

Fix $\varepsilon > 0$ and $\mathbf{H} = 1$. $\exists$ $\mathbf{C} \geq 1$ such that for every $\mathbf{1}$-disk $\mathbf{D} \subset \mathbf{R}^3$ and every $p \in \mathbf{D}$ with $\mathbf{dist}_{\mathbf{D}}(p, \partial\mathbf{D}) \geq \varepsilon$,

$$|\mathbf{A}_{\mathbf{D}}|(p) \leq \mathbf{C}.$$

What does a uniform bound on $|\mathbf{A}|$ imply?

- In general, a neighborhood of a point $p \in \mathbf{M}$ is always a graph over $T_p\mathbf{M}$. However, the size of such neighborhood depends on p.
- If $\sup_{\mathbf{M}} |\mathbf{A}| = \sup_{\mathbf{M}} |d\xi| \leq \mathbf{C}$ then the size of such neighborhood only depends on $\mathbf{C}$ and NOT on p:

$$d_{\mathbf{S}^2}(\xi(p), \xi(q)) \leq \int_{\gamma_{p,q}} |\nabla \xi| \leq length(\gamma_{p,q}) \sup_{\gamma_{p,q}} |\mathbf{A}| \leq R\mathbf{C},$$

if $q \in \mathcal{B}_R(p)$. Take $R\mathbf{C} < \frac{\pi}{10}$.

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  if $q \in \mathcal{B}_R(p)$. Take $R\mathbf{C} < \frac{\pi}{10}$.
- Let $\mathbf{u}$ be such graph then $\left(\xi = \frac{1}{\sqrt{1+|\nabla u|^2}}(-u_x, -u_y, 1) \text{ and } \frac{|\mathrm{Hess}(u)|}{(1+|\nabla u|^2)^{\frac{3}{2}}} \leq |\mathbf{A}|\right)$
  - $\|\mathbf{u}\|_{C^2} \leq 10\mathbf{C}$
  - if $\mathbf{u}$ is a CMC graph then $\operatorname{div} \frac{\nabla \mathbf{u}}{\sqrt{1+|\nabla \mathbf{u}|^2}} = \mathbf{H} \implies$ $\|\mathbf{u}\|_{C^{2,\alpha}}$ is uniformly bounded independently of p.

What does a uniform bound on $|\mathbf{A}|$ imply?

- $\sup_{\mathbf{M}_n} |\mathbf{A}| \leq C_1$ uniformly $\implies$ nearby a point $P \in \mathbf{U}$ we have a sequence of graphs $\mathbf{u}_n$ with $\|\mathbf{u}_n\|_{C^{2,\alpha}}$ uniformly bounded.
- Arzela-Ascoli $\implies$ subsequence converging C^2 to a minimal graph.
- ...

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Well-known compactness result:

Let $\mathbf{U}$ be an open set of $\mathbf{R}^3$ and let $\mathbf{M}_n$ be a sequence of minimal surfaces properly embedded in $\mathbf{U}$. If there exist constants $C_1, C_2 < \infty$ so that

$$\sup_{\mathbf{M}_n} |\mathbf{A}| \leq C_1, \quad \text{Area}(\mathbf{M}_n) < C_2$$

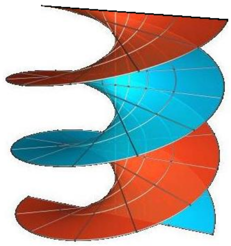
then, up to passing to a subsequence, $\mathbf{M}_n$ converges to a minimal surface $\mathbf{M}$ properly embedded in $\mathbf{U}$. An analogous result is true for $\mathbf{H}$-surfaces.

Theorem (Curvature Estimates for $\mathbf{H}$-Disks, Meeks-T.)

Fix $\varepsilon > 0$ and $\mathbf{H} = 1$. $\exists\ \mathbf{C} \geq 1$ such that for every $\mathbf{1}$-disk $\mathbf{D} \subset \mathbf{R}^3$ and every $p \in \mathbf{D}$ with $\mathbf{dist}_{\mathbf{D}}(p, \partial \mathbf{D}) \geq \varepsilon$,

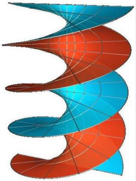
$$|\mathbf{A}_{\mathbf{D}}|(p) \leq \mathbf{C}.$$

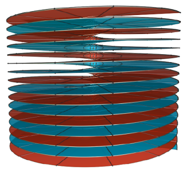
The local estimate on $|\mathbf{A}|$ fails in the minimal case; counterexamples being rescaled helicoids.



The geometry of a minimal and CMC disk around a point with large $|\mathbf{A}|$.

Minimal Case:





Colding-Minicozzi Theory

Non-zero CMC case:

There are no such points. (Meeks, T.)

Definition (Injectivity Radius)

- Given a Riemannian surface $\mathbf{M}$, the injectivity radius function $\mathbf{I}_{\mathbf{M}} \colon \mathbf{M} \to (0, \infty]$ is defined by: $\mathbf{I}_{\mathbf{M}}(\mathbf{p}) = \sup\{R > 0 \mid \exp_{\mathbf{p}} \colon B(R) \subset T_{\mathbf{p}}\mathbf{M} \to \mathbf{M}$ is a diffeomorphism.$\}$
- The injectivity radius of $\mathbf{M}$ is the infimum of $\mathbf{I}_{\mathbf{M}}$.

Theorem

Let $\mathbf{M} \looparrowright \mathbf{R^3}$ be a complete surface.

$$\sup_{\mathbf{M}} |\mathbf{A}| < \infty \implies \mathbf{I}_{\mathbf{M}} > 0.$$

Theorem (Meeks-T.)

Let $\mathbf{M} \subset \mathbf{R^3}$ be a complete, connected $\mathbf{H}$-surface with $\mathbf{H} > 0$.

$$\sup_{\mathbf{M}} |\mathbf{A}| < \infty \iff \mathbf{I}_{\mathbf{M}} > 0.$$

(Note: small geodesic balls are disks.)

Theorem (Meeks-T.)

Let $\mathbf{T^3}$ be a flat 3-torus. There exists a constant $C := C(\mathbf{T^3}, \mathbf{H}, \mathbf{g})$ such that the following holds. If $\mathbf{M}$ is a closed (compact with no boundary) $\mathbf{H}$-surface in $\mathbf{T^3}$, $\mathbf{H} > 0$, of genus $\mathbf{g}$ then

$$\mathrm{Area}(\mathbf{M}) \leq C.$$

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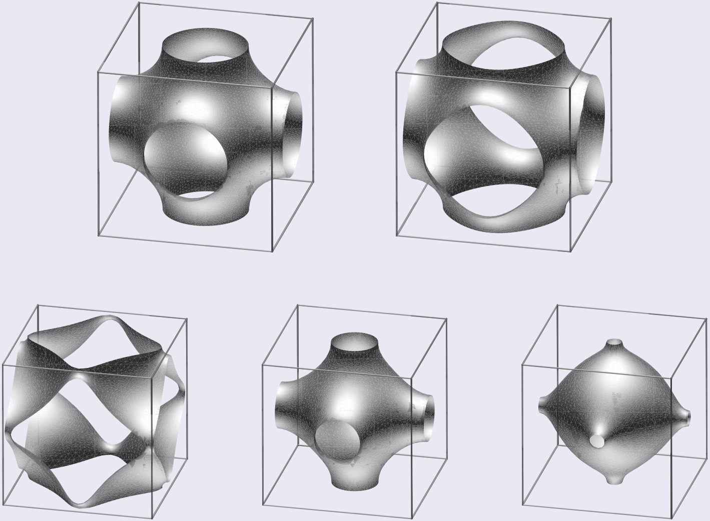
Remark

In contrast to this theorem, Traizet proved that every flat three-torus $\mathbf{T^3}$ admits for every positive integer $\mathbf{g}$ with $\mathbf{g} \neq 2$, closed minimal surfaces of genus $\mathbf{g}$ that are embedded in $\mathbf{T^3}$ with arbitrarily large area.

Examples of triply-periodic H-surfaces.

Images by Grosse-Brauckmann



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$$\mathrm{Area}(\mathbf{M}) \leq C.$$

Proof

Arguing by contradiction, let $\mathbf{M}_n$ be a sequence of compact $\mathbf{H}$-surfaces in $\mathbf{T^3}$, $\mathbf{H} > 0$, of genus $\mathbf{g}$ with

$$\mathrm{Area}(\mathbf{M}_n) > n.$$

Note that $\mathbf{M}_n$ separates $\mathbf{T^3}$ into two components, one of which is mean convex.

Claim

$\lim_{n\to\infty} \inf_{\mathbf{M}_n} \mathbf{I}_{\mathbf{M}_n} = 0$

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$$\lim_{n\to\infty} \inf_{\mathbf{M}_n} \mathbf{I}_{\mathbf{M}_n} > \delta > 0, \text{ then}$$

- by curvature estimates for **H**-disks there exists a constant **C** such that

$$\sup_n \sup_{\mathbf{M}_n} |\mathbf{A}| < \mathbf{C}.$$

Proof

- by curvature estimates for $\mathbf{H}$-disks there exists a constant $\mathbf{C}$ such that

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- by the 1-sided regular neighborhood property $\mathbf{M}_n$ has a regular neighborhood $\mathbf{N}(n, \mathbf{C})$ on its mean convex side and there exists $\varepsilon(\mathbf{C})$ such that

$$\varepsilon \mathrm{Area}(\mathbf{M}_n) < \mathrm{Volume}(\mathbf{N}(n, \mathbf{C}))$$

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$$\varepsilon \mathrm{Area}(\mathbf{M}_n) < \mathrm{Volume}(\mathbf{N}(n, \mathbf{C})) < \mathrm{Volume}(\mathbf{T^3}).$$

- Contradiction:

$$n < \mathrm{Area}(\mathbf{M}_n) < \frac{\mathrm{Volume}(\mathbf{N}(n, \mathbf{C}))}{\varepsilon} < \frac{\mathrm{Volume}(\mathbf{T^3})}{\varepsilon}.$$

Definition

We say that the $|\mathbf{A}|$ of the $\mathbf{M}_n$ blows-up at $p \in \mathbf{T^3}$ if there is a sequence of points $p_n \in \mathbf{M}_n$ such that

$$p_n \to p \quad \text{and} \quad |\mathbf{A}|(p_n) \to \infty.$$

Blow-up points or **singular points**.

Note that $\mathbf{I}_{\mathbf{M}_n}(p_n) \to 0$. (Injectivity radius goes to zero)

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By passing to a subsequence we may assume that there is a relatively closed subset $\boldsymbol{\Delta} \subset \mathbf{T^3}$ such that the $|\mathbf{A}|$ of the $\mathbf{M}_n$ blows-up at each $p \in \boldsymbol{\Delta}$.

Note that by the previous claim $\boldsymbol{\Delta} \neq \emptyset$.

Let $\mathbf{B}$ be a compact set of $\mathbf{T^3} - \mathbf{\Delta}$. Then, by the 1-sided regular neighborhood property, there exists a constant $\varepsilon(\mathbf{B}) > 0$ such that

$$\mathrm{Area}(\mathbf{M}_n \cap \mathbf{B}) < \frac{\mathrm{Volume}(\mathbf{T^3})}{\varepsilon(\mathbf{B})}.$$

Claim

$\mathbf{\Delta}$ does not consist of a finite number of points $\{Q_1, \ldots, Q_m\}$.

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Proof

- Fix $\delta > 0$ such that $4e^{-\delta \mathbf{H}} \geq 2$, $\mathbb{B}(Q_i, 2\delta)$ is an open ball in $\mathbf{T^3}$ for any $i := 1, \ldots, m$ and $\mathbb{B}(Q_i, 2\delta) \cap \mathbb{B}(Q_j, 2\delta) = \emptyset$ for $i \neq j$;
- By the 1-sided regular neighborhood property, there exists $\varepsilon(\delta)$ such that

$$\mathrm{Area}(\mathbf{M}_n \cap [\mathbf{T^3} - \bigcup_{i=1}^{m} \mathbb{B}(Q_i, \delta)]) < \frac{\mathrm{Volume}(\mathbf{T^3})}{\varepsilon}.$$

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- We need to estimate $\mathrm{Area}(\mathbf{M}_n \bigcap [\bigcup_{i=1}^{m} \mathbb{B}(Q_i, \delta)])$.

Monotonicity Formula

$$\frac{d}{dr}\left(\frac{\mathrm{Area}(\mathbf{M}\cap\mathbb{B}(P,r))}{r^2}\right) \geq -r^{-2}\int_{\mathbf{M}\cap\mathbb{B}(r)} |\mathbf{H}|\,dA \implies$$

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$$\frac{d}{dr}\left[e^{\mathbf{H}r}\left(\frac{\mathrm{Area}(\mathbf{M}_n\cap\mathbb{B}(Q_i,r))}{r^2}\right)\right] \geq 0 \implies \dots$$

$$\dots \implies \frac{\mathrm{Area}(\mathbf{M}_n\cap[\mathbb{B}(Q_i,2\delta)-\mathbb{B}(Q_i,\delta)])}{4} \geq$$

$$\left(e^{-\delta\mathbf{H}}-\frac{1}{4}\right)\mathrm{Area}(\mathbf{M}_n\cap\mathbb{B}(Q_i,\delta))$$

Recall

$4e^{-\delta \mathbf{H}} \geq 2$

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$$\left(e^{-\delta \mathbf{H}} - \frac{1}{4}\right) \mathrm{Area}(\mathbf{M}_n \cap \mathbb{B}(Q_i, \delta)) > \frac{1}{4} \mathrm{Area}(\mathbf{M}_n \cap \mathbb{B}(Q_i, \delta))$$

Recall

$$\text{Area}(\mathbf{M}_n \cap [\mathbf{T^3} - \bigcup_{i=1}^{m} \mathbb{B}(Q_i, \delta)]) < \frac{\text{Volume}(\mathbf{T^3})}{\varepsilon}.$$

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$$< \frac{\mathrm{Volume}(\mathbf{T}^3)}{\varepsilon} \implies$$

Contradiction!

$$n \leq \mathrm{Area}(\mathbf{M}_n) < (m+1)\frac{\mathrm{Volume}(\mathbf{T}^3)}{\varepsilon}.$$

Claim QED

$\mathbf{\Delta}$ does not consist of a finite number of points $\{Q_1, \ldots, Q_m\}$.

Claim

$\mathbf{\Delta}$ does consist of a finite number of points.

What does $\mathbf{M}_n$ look like nearby a point P in $\mathbf{\Delta}$?

Let $P \in \mathbf{\Delta}$. Then there exists a sequence of points P_n such that $\lim_{n\to\infty} \mathbf{I}_{\mathbf{M}_n}(P_n) = 0$. Let

$$\mathbf{\Sigma}_n := \frac{1}{\mathbf{I}_{\mathbf{M}_n}(P_n)}[\mathbf{M}_n - P_n].$$

(Not exactly)

Let $\mathbf{\Sigma}_n := \frac{1}{\mathbf{I}_{\mathbf{M}_n}(P_n)}[\mathbf{M}_n - P_n]$. Note that $\mathbf{I}_{\mathbf{\Sigma}_n}(\vec{0}) = 1$ and $\mathbf{H}_n \to 0$.

Claim

$\mathbf{\Sigma}_n$ converges to a catenoid or to a complete properly embedded minimal surface of positive genus.

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Claim

Σ_n converges to a catenoid or to a complete properly embedded minimal surface of positive genus.

Case 1: Σ_n has uniformly bounded $|\mathbf{A}|$ in $\mathbf{R}^3$. (No blow-up points.)

Case 2: Σ_n does NOT have uniformly bounded $|\mathbf{A}|$ in $\mathbf{R}^3$.

Case 1: $\boldsymbol{\Sigma}_n$ has uniformly bounded $|\mathbf{A}|$ in $\mathbf{R}^3$.

- "Well-known Compactness Theorem";
- Classification of complete properly embedded minimal surfaces of finite genus;

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- "Well-known Compactness Theorem";
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$$\Downarrow$$

$\mathbf{\Sigma}_n$ converges to a complete properly embedded minimal surfaces of finite genus $\mathbf{\Sigma}$:

- **a flat plane or a helicoid** (**Colding-Minicozzi**, **Meeks-Rosenberg**);
- **a catenoid** (**Lopez-Ros**);
- **a Riemann minimal example** (**Meeks-Perez-Ros**);
- a surface with positive genus. (Not too many of these!)

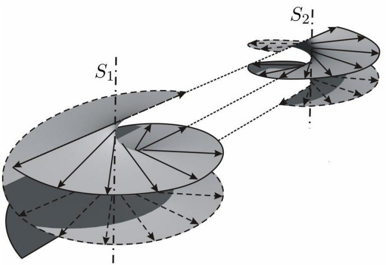
- ~~**a flat plane or a helicoid** (**Colding-Minicozzi**, **Meeks-Rosenberg**);~~
- **a catenoid** (**Lopez-Ros**);
- **a Riemann minimal example** (**Meeks-Perez-Ros**);
- a surface with positive genus.

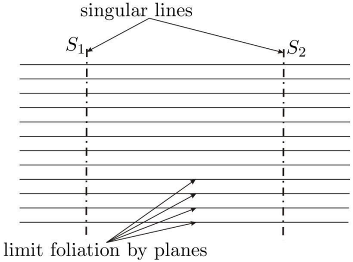
Remark

The occurrence of a plane or a helicoid can be immediately ruled out because $\mathbf{I}_{\boldsymbol{\Sigma}}(\vec{0}) = 1$.

Case 2: $\mathbf{\Sigma}_n$ does NOT have uniformly bounded $|\mathbf{A}|$ in $\mathbf{R}^3$.

$\mathbf{\Sigma}_n$ "converges" to parking garage structure.





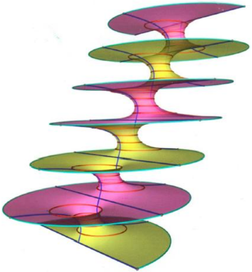
Claim

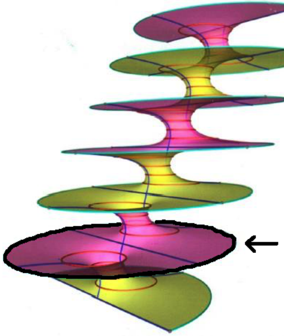
$\boldsymbol{\Sigma}_n$ converges to a catenoid or to a complete properly embedded minimal surface of positive genus.

So far we have:

- a catenoid or a surface with positive genus;
- a Riemann minimal example;
- a parking garage structure.

We need to rule out the occurrence of a Riemann minimal example and of a parking garage structure.



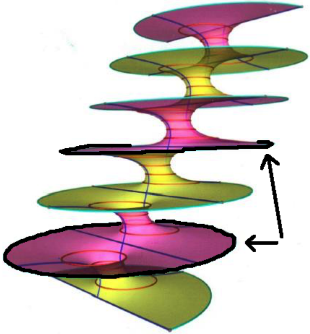


Claim

This curve is not homotopically trivial in the local picture and it is not homotopically trivial in $\mathbf{M}_n$.

Proof

Recall that $\mathbf{M}_n$ separates $\mathbf{T}^3$ into two components, one of which is mean convex. Contradiction: solving Plateau Problem contradicts Convex Hull Property for minimal surfaces.



Claim

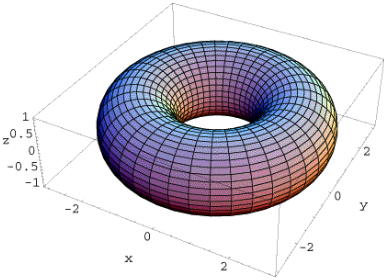
These curves do not bound an annulus in the local picture and they do not bound an annulus in $\mathbf{M}_n$.

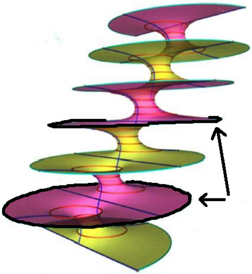
Proof

Recall that $\mathbf{M}_n$ separates $\mathbf{T^3}$ into two components, one of which is mean convex. Contradiction: solving Plateau Problem contradicts Convex Hull Property for minimal surfaces.

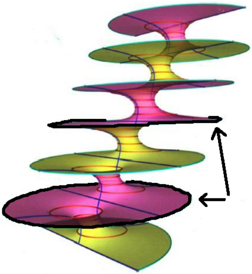
Claim

Let Σ be a compact Riemann surface of genus g and let Γ be a collection of simple closed curves in Σ that are not homotopically trivial and are pair-wise disjoint. If the number of curves in Γ is greater than $3g - 2$ then there exists at least a pair of distinct curves that bounds an annulus in Σ.

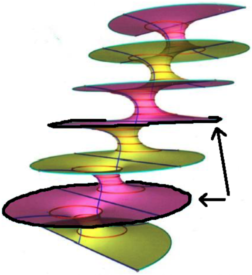




- The genus of $\mathbf{M}_n$ is bounded uniformly.



- The genus of $\mathbf{M}_n$ is bounded uniformly.
- The numbers of curves in $\mathbf{M}_n$ that are not homotopically trivial is becoming arbitrarily large.



- The genus of $\mathbf{M}_n$ is bounded uniformly.
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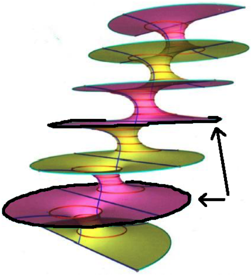

- The genus of $\mathbf{M}_n$ is bounded uniformly.
- The numbers of curves in $\mathbf{M}_n$ that are not homotopically trivial is becoming arbitrarily large.
- Contradiction: by the claim there must exist a pair that bounds an annulus in $\mathbf{M}_n$.

This rules out the appearance of a Riemann minimal example and the exact same argument works to rule out a minimal parking garage structure.

Claim QED

Σ_n converges to a catenoid or to a properly embedded minimal surface of positive genus.

Recall that we are trying to prove that the number of singular points, that is points in Δ, is finite.

Since the genus is additive, the number of singular points where a catenoid does NOT appear is bounded by the genus

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Recall that we are trying to prove that the number of singular points, that is points in $\boldsymbol{\Delta}$, is finite.

Since the genus is additive, the number of singular points where a catenoid does NOT appear is bounded by the genus and thus it suffices to **bound the number of catenoids appearing**.

Given $p \in \mathbf{\Delta}$ let $\mathbf{\Gamma}_n(p)$ be a sequence of closed loops converging to the closed geodesic of the catenoid forming at p.

$\mathbf{\Gamma}_n(p)$ is NOT homotopically trivial in $\mathbf{M}_n$ because a catenoid has non-zero flux and flux is a homological invariant.

Suppose $p, q \in \mathbf{\Delta}$ and $\mathbf{\Gamma}_n(p) \cup \mathbf{\Gamma}_n(q)$ bounds an annulus $\mathbf{A}_n(p, q)$ (and eventually they will because the genus is bounded).

Claim

The number m of such annuli is bounded.

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- $\mathbf{A}_n(p,q)$ lifts to $\mathbf{R^3}$ and it is large;
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- Therefore the number of such annuli is bounded.

$$\varepsilon m \leq \sum_{i=1}^{m} \mathrm{Volume}(\mathbf{V}_n(p_i, q_i)) \leq \mathrm{Volume}(\mathbf{T^3}).$$

Claim

The number of singular points, that is points in $\mathbf{\Delta}$, is finite.

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- If $\mathbf{\Delta}$ is infinite then there is an arbitrarily large number of curves $\mathbf{\Gamma}_n(p)$ that are not homotopically trivial;
- Since the genus of $\mathbf{M}_n$ is uniformly bounded, this creates an arbitrarily large number of curves bounding annuli;
- This contradicts the fact that the number of such annuli is bounded.

Theorem (Meeks-T.)

Let $\mathbf{T^3}$ be a flat 3-torus. There exists a constant $C := C(\mathbf{T^3}, \mathbf{H}, \mathbf{g})$ such that the following holds. If $\mathbf{M}$ is a closed (compact with no boundary) $\mathbf{H}$-surface in $\mathbf{T^3}$, $\mathbf{H} > 0$, of genus $\mathbf{g}$ then

$$\mathrm{Area}(\mathbf{M}) \leq C.$$

Corollary

Let $\mathbf{T^3}$ be a flat 3-torus. If $\mathbf{M}_n$ is a sequence of compact $\mathbf{H}$-surfaces in $\mathbf{T^3}$, $\mathbf{H} > 0$, of genus at most $\mathbf{g}$ then, up to a subsequence, it converges to a non-empty, possibly disconnected, strongly Alexandrov embedded surface $\mathbf{M}_\infty$ of constant mean curvature $\mathbf{H}$ and genus at most $\mathbf{g}$. The convergence is smooth away from a finite set of points, $\mathbf{\Delta}$.
The set $\mathbf{\Delta}$ is a subset of the set of points where $\mathbf{M}_\infty$ is not embedded.

A closed immersed surface $f : \mathbf{\Sigma} \to \mathbf{N}$ of positive mean curvature in $\mathbf{N}$ is called *strongly Alexandrov embedded* if f extends on the mean convex side of its image to an immersion of a compact three-manifold $\mathbf{W}$ with $\mathbf{\Sigma} = \partial \mathbf{W}$, where the extended immersion is injective on the interior of $\mathbf{W}$.

Corollary

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The set $\boldsymbol{\Delta}$ is a subset of the set of points where $\mathbf{M}_\infty$ is not embedded.

- In the previous arguments, to bound the number of singular points we did not use that the area was going to infinity;
- Area estimate and bounded genus imply that the singularities are removable;
- Density arguments show that $\boldsymbol{\Delta}$ is a subset of the set of points where $\mathbf{M}_\infty$ is not embedded;
- Density arguments show that the singular points can only be catenoid singular points.

A few more details.

What does $\mathbf{M}_n$ look like nearby a point P in $\mathbf{\Delta}$?

Let $P \in \mathbf{\Delta}$. Then there exists a sequence of points P_n such that $\lim_{n\to\infty} \mathbf{I}_{\mathbf{M}_n}(P_n) = 0$.
In fact, the sequence of points P_n converging to P can be taken to be a sequence of points of **almost minimal injectivity radius**.

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In fact, the sequence of points P_n converging to P can be taken to be a sequence of points of **almost minimal injectivity radius**.
Namely, there exists a sequence of numbers $\delta_n > 0$ such that

1. $\lim_{n\to\infty} \mathbf{I}_{\mathbf{M}_n}(P_n) = 0$ and $\lim_{n\to\infty} \delta_n = 0$;
2. $\lim_{n\to\infty} \frac{\delta_n}{\mathbf{I}_{\mathbf{M}_n}(P_n)} = \infty$;
3. Genus of $\mathcal{B}_{\delta_n}(P_n)$ is at most $\mathbf{g}$;
4. $\sup_{\mathcal{B}_{\delta_n}(P_n)} \mathbf{I}_{\mathbf{M}_n}(Q) \geq 4\mathbf{I}_{\mathbf{M}_n}(P_n)$.

Let $\mathbf{\Sigma}_n := \frac{1}{\mathbf{I}_{\mathbf{M}_n}(P_n)}[\mathcal{B}_{\delta_n}(P_n) - P_n]$.

Claim

$\mathbf{\Sigma}_n$ converges to a catenoid or to a properly embedded minimal surface of finite topology, positive genus and with more than three ends.

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1. $\lim_{n\to\infty} \mathbf{I}_{\mathbf{M}_n}(P_n) = 0$ and $\lim_{n\to\infty} \delta_n = 0$;
2. $\lim_{n\to\infty} \frac{\delta_n}{\mathbf{I}_{\mathbf{M}_n}(P_n)} = \infty$;
3. Genus of $\mathcal{B}_{\delta_n}(P_n)$ is zero;
4. $\sup_{\mathcal{B}_{\delta_n}(P_n)} \mathbf{I}_{\mathbf{M}_n}(Q) \geq 4\mathbf{I}_{\mathbf{M}_n}(P_n)$.

What are the properties of $\mathbf{\Sigma}_n := \frac{1}{\mathbf{I}_{\mathbf{M}_n}(P_n)}[\mathcal{B}_{\delta_n}(P_n) - P_n]$?

1. Since $\lim_{n\to\infty} \frac{\delta_n}{\mathbf{I}_{\mathbf{M}_n}(P_n)} = \infty$, these surfaces are becoming arbitrarily large and the ambient space is becoming $\mathbf{R}^3$;
2. $\mathbf{I}_{\mathbf{\Sigma}_n}(0) = 1$ and the injectivity radius of $\mathbf{\Sigma}_n$ is bounded from below by $\frac{1}{2}$;
3. $\mathbf{\Sigma}_n$ has genus at most $\mathbf{g}$;
4. $\lim_{n\to\infty} \mathbf{H}_n = \lim_{n\to\infty} \mathbf{I}_{\mathbf{M}_n}(P_n)\mathbf{H} = 0$.