Compactness issues for Willmore surfaces

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1. Willmore functional, Willmore surfaces
2. The conservation Laws and first compactness result
3. Control of the conformal factor
4. Quantization of the energy
5. Compactness of Willmore surfaces with bounded conformal class

contents

How to measure the bending of a compact surface ?

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and that equality characterizes the round sphere. Then we call critical point of W, Willmore surfaces.

Theorem (Blaschke 1920)

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where $\mathring{A}$ is the traceless part of the second fundamental form A.
$|\mathring{A}_\phi|^2 \, dv$ is conformally invariant.

Theorem (Li-Yau 82)

Let Σ be a closed surface of $\mathbb{R}^3$, $p \in \Sigma$ with density $\theta(p)$ then

$$\int_\Sigma |H|^2 d\sigma \geq \theta(p) 4\pi$$

with equality if Σ is conformally minimal.

- When $\Sigma = S^2$, there is a complete classification of Bryant(1986) in $\mathbb{R}^3$ which insures notably that the level of energy are $k4\pi$, except for $k = 2, 3, 5, 7$ and 9 (Michelat 19).

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- For higher genus we know that the minimum is achieved (Bauer-Kuwert 03, Rivière 10).
- Works using integrable system : Dorfmeister & al

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- First problem Δ_g depends on the immersion Φ, since $g = \Phi^*(\xi)$. We can solve this problem by choosing a conformal parametrization with respect to a "reference metric" h. If $e^{2\lambda} h = \Phi^*(\xi)$ then $\Delta_g = e^{-2\lambda}\Delta_h$.

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- The cubic term is not controlled *a priori*.

Uniformization of surfaces : We can always choose our parametrization Φ conformal with respect to a metric with constant curvature, i.e. $\Phi^*(\xi) = e^{2\lambda} h$ such that the Gauss curvature of (Σ, h) equal 1,0 or -1.

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Whats happen when the conformal degenerate ?

$K = 1$: nothing.

$K = 0$: long and thin tori.

$K = -1$, Deligne-Mumford's description : The surface splits in **thick parts** : compact surface with small disc remove joined by **thin parts**, which are conformally equivalent to a disc with a small disc remove.

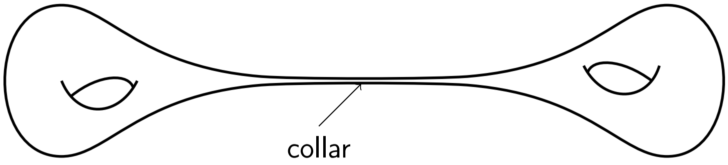

Figure – Genus 2 surface degenerating with formation of a *collar*.

Let Σ a smooth compact surface equipped with a reference smooth metric g_0. One defines the Sobolev spaces $W^{k,p}(\Sigma,\mathbb{R}^3)$ of measurable maps from Σ into $\mathbb{R}^3$ into the following way

$$W^{k,p}(\Sigma,\mathbb{R}^3)=\left\{\vec{f}:\Sigma\to\mathbb{R}^3\text{measurable}\text{s.t.}\sum_{l=0}^{k}\int_\Sigma|\nabla^l\vec{f}|_{g_0}^p\,dv_{g_0}<+\infty\right\}.$$

Let $\Phi \in W^{1,\infty}(\Sigma, \mathbb{R}^3)$, if we assume that

$$(1) \qquad C_\Phi^{-1}\, g_0(X,X) \leq \langle d\Phi(X), d\Phi(X)\rangle \leq C_\Phi\, g_0(X,X).$$

Then we can define the Gauss map as being the following measurable map in $L^\infty(\Sigma)$,

$$\vec{n}_\Phi = \frac{\Phi_x \wedge \Phi_y}{|\Phi_x \wedge \Phi_y|}.$$

We then introduce the space $\mathcal{E}_\Sigma$ of **weak immersions** of Σ with bounded second fundamental form as follow :

$$\mathcal{E}_\Sigma = \left\{ \begin{array}{c} \Phi \in W^{1,\infty}(\Sigma) \text{ which satisfies (1) for some } C_\Phi > 0 \\ \\ \text{and } \int_\Sigma |d\vec{n}_\Phi|^2_{g_\Phi} \, dvol_\Phi < +\infty \end{array} \right\},$$

where $g_\Phi = \Phi^*\xi$.

contents

Let Ω an open subset of $\mathbb{R}^s$ and M a submanifold of $\mathbb{R}^m$. Suppose that

$$L : \{(x,p,q)\,|(x,p) \in \Omega \times M,\ q \in T_pM \otimes T_x^*\Omega\} \to \mathbb{R}$$

is a continuously differentiable function. Then we can define the following action on $C^1(\Omega, M)$

$$\mathcal{L}(u) = \int_\Omega L(x, u(x), du(x))\, dx$$

Theorem (Noether 1915)

Let X be a Lipschitz tangent vector field on M which is an infinitesimal symmetry for the action $\mathcal{L}$. If u is a critical point of $\mathcal{L}$, then

$$\sum_{\alpha=1}^{s} \frac{\partial}{\partial x^\alpha}\left(X^j(u)\frac{\partial L}{\partial q_\alpha^j}(x,u,du)\right) = 0.$$

Theorem (Rivière 08, Bernard 12)

Let $\Phi : \Sigma \to \mathbb{R}^3$ be a smooth immersion of an oriented surface Σ. One introduces the quantities

$$\vec{\mathcal{W}} = \Delta_\perp \vec{H} + 2(H^2 - K)\vec{H}$$

and

$$\vec{T} = 2\nabla\vec{H} - 3H\nabla\vec{n} + \vec{H} \wedge \nabla^\perp\vec{n}$$

Then translation, dilation and rotation invariance implies the three conservation laws :

$$\begin{cases} div(\vec{T}) = -\vec{\mathcal{W}} \\ div(\vec{T}.\Phi) = -\vec{\mathcal{W}}.\Phi \\ div(\vec{T} \wedge \Phi + 2\vec{H} \wedge \nabla\Phi) = -\vec{\mathcal{W}} \wedge \Phi \end{cases}$$

Let $\Phi : \mathbb{D} \to \mathbb{R}^3$ be a conformal weak immersion, then its image is Willmore if and only if

$$\operatorname{div}(2\nabla \vec{H} - 3H\nabla\vec{n} + \vec{H} \wedge \nabla^{\perp}\vec{n}) = 0$$

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moreover there exists S and $\vec{R}$ such that

$$\begin{cases} \nabla^{\perp} S = \vec{T}.\Phi \\ \nabla^{\perp} \vec{R} = \vec{T} \wedge \Phi + \vec{H} \wedge \nabla \Phi \end{cases}$$

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Which implies that S and $\vec{R}$ satisfy the following system :

$$\begin{cases} \Delta S = -\nabla\vec{n}.\nabla^{\perp}\vec{R} \\ \Delta\vec{R} = \nabla\vec{n}.\nabla^{\perp}\vec{R} + \nabla^{\perp}S.\nabla\vec{n} \\ \Delta\Phi = \frac{1}{2}(\nabla^{\perp}S.\nabla\Phi + \nabla^{\perp}\vec{R} \wedge \nabla\Phi) \end{cases}$$

Lemma (Compactness by compensation, Wente, 1969)

Let $a, b \in W^{1,2}(\mathbb{D})$ and $\phi \in W_0^{1,1}(\mathbb{D})$ is a solution of

$$\Delta\phi = a_x b_y - a_y b_x = \nabla a \nabla^{\perp} b,$$

then

$$\|\phi\|_\infty + \|\nabla\phi\|_2 \leq C\|\nabla a\|_2\|\nabla b\|_2,$$

where C is a universal constant.

Definition

Let D be a domain of $\mathbb{R}^k$, $p \in (1, +\infty)$ and $q \in [1, +\infty]$. The Lorentz space $L^{p,q}(D)$ is the set of measurable functions $f : D \to \mathbb{R}$ such that

$$|f|_{p,q} = \left(\int_0^{+\infty} \left(t^{\frac{1}{p}} f^*(t) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} < +\infty \text{ if } q < +\infty$$

or

$$|f|_{p,\infty} = \sup \left(t^{\frac{1}{p}} f^*(t) \right) \text{ if } q = +\infty$$

where f^* the decreasing rearrangement of f.

In short : Lorentz space are sophistication of L^p space such that $L^{2,1} \subset L^2 \subset L^{2,\infty}$ and $L^{2,1}$ and $L^{2,\infty}$ are duals and $1/r \in L^{2,\infty}$ in dimension 2. Moreover $\Delta u \in L^1 \implies \nabla u \in L^{2,\infty}$.

Lemma (Improved Wente)

Let $a, b \in W^{1,2}(\mathbb{D})$ *and* $\phi \in W_0^{1,1}(\mathbb{D})$ *is a solution of*

$$\Delta\phi = a_x b_y - a_y b_x,$$

then

$$\|\nabla\phi\|_{2,1} \leq C\|\nabla a\|_2 \|\nabla b\|_2.$$

Theorem (ε-regularity, Rivière 08)

There exists a constant $\varepsilon_0 > 0$ and for any $A > 0$ a sequence of positive numbers $C_l(A) > 0$ for $l \in \mathbb{N}^*$ such that for any weak immersion Φ be a weak conformal Willmore immersion from $\mathbb{D}$ into $\mathbb{R}^3$ satisfying

$$\int_{\mathbb{D}} |\nabla \vec{n}|^2 dz \leq \varepsilon_0$$

and

$$\|d\lambda\|_{L^{2,\infty}(\mathbb{D})} \leq A$$

then, for every $l \in \mathbb{N}^*$,

$$(2) \qquad \|e^{-\lambda}\nabla^l \Phi\|_{L^\infty(D_{\frac{1}{2}})} \leq C_l(A) \left(\int_D |\nabla \vec{n}_\Phi|^2 \, dz + 1 \right)^{\frac{1}{2}},$$

where $e^{2\lambda} = \frac{|\nabla \Phi|^2}{2}$.

Theorem (weak compactness, Rivière 08)

Let Φ_k be a sequence of weak conformal Willmore immersions from (Σ, h) into $\mathbb{R}^3$ with L^2-bounded second fundamental form satisfying

$$\|d\lambda_k\|_{L^{2,\infty}(\Sigma)} \leq A$$

then there exists a Mobius transformation of $\mathbb{R}^3$ Θ_k ,$\{a_1, \ldots, a_p\}$ and Φ_∞ a weak willmore (possibly branched) immersion such that

$$\Theta_k \circ \Phi_k \to \Phi_\infty \text{ in } C^2_{loc}(\Sigma \setminus \{a_1, \ldots, a_p\}).$$

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Natural question : Is all the energy loose is due to the concentration of bubbles ?

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$$\lim_{n \to +\infty} W(\Phi_n) = W(\Phi_\infty) + \sum_{i=1}^{p} W(\omega_i) \,?$$

contents

Lemma (Hélein, Rivière)

Let (Σ, h) a closed surface with constant curvature and $\Phi : (\Sigma, h) \to \mathbb{R}^3$ a conformal weak immersion such that

$$\int_\Sigma |d\vec{n}|^2 \, dz \leq A$$

then

$$\|\nabla\lambda\|_{2,\infty} \leq C$$

where λ is such that $e^{2\lambda} = \frac{|\nabla\Phi|^2}{2}$ and C depends only on A and h.

Proof :
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But

$$\| e^{2\lambda} K_\Phi - K_h \|_1 \leq \int_\Sigma (|\nabla_h \vec{n}|_h^2 + |K_h|) dv_h \leq C$$

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Moreover using interpolation theory we have

$$\|\nabla \lambda\|_{2,\infty} \leq \|\nabla G_h\|_{2,\infty} \|e^{2\lambda} K_\Phi - K_h\|_1$$

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Finally remarking that the singularity of ∇G_h is like $1/r$ we conclude.

□

contents

Theorem (Bernard-Rivière 12)

Let $\Phi_k : (\Sigma, h_k) \to \mathbb{R}^3$ a sequence of weak Willmore immersion with bounded energy and **bounded conformal class** *then there exist a $\Phi_\infty : (\Sigma, h) \to \mathbb{R}^3$ a weak Willmore branched immersion and $\omega_i : S^2 \to \mathbb{R}^3$ some non trivial Willmore (possibly branched) immersion such that*

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- Michelat-Rivière 17, the bubbles are conformally minimal, **the energy is quantized by 4π.**

When the conformal class degenerate, we need to control the conformal factor ! That is to say to control the Green function of the Laplacian independently of the conformal class.

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Theorem (Laurain-Rivière 13)

Let (Σ, c_n) a sequence of Riemann surfaces, then in a "good atlas" $\|\nabla G_n\|_{L^{2,\infty}}$ is uniformly bounded independently of the conformal class.

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Main-tool : Uniform estimate on harmonic function on degenerating annuli.

Theorem (L-Rivière 16)

Let $\Phi_n : (\Sigma, h_n) \to \mathbb{R}^3$ a sequence of weak conformal Willmore immersion with bounded energy and such that on any sequences of shrinking geodesic γ_n of (Σ, h_n),

$$\lim_n \frac{1}{|\gamma_n|^{\frac{1}{2}}} \int_{\gamma_n} (2\partial_\nu \vec{H}_n - 3H_n\partial_\nu \vec{n}_n - \vec{H}_n \wedge \partial_\tau \vec{n}_n) \wedge \Phi_n + 2H_n \partial_\tau \Phi_n \, d\sigma = 0$$

then $W(\Phi_n)$ satisfies an energy identity, i.e. there exists a $\Phi_\infty : (\Sigma, h_\infty) \to \mathbb{R}^3$ a weak Willmore branched immersion and $\omega_i : S^2 \to \mathbb{R}^3$ some non trivial Willmore (possibly branched) immersion such that

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Where come from this residue ?

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Theorem (Zhu(10))

Let $u_n : (\Sigma_n, h_n, c_n) \to N$ be a sequence of harmonic maps with uniformly bounded energy $E(u_n, \Sigma_n) \leq \Lambda < \infty$, where (Σ_n, h_n, c_n) is a sequence of closed hyperbolic Riemann surfaces of genus $g > 1$, if we set

$$\alpha_i^n = \int_{\gamma_n^i} \Phi(u_n)\, d\sigma$$

where γ_n^i is a set of pinching geodesic of length l_n^i and $\Phi(u_n)$ the Hopf differential. Then

$$\lim_n E(u_n) = E(u_\infty) + \sum_{j=1}^{p} E(\omega^j) + \sum_{i=1}^{g} \lim_n \frac{\pi^2 \Re(\alpha_n^i)}{l_n^i}$$

Integration of the conservation laws in the collar region. We are on $\mathbb{D} \setminus B(0,\varepsilon)$ and

$$\vec{T} = 2\nabla\vec{H} - 3H\nabla\vec{n} + \vec{H} \wedge \nabla^{\perp}\vec{n}$$

$$\begin{cases} div(\vec{T}) = 0 \\ div(\vec{T}.\Phi) = 0 \\ div(\vec{T} \wedge \Phi + \vec{H} \wedge \nabla\Phi) = 0 \end{cases}$$

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Then there exists $\vec{L}, S$ and $\vec{R}$ such that

$$\begin{cases} \vec{T} = \nabla^{\perp} \vec{L} + \vec{c}_{tra} \nabla \log(\rho) \\ \vec{T}.\Phi = \nabla^{\perp} S + c_{dil} \nabla \log(\rho) \\ \vec{T} \wedge \Phi + \vec{H} \wedge \nabla \Phi = \nabla^{\perp} \vec{R} + \vec{c}_{rot} \nabla \log(\rho) \end{cases}$$

Where

$$\begin{cases} \vec{c}_{tra} = \int_{\partial\mathbb{D}} \vec{T}.\nu \; d\sigma \\ c_{dil} = \int_{\partial\mathbb{D}} \langle \vec{T}, \Phi \rangle.\nu \; d\sigma \\ \vec{c}_{rot} = \int_{\partial\mathbb{D}} (\vec{T} \wedge \partial_\tau \vec{n}) \wedge \Phi + 2H\partial_\tau \Phi \; d\sigma \\ \vec{c}_{inv} = \int_{\partial\mathbb{D}} |\Phi|^2 \vec{T}.\nu + 2\langle \Phi, \vec{T} \rangle \Phi + 4\Phi \times \mathring{A} \nabla^\perp \Phi \, d\sigma \end{cases}$$

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- $\vec{c}_{rot}$ vanishes on an Hopf torus if and only if it is a minimal torus, i.e. the Clifford torus. L-Rivière 16.

contents

When concentration occurs, branched points can appear so it is very important to have a precise description

Theorem (Bernard-Rivière 11)

Let $\Phi : \mathbb{D} \setminus \{0\}$ be a Willmore immersion with finite total curvature. Then there exists $\theta_0 \in \mathbb{N}^*$ and $\vec{A} \neq 0$ such that

$$\Phi(z) = \Re(\vec{A} z^{\theta_0}) + O(|z|^{\theta_0+1} \log |z|)$$

Moreover if $\theta_0 = 1$, then

$$\vec{H} = \vec{c}_{tra} \log |z| + O(|z| \log |z|)$$

else if $\theta_0 \geq 2$, there exists $m \leq \theta_0 - 1$ and $\vec{C} \neq 0$ such that

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See also (Kuwert-Schätzle 2005).

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$\vec{\Phi}_k : \Sigma \to \mathbb{R}^3$ a sequence of conformal Willmore immersions such that $[\Phi_k^*(\xi)]$, the conformal class of the pullback metric, remains in a compact set of the moduli space and

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- In fact assuming that the limit is not branched we can achieved $\beta_g^m + 12\pi$.

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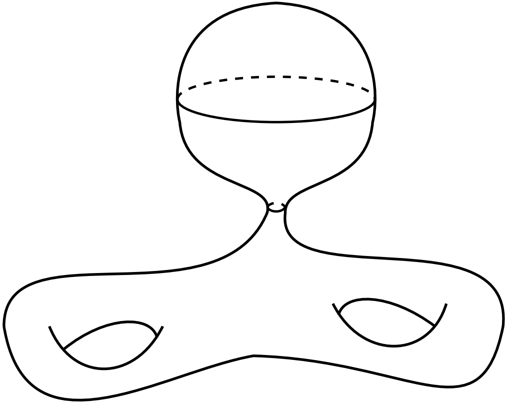
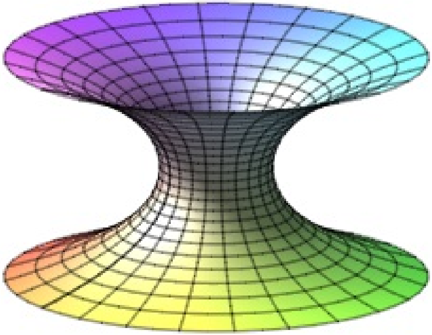
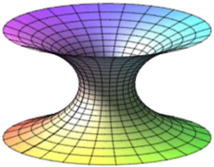
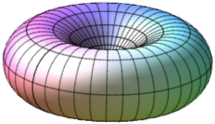
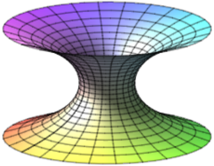


Figure – One compact bubble









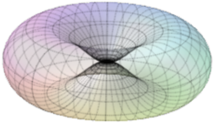


Figure – $\vec{c}_{tra} \neq 0$

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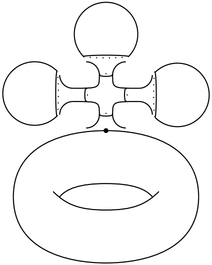

Concentration of a minimal surface

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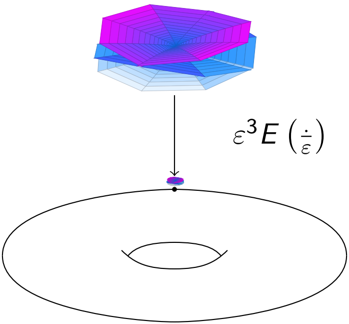

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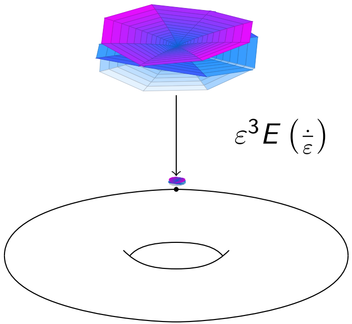

Is this possible to glue an Enneper on Chen-Gackstatter surface ?

Theorem (Marque, 2019)

Let $\Phi \in \mathcal{E}(\mathbb{D})$ be a weak Willmore immersion. Then there exists ε_0 such that if

$$\|H\nabla\Phi\|_{L^2(\mathbb{D})} \leq \varepsilon_0$$

then for any $r < 1$

$$\|H\nabla\Phi\|_{L^\infty(\mathbb{D}_r)} \leq C\|H\nabla\Phi\|_{L^2(\mathbb{D})},$$

and

$$\|\nabla\Phi\|_{W^{3,p}(\mathbb{D}_r)} \leq C\|\nabla\Phi\|_{L^2(\mathbb{D})}$$

for all $p < \infty$.

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- (3) + sharper Wente $\implies \nabla S, \nabla \vec{R} \in L^{2,1}$ "strong L^2".
- duality of $L^{2,1}$ and $L^{2,\infty}$.

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The only minimal torus with 12π total curvature is the Chen-Gackstatter torus.

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Sequences of Willmore tori with energy $W \leq 12\pi$ are pre-compact.

Theorem (Bryant 1984, Marque 2019)

Willmore immersions with an energy greater than 16π are not compact.

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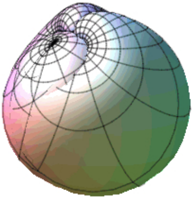

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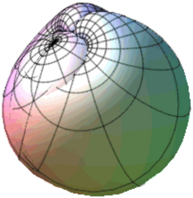

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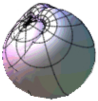

Thank you for your attention !