IMPROVED MOSER-TRUDINGER-ONOFRI INEQUALITY UNDER CONSTRAINTS

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Abstract. A classical result of Aubin states that the constant in Moser-Trudinger-Onofri inequality on $\mathbb{S}^2$ can be imporved for furnctions with zero first order moments of the area element. We generalize it to higher order moments case. These new inequalities bear similarity to a sequence of Lebedev-Milin type inequalities on $\mathbb{S}^1$ coming from the work of Grenander-Szego on Toeplitz determinants (as pointed out by Widom). We also discuss the related sharp inequality by the method of perturbation.

1. Introduction

Let (M,g) be a smooth compact Riemann surface without boundary. For an integrable function u on M, we denote

$$\overline{u}=\frac{1}{\mu(M)}\int_M ud\mu. \tag{1.1}$$

Here μ is the measure associated with the Riemannian metric g.

The classical Moser-Trudinger inequality (see [ChY2, F, M]) tells us that for every $u\in H^1(M)\setminus\{0\}$ with $\overline{u}=0$, we have

$$\int_M e^{4\pi\frac{u^2}{\|\nabla u\|^2_{L^2(M)}}}d\mu\leq c(M,g). \tag{1.2}$$

Here $c(M,g)$ is a positive constant independent of u.

A direct consequence of (1.2) is the following Moser-Trudinger-Onofri inequality: for every $u\in H^1(M)$ with $\overline{u}=0$, we have

$$\log\int_M e^{2u}d\mu\leq\frac{1}{4\pi}\|\nabla u\|^2_{L^2(M)}+c_1(M,g). \tag{1.3}$$

(1.3) seems to attract more interest than (1.2) in view of its close relation to Gauss curvature equation and spectral geometry through the classical Polyakov formula (see for example [On, OsPS]).

On the standard sphere, it is found in [A, corollary 2 on p159] that for $u\in H^1(\mathbb{S}^2)$ with $\overline{u}=0$ and $\int_{\mathbb{S}^2}x_ie^{2u(x)}d\mu(x)=0$ for $i=1,2,3$, the constant $\frac{1}{4\pi}$ in (1.3) can be lowered i.e. for any $\varepsilon>0$, we have

$$\log\left(\frac{1}{4\pi}\int_{\mathbb{S}^2}e^{2u}d\mu\right)\leq\left(\frac{1}{8\pi}+\varepsilon\right)\|\nabla u\|^2_{L^2}+c_\varepsilon. \tag{1.4}$$

Here c_ε is a constant depending on ε only.

A closely related question is to find the best constant in (1.3) and (1.4). In [On], the best constant $c_1(M,g)$ for (1.3) is found on the standard $\mathbb{S}^2$. More precisely it is showed that for $u \in H^1(\mathbb{S}^2)$ with $\overline{u} = 0$, we have

$$\log\left(\frac{1}{4\pi}\int_{\mathbb{S}^2} e^{2u} d\mu\right) \leq \frac{1}{4\pi}\|\nabla u\|_{L^2}^2. \tag{1.5}$$

For (1.4), it is proved recently in [GuM] that the best constant c_ε is 0. In another word, for $u \in H^1(\mathbb{S}^2)$ with $\overline{u} = 0$ and $\int_{\mathbb{S}^2} x_i e^{2u(x)} d\mu(x) = 0$ for $i = 1, 2, 3$, we have

$$\log\left(\frac{1}{4\pi}\int_{\mathbb{S}^2} e^{2u} d\mu\right) \leq \frac{1}{8\pi}\|\nabla u\|_{L^2}^2. \tag{1.6}$$

This confirms a conjecture in [ChY1].

To motivate our discussion, let us look at some research on $\mathbb{S}^1$ which has similar spirit as above. For convenience we let D be the unit disk in $\mathbb{R}^2$. For any $u \in H^1(D)$ with $\int_{\mathbb{S}^1} u d\theta = 0$, the Lebedev-Milin inequality (see [D, chapter 5]) tells us

$$\log\left(\frac{1}{2\pi}\int_{\mathbb{S}^1} e^{u} d\theta\right) \leq \frac{1}{4\pi}\|\nabla u\|_{L^2(D)}^2. \tag{1.7}$$

This should be compared to (1.5).

On the other hand, as observed in [Wi], we have a sequence of Lebedev-Milin type inequalities following from the work of Grenander-Szego [GrS] on Toeplitz determinants. More precisely for any integer $m \geq 0$, $u \in H^1(D)$ with $\int_{\mathbb{S}^1} u d\theta = 0$ and $\int_{\mathbb{S}^1} e^u e^{ik\theta} d\theta = 0$ for $k = 1, \cdots, m$, we have

$$\log\left(\frac{1}{2\pi}\int_{\mathbb{S}^1} e^{u} d\theta\right) \leq \frac{1}{4\pi(m+1)}\|\nabla u\|_{L^2(D)}^2. \tag{1.8}$$

For $m = 0$, (1.8) is just (1.7). For $m = 1$, (1.8) is proved in [OsPS, section 2]. These inequalities should be compared to (1.6). Note that $\cos k\theta$ and $\sin k\theta$ are eigenfunctions of $-\Delta_{\mathbb{S}^1}$ with eigenvalue k^2. So (1.8) actually tells us we can improve the coefficient of $\|\nabla u\|_{L^2(D)}^2$ further if e^u is perpendicular to more eigenfunctions of $-\Delta_{\mathbb{S}^1}$. For a while, people wonder whether we have similar improvements of (1.4) or (1.6) on $\mathbb{S}^2$. The main aim of this note is to confirm this guess.

To state the main results, we need some notations. For any nonnegative integer k, we denote

$$\mathcal{P}_k = \left\{\text{all polynomials on } \mathbb{R}^3 \text{ with degree at most } k\right\}; \tag{1.9}$$

$$\mathring{\mathcal{P}}_k = \left\{p \in \mathcal{P}_k : \int_{\mathbb{S}^2} p d\mu = 0\right\}; \tag{1.10}$$

$$H_k = \left\{\text{all degree } k \text{ homogeneous polynomials on } \mathbb{R}^3\right\}; \tag{1.11}$$

$$\mathcal{H}_k = \left\{h \in H_k : \Delta_{\mathbb{R}^3} h = 0\right\}. \tag{1.12}$$

It is known that

$$\mathcal{H}_k|_{\mathbb{S}^2} = \left\{h|_{\mathbb{S}^2} : h \in \mathcal{H}_k\right\} \tag{1.13}$$

is exactly the eigenspace of $-\Delta_{\mathbb{S}^2}$ associated with eigenvalue $k(k+1)$. Moreover

$$\left.\mathring{\mathcal{P}}_k\right|_{\mathbb{S}^2} = \bigoplus_{i=1}^{k} \mathcal{H}_i|_{\mathbb{S}^2}. \tag{1.14}$$

We refer the reader to [SW, chapter IV] for these facts.

Definition 1.1. *Let $m \in \mathbb{N}$, we denote*

$$\begin{aligned} & \mathcal{N}_m \\ = \; & \Big\{ N \in \mathbb{N} : \exists x_1, \cdots, x_N \in \mathbb{S}^2 \text{ and } \nu_1, \cdots, \nu_N \in [0,\infty) \text{ s.t. } \nu_1 + \cdots + \nu_N = 1 \\ & \text{and for any } p \in \overset{\circ}{\mathcal{P}}_m,\ \nu_1 p(x_1) + \cdots + \nu_N p(x_N) = 0. \Big\} \\ = \; & \Big\{ N \in \mathbb{N} : \exists x_1, \cdots, x_N \in \mathbb{S}^2 \text{ and } \nu_1, \cdots, \nu_N \in [0,\infty) \text{ s.t. for any } p \in \mathcal{P}_m, \\ & \nu_1 p(x_1) + \cdots + \nu_N p(x_N) = \frac{1}{4\pi} \int_{\mathbb{S}^2} p \, d\mu. \Big\}. \end{aligned} \tag{1.15}$$

The smallest number in $\mathcal{N}_m$ is denoted as N_m i.e. $N_m = \min \mathcal{N}_m$.

The importance of N_m lies in the following theorem.

Theorem 1.1. *Assume $u \in H^1(\mathbb{S}^2)$ such that $\int_{\mathbb{S}^2} u\, d\mu = 0$ (here μ is the standard measure on $\mathbb{S}^2$) and for every $p \in \overset{\circ}{\mathcal{P}}_m$, $\int_{\mathbb{S}^2} p e^{2u} d\mu = 0$, then for any $\varepsilon > 0$, we have*

$$\log \int_{\mathbb{S}^2} e^{2u} d\mu \le \left(\frac{1}{4\pi N_m} + \varepsilon \right) \|\nabla u\|_{L^2}^2 + c_\varepsilon. \tag{1.16}$$

It is worth pointing out that the coefficient $\frac{1}{4\pi N_m} + \varepsilon$ is almost optimal (see Lemma 3.1).

The condition in (1.15) is the same as saying the cubature formula

$$\frac{1}{4\pi} \int_{\mathbb{S}^2} f \, d\mu \approx \nu_1 f(x_1) + \cdots + \nu_N f(x_N) \tag{1.17}$$

for functions f on $\mathbb{S}^2$ has nonnegative weights and degree of precision m (here we use the terminology in [HSW]). Various cubature formulas are of great practical importance in scientific computing and have been extensively studied in the literature (see the review articles [Co, HSW] and the references therein). In particular, the number N_m is discussed in [HSW, chapter 4]. It follows from [Co, theorem 7.1] or [HSW, theorem 4] that

$$N_m \ge \left(\left[\frac{m}{2} \right] + 1 \right)^2. \tag{1.18}$$

Here $[t]$ denotes the largest integer less than or equal to t. In our case when all the weights ν_i's are nonnegative, a simple proof of (1.18) is given on [HSW, p1203]. In general, finding the exact values of N_m for all m's is still an open problem.

On the other hand, it is straightforward to see that $N_1 = 2$ (see Example 4.1). Hence (1.4) follows from Theorem 1.1. It is also well known in numerical analysis community that $N_2 = 4$ (we provide an elementary proof of this fact in Lemma 4.1 for reader's convenience). As a consequence, we have

Corollary 1.1. *Assume $u \in H^1(\mathbb{S}^2)$ such that $\int_{\mathbb{S}^2} u\, d\mu = 0$ and for every $p \in \overset{\circ}{\mathcal{P}}_2$, $\int_{\mathbb{S}^2} p e^{2u} d\mu = 0$, then for any $\varepsilon > 0$, we have*

$$\log \int_{\mathbb{S}^2} e^{2u} d\mu \le \left(\frac{1}{16\pi} + \varepsilon \right) \|\nabla u\|_{L^2}^2 + c_\varepsilon. \tag{1.19}$$

In Section 2, we will derive some extensions of the concentration compactness principle in dimension 2. These refinements will be used in Section 3 to prove our main theorem. In Section 4, we discuss some elementary facts about N_m. In particular we will show $N_2 = 4$. In Section 5, we will make a first effort toward related sharp inequalities generalizing (1.6). In Section 6, we will show our approach gives a new way to prove the sequence of Lebedev-Milin type inequalities on the unit circle.

2. Refinements of concentration compactness principle in dimension 2

In this section, we will extend the concentration compactness principle in dimension 2 developed in [L, section I.7]. These extensions will be crucial in the derivation of Theorem 1.1.

We start from a basic consequence of Moser-Trudinger inequality (1.2).

Lemma 2.1. *For any $u \in H^1(M)$ and $a > 0$, we have*

$$\int_M e^{au^2} d\mu < \infty. \tag{2.1}$$

Proof. Without losing of generality, we can assume u is nonnegative and unbounded. For $b > 0$, let $v = (u-b)^+$, then

$$\|\nabla v\|_{L^2}^2 = \int_{u>b} |\nabla u|^2 \, d\mu \to 0$$

as $b \to \infty$. Let $w = v - \overline{v}$, then

$$0 \le u \le v + b = w + \overline{v} + b.$$

Hence

$$u^2 \le 2w^2 + 2(\overline{v}+b)^2.$$

We have

$$e^{au^2} \le e^{2a(\overline{v}+b)^2} e^{2aw^2} \le e^{2a(\overline{v}+b)^2} e^{4\pi \frac{w^2}{\|\nabla w\|_{L^2}^2}}$$

when b is large enough. It follows that

$$\int_M e^{au^2} d\mu \le c e^{2a(\overline{v}+b)^2} < \infty.$$

∎

Next we prove a localized version of [L, Theorem I.6].

Lemma 2.2. *Assume $u_i \in H^1(M)$ such that $\overline{u_i} = 0$ and $\|\nabla u_i\|_{L^2} \le 1$. We also assume $u_i \rightharpoonup u$ weakly in $H^1(M)$, $u_i \to u$ a.e. and*

$$|\nabla u_i|^2 \, d\mu \to |\nabla u|^2 \, d\mu + \sigma \tag{2.2}$$

in measure. If $K \subset M$ is a compact subset with $\sigma(K) < 1$, then for any $1 \le p < \frac{1}{\sigma(K)}$, we have $e^{4\pi u_i^2}$ is bounded in $L^p(K)$ i.e.

$$\sup_i \int_K e^{4\pi p u_i^2} d\mu < \infty. \tag{2.3}$$

Proof. For basics about measure theory we refer the readers to [EG]. Let $v_i = u_i - u$, then $v_i \rightharpoonup 0$ weakly in $H^1(M)$, $v_i \to 0$ in $L^2(M)$. For any $\varphi \in C^\infty(M)$, we have

$$\begin{aligned}
&\|\nabla(\varphi v_i)\|_{L^2}^2 \\
= &\int_M \left(|\nabla\varphi|^2 v_i^2 + 2\varphi v_i \nabla\varphi\cdot\nabla v_i + \varphi^2|\nabla v_i|^2\right) d\mu \\
= &\int_M |\nabla\varphi|^2 v_i^2 d\mu + 2\int_M \varphi v_i\nabla\varphi\cdot\nabla v_i d\mu \\
&+\int_M \left(\varphi^2|\nabla u_i|^2 - 2\varphi^2\nabla u\cdot\nabla u_i + \varphi^2|\nabla u|^2\right) d\mu \\
\to &\int_M \varphi^2 d\sigma
\end{aligned}$$

as $i\to\infty$. Assume $1 \le p_1 < \frac{1}{\sigma(K)}$, then $\sigma(K) < \frac{1}{p_1}$. Hence there exists $\varphi \in C^\infty(M)$ such that $\varphi|_K = 1$ and $\int_M \varphi^2 d\sigma < \frac{1}{p_1}$. It follows that for i large enough,

$$\|\nabla(\varphi v_i)\|_{L^2}^2 < \frac{1}{p_1}.$$

Hence

$$\begin{aligned}
\int_K e^{4\pi p_1 (v_i - \overline{\varphi v_i})^2} d\mu &\le \int_M e^{4\pi p_1(\varphi v_i - \overline{\varphi v_i})^2} d\mu \\
&\le \int_M e^{4\pi\frac{(\varphi v_i - \overline{\varphi v_i})^2}{\|\nabla(\varphi v_i)\|_{L^2}^2}} d\mu \\
&\le c(M,g).
\end{aligned}$$

To continue, we observe that for any $\varepsilon > 0$,

$$\begin{aligned}
u_i^2 &= \left((v_i - \overline{\varphi v_i}) + u + \overline{\varphi v_i}\right)^2 \\
&= (v_i - \overline{\varphi v_i})^2 + 2(v_i - \overline{\varphi v_i})(u + \overline{\varphi v_i}) + (u + \overline{\varphi v_i})^2 \\
&\le (1+\varepsilon)(v_i - \overline{\varphi v_i})^2 + \left(1+\varepsilon^{-1}\right)(u + \overline{\varphi v_i})^2 \\
&\le (1+\varepsilon)(v_i - \overline{\varphi v_i})^2 + 2\left(1+\varepsilon^{-1}\right)u^2 + 2\left(1+\varepsilon^{-1}\right)\overline{\varphi v_i}^2.
\end{aligned}$$

Hence

$$e^{4\pi u_i^2} \le e^{4\pi(1+\varepsilon)(v_i - \overline{\varphi v_i})^2} e^{8\pi\left(1+\varepsilon^{-1}\right)u^2} e^{8\pi\left(1+\varepsilon^{-1}\right)\overline{\varphi v_i}^2}.$$

Given $1 \le p < \frac{1}{\sigma(K)}$, we can choose a $p_1 \in \left(p, \frac{1}{\sigma(K)}\right)$. There exists a $\varepsilon > 0$ such that $\frac{p_1}{1+\varepsilon} > p$. Note that $e^{4\pi(1+\varepsilon)(v_i - \overline{\varphi v_i})^2}$ is bounded in $L^{\frac{p_1}{1+\varepsilon}}(K)$, $e^{8\pi\left(1+\varepsilon^{-1}\right)u^2} \in L^q(K)$ for any $q < \infty$ (by Lemma 2.1) and $e^{8\pi\left(1+\varepsilon^{-1}\right)\overline{\varphi v_i}^2} \to 1$ as $i \to \infty$, it follows from Holder's inequality that $e^{4\pi u_i^2}$ is bounded in $L^p(K)$. ∎

Corollary 2.1. *With the same assumption as in Lemma 2.2, let*

$$\kappa = \max_{x\in M} \sigma(\{x\}) \le 1. \tag{2.4}$$

1. *If $\kappa < 1$, then for any $1 \le p < \frac{1}{\kappa}$, $e^{4\pi u_i^2}$ is bounded in $L^p(M)$. In particular, $e^{4\pi u_i^2} \to e^{4\pi u^2}$ in $L^1(M)$.*
2. *If $\kappa = 1$, then $\sigma = \delta_{x_0}$ for some $x_0 \in M$, $u = 0$ and after passing to a subsequence,*

$$e^{4\pi u_i^2} \to 1 + c_0\delta_{x_0} \tag{2.5}$$

in measure for some $c_0 \geq 0$.

Proof. First we assume $\kappa < 1$. Let $1 \leq p < \frac{1}{\kappa}$, then for any $x \in M$, $\sigma(x) < \frac{1}{p}$. Hence for some $r_x > 0$ small, we have $\sigma\left(\overline{B_{r_x}(x)}\right) < \frac{1}{p}$. By the compactness of M, we see

$$M = \bigcup_{i=1}^{N} B_{r_i}(x_i).$$

Here $r_i = r_{x_i}$. Then

$$M = \bigcup_{i=1}^{N} \overline{B_{r_i}(x_i)}.$$

It follows from the Lemma 2.2 that

$$\sup_j \int_{\overline{B_{r_i}(x_i)}} e^{4\pi p u_j^2} d\mu < \infty.$$

Summing up, we get

$$\sup_j \int_M e^{4\pi p u_j^2} d\mu < \infty.$$

Next we assume $\kappa = 1$. Since

$$\int_M |\nabla u|^2 \, d\mu + \sigma(M) \leq 1,$$

and $\overline{u} = 0$, we see $u = 0$ and $\sigma = \delta_{x_0}$ for some $x_0 \in M$. For $r > 0$ small, we know $e^{4\pi u_i^2}$ is bounded in $L^q(M \backslash B_r(x_0))$ for any $q < \infty$, hence $e^{4\pi u_i^2} \to 1$ in $L^1(M \backslash B_r(x_0))$. It follows that after passing to a subsequence, $e^{4\pi u_i^2} \to 1 + c_0 \delta_{x_0}$ in measure for some $c_0 \geq 0$. ∎

Now we are ready to derive the main refinement of the earlier concentration compactness principle.

Proposition 2.1. *Assume* $\alpha > 0$, $m_i > 0$, $m_i \to \infty$, $u_i \in H^1(M)$ *such that* $\overline{u_i} = 0$, $\|\nabla u_i\|_{L^2} = 1$ *and*

$$\log \int_M e^{2m_i u_i} d\mu \geq \alpha m_i^2. \tag{2.6}$$

We also assume $u_i \rightharpoonup u$ *weakly in* $H^1(M)$, $|\nabla u_i|^2 d\mu \to |\nabla u|^2 d\mu + \sigma$ *in measure and*

$$\frac{e^{2m_i u_i}}{\int_M e^{2m_i u_i} d\mu} \to \nu \tag{2.7}$$

in measure. Let

$$\{x \in M : \sigma(x) \geq 4\pi\alpha\} = \{x_1, \cdots, x_N\}, \tag{2.8}$$

then

$$\nu = \sum_{i=1}^{N} \nu_i \delta_{x_i}, \tag{2.9}$$

here $\nu_i \geq 0$ *and* $\sum_{i=1}^{N} \nu_i = 1$.

Proof. First we claim that if K is a compact subset of M with $\sigma(K) < 4\pi\alpha$, then $\nu(K) = 0$. Indeed, we can find another compact set K_1 such that $K \subset \operatorname{int} K_1$ and $\sigma(K_1) < 4\pi\alpha$. Fix a number p such that

$$\frac{1}{4\pi\alpha} < p < \frac{1}{\sigma(K_1)},$$

then Lemma 2.2 tells us

$$\int_{K_1} e^{4\pi p u_i^2} d\mu \le c,$$

here c is a constant independent of i. Using

$$2m_i u_i \le 4\pi p u_i^2 + \frac{m_i^2}{4\pi p},$$

we see

$$\int_{K_1} e^{2m_i u_i} d\mu \le c e^{\frac{m_i^2}{4\pi p}}.$$

It follows that

$$\frac{\int_{K_1} e^{2m_i u_i} d\mu}{\int_M e^{2m_i u_i} d\mu} \le c e^{\left(\frac{1}{4\pi p} - \alpha\right) m_i^2}.$$

Hence

$$\nu(K) \le \nu(\operatorname{int} K_1) \le \liminf_{i \to \infty} \frac{\int_{K_1} e^{2m_i u_i} d\mu}{\int_M e^{2m_i u_i} d\mu} = 0.$$

It follows that $\nu(K) = 0$.

If $\sigma(x) < 4\pi\alpha$, then for some $r_x > 0$ small, we have $\sigma\left(\overline{B_{r_x}(x)}\right) < 4\pi\alpha$. It follows from the claim that $\nu\left(\overline{B_{r_x}(x)}\right) = 0$. Hence

$$\nu(M \backslash \{x_1, \cdots, x_N\}) = 0.$$

In another word, $\nu = \sum_{i=1}^N \nu_i \delta_{x_i}$ with $\nu_i \ge 0$ and $\sum_{i=1}^N \nu_i = 1$. ∎

3. Proof of Theorem 1.1

Let $f_1, \cdots, f_L \in C(M)$ and $\alpha > 0$ be given. Here is our strategy to show for any $u \in H^1(M)$ with $\overline{u} = 0$ and $\int_M f_i e^{2u} d\mu = 0$ for $1 \le i \le L$, we have

$$\log \int_M e^{2u} d\mu \le \alpha \|\nabla u\|_{L^2}^2 + c. \tag{3.1}$$

This will be proven by contradiction argument. If it is not the case, then there exists $v_i \in H^1(M)$, $\overline{v_i} = 0$, $\int_M f_j e^{2v_i} d\mu = 0$ for $1 \le j \le L$, such that

$$\log \int_M e^{2v_i} d\mu - \alpha \|\nabla v_i\|_{L^2}^2 \to \infty \tag{3.2}$$

as $i \to \infty$. Then $\log \int_M e^{2v_i} d\mu \to \infty$. Since

$$\log \int_M e^{2v_i} d\mu \le \frac{1}{4\pi} \|\nabla v_i\|_{L^2}^2 + c(M, g), \tag{3.3}$$

we see $\|\nabla v_i\|_{L^2} \to \infty$. Let $m_i = \|\nabla v_i\|_{L^2}$ and $u_i = \frac{v_i}{m_i}$, then $m_i \to \infty$, $\|\nabla u_i\|_{L^2} = 1$, $\overline{u_i} = 0$. After passing to a subsequence, we have

$$\begin{aligned} u_i &\rightharpoonup u \text{ weakly in } H^1(M); \\ \log \int_M e^{2m_i u_i} d\mu - \alpha m_i^2 &\to \infty, \\ |\nabla u_i|^2 d\mu &\to |\nabla u|^2 d\mu + \sigma \text{ in measure}, \\ \frac{e^{2m_i u_i}}{\int_M e^{2m_i u_i} d\mu} &\to \nu \text{ in measure}. \end{aligned}$$

Let

$$\{x \in M : \sigma(x) \geq 4\pi\alpha\} = \{x_1, \cdots, x_N\}, \tag{3.4}$$

then it follows from Proposition 2.1 that

$$\nu = \sum_{i=1}^N \nu_i \delta_{x_i}, \tag{3.5}$$

here $\nu_i \geq 0$ and $\sum_{i=1}^N \nu_i = 1$. On the other hand we have

$$\int_M f_j d\nu = 0$$

for $1 \leq j \leq L$. In another word, we have

$$\begin{aligned} 4\pi\alpha N &\leq 1; && (3.6) \\ \sum_{i=1}^N \nu_i f_j(x_i) &= 0 && (3.7) \end{aligned}$$

for $1 \leq j \leq L$. We hope to get contradiction from these inequalities.

Proof of Theorem 1.1. Let $\alpha = \frac{1}{4\pi N_m} + \varepsilon$. If (1.16) is not true, then the above discussion gives us $x_1, \cdots, x_N \in \mathbb{S}^2$, $\nu_1, \cdots, \nu_N \geq 0$ such that $\sum_{i=1}^N \nu_i = 1$ and for any $p \in \overset{\circ}{\mathcal{P}}_m$, $\nu_1 p(x_1) + \cdots + \nu_N p(x_N) = 0$. Moreover $4\pi\alpha N \leq 1$. In particular, $N \in \mathcal{N}_m$ and hence $N \geq N_m$. It follows that

$$\alpha \leq \frac{1}{4\pi N} \leq \frac{1}{4\pi N_m}.$$

This contradicts with the choice of α. ∎

Next we want to show the constant $\frac{1}{4\pi N_m} + \varepsilon$ in (1.16) is almost sharp.

Lemma 3.1. *Assume $m \in \mathbb{N}$. If $a \geq 0$ and $c \in \mathbb{R}$ such that for any $u \in H^1(\mathbb{S}^2)$ with $\overline{u} = 0$ and $\int_{\mathbb{S}^2} pe^{2u} d\mu = 0$ for every $p \in \overset{\circ}{\mathcal{P}}_m$, we have*

$$\log \int_{\mathbb{S}^2} e^{2u} d\mu \leq a \|\nabla u\|_{L^2}^2 + c, \tag{3.8}$$

then $a \geq \frac{1}{4\pi N_m}$.

Proof. First we note that we can rewrite the assumption as for any $u \in H^1(\mathbb{S}^2)$ with $\int_{\mathbb{S}^2} pe^{2u} d\mu = 0$ for every $p \in \overset{\circ}{\mathcal{P}}_m$, we have

$$\log \int_{\mathbb{S}^2} e^{2u} d\mu \leq a \|\nabla u\|_{L^2}^2 + 2\overline{u} + c. \tag{3.9}$$

Assume $N \in \mathbb{N}$, $x_1, \cdots, x_N \in \mathbb{S}^2$ and $\nu_1, \cdots, \nu_N \in [0, \infty)$ s.t. $\nu_1 + \cdots + \nu_N = 1$ and for any $p \in \overset{\circ}{\mathcal{P}}_m$, $\nu_1 p(x_1) + \cdots + \nu_N p(x_N) = 0$. We will prove $a \geq \frac{1}{4\pi N}$. Lemma 3.1 follows. Without losing of generality we can assume $\nu_i > 0$ for $1 \leq i \leq N$ and $x_i \neq x_j$ for $1 \leq i < j \leq N$.

To continue let us fix some notations. For $x, y \in \mathbb{S}^2$, we denote $\overline{xy}$ as the geodesic distance between x and y on $\mathbb{S}^2$. For $r > 0$ and $x \in \mathbb{S}^2$, we denote $B_r(x)$ as the geodesic ball with radius r and center x i.e. $B_r(x) = \left\{ y \in \mathbb{S}^2 : \overline{xy} < r \right\}$.

Let $\delta > 0$ be small enough such that for $1 \leq i < j \leq N$, $\overline{B_{2\delta}(x_i)} \cap \overline{B_{2\delta}(x_j)} = \emptyset$. For $0 < \varepsilon < \delta$, we let

$$\phi_\varepsilon(t) = \begin{cases} 2\log\frac{\delta}{\varepsilon}, & 0 < t < \varepsilon; \\ 2\log\frac{\delta}{t}, & \varepsilon < t < \delta; \\ 0, & t > \delta. \end{cases}$$

If $b \in \mathbb{R}$, then we write

$$\phi_{\varepsilon,b}(t) = \begin{cases} \phi_\varepsilon(t) + b, & 0 < t < \delta; \\ b\left(2 - \frac{t}{\delta}\right), & \delta < t < 2\delta; \\ 0, & t > 2\delta. \end{cases}$$

Let

$$v(x) = \sum_{i=1}^N \phi_{\varepsilon, \frac{1}{2}\log\nu_i}(\overline{xx_i}), \tag{3.10}$$

then

$$\begin{aligned} \int_{\mathbb{S}^2} e^{2v} d\mu &= \sum_{i=1}^N \int_{B_\delta(x_i)} e^{2\phi_\varepsilon(\overline{xx_i}) + \log\nu_i} d\mu + O(1) \\ &= 2\pi \int_0^\delta e^{2\phi_\varepsilon(r)} \sin r dr + O(1) \\ &= 2\pi\delta^4\varepsilon^{-2} + O\left(\log\frac{1}{\varepsilon}\right) \end{aligned} \tag{3.11}$$

as $\varepsilon \to 0^+$.

Note that since $\dim\left(\overset{\circ}{\mathcal{P}}_m\Big|_{\mathbb{S}^2}\right) = m^2 + 2m$, we can fix $p_1, \cdots, p_{m^2+2m} \in \overset{\circ}{\mathcal{P}}_m$ such that $p_1|_{\mathbb{S}^2}, \cdots, p_{m^2+2m}|_{\mathbb{S}^2}$ is a base for $\overset{\circ}{\mathcal{P}}_m\Big|_{\mathbb{S}^2}$. For $1 \leq j \leq m^2 + 2m$, we have

$$\int_{\mathbb{S}^2} e^{2v} p_j d\mu = O\left(\log\frac{1}{\varepsilon}\right) \tag{3.12}$$

as $\varepsilon \to 0^+$. Indeed,

$$\begin{aligned} &\int_{\mathbb{S}^2} e^{2v} p_j d\mu \\ = &\sum_{i=1}^N \nu_i \int_{B_\delta(x_i)} e^{\phi_\varepsilon(\overline{xx_i})} p_j(x) d\mu(x) + O(1) \\ = &\sum_{i=1}^N \left(\nu_i p_j(x_i) \int_{B_\delta(x_i)} e^{\phi_\varepsilon(\overline{xx_i})} d\mu(x) + \int_{B_\delta(x_i)} e^{\phi_\varepsilon(\overline{xx_i})} O\left(\overline{xx_i}^2\right) d\mu(x) \right) + O(1), \end{aligned}$$

here we have used the Talyor expansion of p_j near x_i and the vanishing of integral of first order terms by symmetry. Using

$$\sum_{i=1}^{N} \nu_i p_j\left(x_i\right)=0,$$

we see

$$\int_{\mathbb{S}^2} e^{2v} p_j d\mu=O\left(\log\frac{1}{\varepsilon}\right).$$

To get a test function satisfying orthogornality condition, we need to do some corrections. We first claim that there exists $\psi_1,\cdots,\psi_{m^2+2m}\in C_c^\infty\left(\mathbb{S}^2\backslash\bigcup_{i=1}^N\overline{B_{2\delta}\left(x_i\right)}\right)$ such that the determinant

$$\det\left[\int_{\mathbb{S}^2}\psi_j p_k d\mu\right]_{1\le j,k\le m^2+2m}\neq 0. \tag{3.13}$$

Indeed, here is one way to construct these functions. Fix a nonzero smooth function $\eta\in C_c^\infty\left(\mathbb{S}^2\backslash\bigcup_{i=1}^N\overline{B_{2\delta}\left(x_i\right)}\right)$, then $\eta p_1,\cdots,\eta p_{m^2+2m}$ are linearly independent. It follows that the matrix

$$\left[\int_{\mathbb{S}^2}\eta^2 p_j p_k d\mu\right]_{1\le j,k\le m^2+2m}$$

is positive definite and has positive determinant. Then $\psi_j=\eta^2 p_j$ satisfies the claim.

It follows from (3.13) that we can find $\beta_1,\cdots,\beta_{m^2+2m}\in\mathbb{R}$ such that

$$\int_{\mathbb{S}^2}\left(e^{2v}+\sum_{j=1}^{m^2+2m}\beta_j\psi_j\right)p_k d\mu=0 \tag{3.14}$$

for $k=1,\cdots,m^2+2m$. Moreover

$$\beta_j=O\left(\log\frac{1}{\varepsilon}\right) \tag{3.15}$$

as $\varepsilon\to 0^+$. As a consequence we can find a constant $c_1>0$ such that

$$\sum_{j=1}^{m^2+2m}\beta_j\psi_j+c_1\log\frac{1}{\varepsilon}\ge\log\frac{1}{\varepsilon}. \tag{3.16}$$

We define u as

$$e^{2u}=e^{2v}+\sum_{j=1}^{m^2+2m}\beta_j\psi_j+c_1\log\frac{1}{\varepsilon}. \tag{3.17}$$

Note this u will be the test function we use to prove Lemma 3.1.

It follows from (3.14) that $\int_{\mathbb{S}^2}e^{2u}p d\mu=0$ for all $p\in\mathring{\mathcal{P}}_m$. Moreover using (3.11) and (3.15) we see

$$\int_{\mathbb{S}^2}e^{2u}d\mu=2\pi\delta^4\varepsilon^{-2}+O\left(\log\frac{1}{\varepsilon}\right)=2\pi\delta^4\varepsilon^{-2}\left(1+o\left(1\right)\right), \tag{3.18}$$

hence

$$\log \int_{\mathbb{S}^2} e^{2u} d\mu = 2 \log \frac{1}{\varepsilon} + O(1) \tag{3.19}$$

as $\varepsilon \to 0^+$. Calculation shows

$$\overline{u} = o\left(\log \frac{1}{\varepsilon}\right). \tag{3.20}$$

At last we claim

$$\int_{\mathbb{S}^2} |\nabla u|^2 d\mu = 8\pi N \log \frac{1}{\varepsilon} + o\left(\log \frac{1}{\varepsilon}\right). \tag{3.21}$$

Once this is known, we plug u into (3.9) and get

$$2 \log \frac{1}{\varepsilon} \leq 8\pi N a \log \frac{1}{\varepsilon} + o\left(\log \frac{1}{\varepsilon}\right).$$

Divide $\log \frac{1}{\varepsilon}$ on both sides and let $\varepsilon \to 0^+$, we see $a \geq \frac{1}{4\pi N}$.

To derive (3.21), we note that on $\mathbb{S}^2 \backslash \bigcup_{i=1}^{N} \overline{B_{2\delta}(x_i)}$, $|\nabla u| = O(1)$ (here we need to use (3.15) and (3.16)), hence

$$\begin{aligned}
\int_{\mathbb{S}^2} |\nabla u|^2 d\mu &= \sum_{i=1}^{N} \int_{B_{2\delta}(x_i)} |\nabla u|^2 d\mu + O(1) \\
&= \sum_{i=1}^{N} \int_{B_{\delta}(x_i)} |\nabla u|^2 d\mu + O(1) \\
&= \sum_{i=1}^{N} 8\pi \int_{\varepsilon}^{\delta} \frac{r^{-10} \sin r}{\left(\frac{c_1 \log \frac{1}{\varepsilon}}{\nu_i \delta^4} + r^{-4}\right)^2} dr + O(1) \\
&= 8\pi N \log \frac{1}{\varepsilon} + o\left(\log \frac{1}{\varepsilon}\right).
\end{aligned}$$

∎

4. The number N_m

We start with the following basic observation.

Example 4.1. $N_1 = 2$. *It is clear that* $N_1 \geq 2$, *on the other hand, by setting* $\nu_1 = \nu_2 = \frac{1}{2}$ *and* $x_2 = -x_1$, *we see* $N_1 \leq 2$. *Hence* $N_1 = 2$.

Lemma 4.1. $N_2 = 4$.

Proof. Indeed it follows from (1.18) that $N_2 \geq 4$. Here we give a direct proof. Note that $N_2 \geq N_1 = 2$.

If $N_2 = 2$, then we have $\nu_1 x_1 + \nu_2 x_2 = 0$. It implies $\nu_1 = \nu_2 = \frac{1}{2}$. Hence $x_2 = -x_1$. By rotation, we assume $x_1 = (0, 0, 1)$. Let $p(y) = y_1^2$, then

$$\nu_1 p(x_1) + \nu_2 p(x_2) = 0 \neq \frac{1}{4\pi} \int_{\mathbb{S}^2} p d\mu.$$

We get a contradiction.

If $N_2 = 3$, then we have $\nu_1 x_1 + \nu_2 x_2 + \nu_3 x_3 = 0$. It follows that x_1, x_2, x_3 must lie in a plane. By rotation we can assume that plane is the horizontal plane. Let $p = y_3^2$, then

$$\nu_1 p(x_1) + \nu_2 p(x_2) + \nu_3 p(x_3) = 0 \neq \frac{1}{4\pi} \int_{\mathbb{S}^2} p d\mu.$$

This gives us a contradiction.

Hence we only need to find $x_1, x_2, x_3, x_4 \in \mathbb{S}^2$, $\nu_1, \nu_2, \nu_3, \nu_4 \geq 0$ with $\nu_1 + \nu_2 + \nu_3 + \nu_4 = 1$ such that for any $p \in \overset{\circ}{\mathcal{P}}_2$, we have

$$\nu_1 p(x_1) + \nu_2 p(x_2) + \nu_3 p(x_3) + \nu_4 p(x_4) = 0. \tag{4.1}$$

We claim the four vortices of a regular tetrahedron inside the unit sphere with $\nu_i = \frac{1}{4}$ for $1 \leq i \leq 4$ would satisfy the property. Indeed, let

$$\begin{aligned} x_1 &= (0, 0, 1); \\ x_2 &= \left(0, \frac{2\sqrt{2}}{3}, -\frac{1}{3}\right); \\ x_3 &= \left(\sqrt{\frac{2}{3}}, -\frac{\sqrt{2}}{3}, -\frac{1}{3}\right); \\ x_4 &= \left(-\sqrt{\frac{2}{3}}, -\frac{\sqrt{2}}{3}, -\frac{1}{3}\right). \end{aligned}$$

Then we have

$$x_1 + x_2 + x_3 + x_4 = 0.$$

Moreover using

$$\mathcal{H}_2 = \operatorname{span}\left\{ y_1^2 - \frac{|y|^2}{3}, y_2^2 - \frac{|y|^2}{3}, y_1 y_2, y_1 y_3, y_2 y_3 \right\},$$

checking (4.1) for each p in the base verifies the identity. ∎

It remains an interesting question to find N_m for all m's.

5. A sharp inequality by perturbation

In this section we prove a sharp inequality by the perturbation method in the same spirit as [ChY1].

Theorem 5.1. *There exists an $a_0 < \frac{1}{8\pi}$ such that for all $u \in H^1(\mathbb{S}^2)$ satisfying $\int_{\mathbb{S}^2} u d\mu = 0$ and for every $p \in \overset{\circ}{\mathcal{P}}_2$, $\int_{\mathbb{S}^2} p e^{2u} d\mu = 0$, we have*

$$\log\left(\frac{1}{4\pi} \int_{\mathbb{S}^2} e^{2u} d\mu\right) \leq a_0 \|\nabla u\|_{L^2}^2. \tag{5.1}$$

For convenience we denote

$$\mathcal{S}_2 = \left\{ u \in H^1(\mathbb{S}^2) : \overline{u} = 0, \int_{\mathbb{S}^2} p e^{2u} d\mu = 0 \text{ for all } p \in \overset{\circ}{\mathcal{P}}_2 \right\}. \tag{5.2}$$

For a given number $a \in \left(\frac{1}{16\pi}, \frac{1}{8\pi}\right)$, it follows from Corollary 1.1 that for every $u \in \mathcal{S}_2$,

$$\log\left(\frac{1}{4\pi} \int_{\mathbb{S}^2} e^{2u} d\mu\right) \leq a \|\nabla u\|_{L^2}^2 + c_a. \tag{5.3}$$

Let

$$s = s_a = \inf_{u \in \mathcal{S}_2} \left[a \left\| \nabla u \right\|_{L^2}^2 - \log \left(\frac{1}{4\pi} \int_{\mathbb{S}^2} e^{2u} d\mu \right) \right]. \tag{5.4}$$

We claim s is achieved. Indeed if $u_i \in \mathcal{S}_2$ is a minimizing sequence, then

$$a \left\| \nabla u_i \right\|_{L^2}^2 - \log \left(\frac{1}{4\pi} \int_{\mathbb{S}^2} e^{2u_i} d\mu \right) \le c.$$

Here c is a constant independent of i. Choose a number ε with $0 < \varepsilon < a - \frac{1}{16\pi}$. Using Corollary 1.1 we have

$$a \left\| \nabla u_i \right\|_{L^2}^2 \le \log \left(\frac{1}{4\pi} \int_{S^2} e^{2u_i} d\mu \right) + c \le \left(\frac{1}{16\pi} + \varepsilon \right) \left\| \nabla u_i \right\|_{L^2}^2 + c.$$

It follows that

$$\left\| \nabla u_i \right\|_{L^2} \le c.$$

After passing to a subsequence we can find $u \in H^1(\mathbb{S}^2)$ such that $u_i \rightharpoonup u$ weakly in $H^1(\mathbb{S}^2)$. Hence $u_i \to u$ in $L^2(\mathbb{S}^2)$ and we can also assume $u_i \to u$ a.e. For any $b > 0$, we have

$$2bu_i \le 4\pi \frac{u_i^2}{\left\| \nabla u_i \right\|_{L^2}^2} + \frac{b^2 \left\| \nabla u_i \right\|_{L^2}^2}{4\pi}.$$

Hence

$$\int_{\mathbb{S}^2} e^{2bu_i} d\mu \le c e^{\frac{b^2 \left\| \nabla u_i \right\|_{L^2}^2}{4\pi}} \le c.$$

It follows that $e^{2u_i} \to e^{2u}$ in $L^1(\mathbb{S}^2)$. Hence for any $p \in \mathring{\mathcal{P}}_2$, $\int_{\mathbb{S}^2} p e^{2u} d\mu = 0$. It follows that $u \in \mathcal{S}_2$.

$$\begin{aligned} s &\le a \left\| \nabla u \right\|_{L^2}^2 - \log \left(\frac{1}{4\pi} \int_{\mathbb{S}^2} e^{2u} d\mu \right) \\ &\le \liminf_{i \to \infty} \left[a \left\| \nabla u_i \right\|_{L^2}^2 - \log \left(\frac{1}{4\pi} \int_{\mathbb{S}^2} e^{2u_i} d\mu \right) \right] \\ &= s. \end{aligned}$$

Hence u is a minimizer.

Let u_a be a minimizer for (5.4). When no confusion would happen, we simply write u instead of u_a. We will show that if a is close enough to $\frac{1}{8\pi}$, the minimizer u must be identically zero. This would imply Theorem 5.1.

To achieve this aim, we can assume $\frac{5}{48\pi} < a < \frac{1}{8\pi}$. Since u is a minimizer, we see

$$a \left\| \nabla u \right\|_{L^2}^2 - \log \left(\frac{1}{4\pi} \int_{\mathbb{S}^2} e^{2u} d\mu \right) \le 0.$$

Hence applying Corollary 1.1 we get

$$a \left\| \nabla u \right\|_{L^2}^2 \le \log \left(\frac{1}{4\pi} \int_{\mathbb{S}^2} e^{2u} d\mu \right) \le \frac{1}{12\pi} \left\| \nabla u \right\|_{L^2}^2 + c.$$

It implies $\left\| \nabla u \right\|_{L^2}^2 \le c$, a constant independent of a.

Next we claim that as $a \to \frac{1}{8\pi}$, $u_a \rightharpoonup 0$ weakly in $H^1(\mathbb{S}^2)$. Indeed if this is not the case, then we can find a sequence $a_i \to \frac{1}{8\pi}$, $u_i = u_{a_i}$ such that $u_i \rightharpoonup w$ weakly in $H^1(\mathbb{S}^2)$ and $w \ne 0$. We can also assume $u_i \to w$ a.e. It follows from classical

Moser-Trudinger inequality (see (1.2)) that $e^{2u_i} \to e^{2w}$ in $L^1\left(\mathbb{S}^2\right)$. Hence $w \in \mathcal{S}_2$. Since

$$a_i \left\|\nabla u_i\right\|_{L^2}^2 \le \log\left(\frac{1}{4\pi}\int_{\mathbb{S}^2} e^{2u_i} d\mu\right),$$

taking a limit we get

$$\frac{1}{8\pi}\left\|\nabla w\right\|_{L^2}^2 \le \log\left(\frac{1}{4\pi}\int_{\mathbb{S}^2} e^{2w} d\mu\right).$$

It follows from equality case of (1.4) (see [GuM]) that $w = 0$. This gives us a contradiction.

Applying the Moser-Trudinger inequality (1.2) again we see for any $b > 0$, $e^{2bu_a} \to 1$ in $L^q\left(S^2\right)$ for any $q \in [1,\infty)$ as $a \to \frac{1}{8\pi}$. Hence

$$a\left\|\nabla u_a\right\|_{L^2}^2 \le \log\left(\frac{1}{4\pi}\int_{\mathbb{S}^2} e^{2u_a} d\mu\right) \to 0.$$

It follows that $\left\|\nabla u_a\right\|_{L^2} = o(1)$ as $a \to \frac{1}{8\pi}$.

To continue we observe that since

$$\left.\mathring{\mathcal{P}}_2\right|_{\mathbb{S}^2} = \left.\mathcal{H}_1\right|_{\mathbb{S}^2} \oplus \left.\mathcal{H}_2\right|_{\mathbb{S}^2} = \left.\left(\mathcal{H}_1+\mathcal{H}_2\right)\right|_{\mathbb{S}^2},$$

u satisfies the Euler-Langrage equation

$$-a\Delta u - \frac{e^{2u}}{\int_{\mathbb{S}^2} e^{2u} d\mu} = -\frac{1}{4\pi} + \ell e^{2u} + h e^{2u} \tag{5.5}$$

for some $\ell = \ell_a \in \mathcal{H}_1$ and $h = h_a \in \mathcal{H}_2$.

Since $\mathcal{H}_1 + \mathcal{H}_2$ is a finite dimensional vector space, any two norms on it are equivalent. Hence we fix an arbitrary norm on $\mathcal{H}_1 + \mathcal{H}_2$ from now on. We claim that $\ell_a \to 0$ and $h_a \to 0$ as $a \to \frac{1}{8\pi}$. For convenience we write

$$\lambda = \frac{1}{4\pi}\int_{\mathbb{S}^2} e^{2u} d\mu.$$

Note that $\lambda = 1 + o(1)$. The equation becomes

$$-a\Delta u + \frac{1}{4\pi} = e^{2u}\left(\frac{1}{4\pi\lambda} + \ell + h\right). \tag{5.6}$$

Multiplying $\frac{1}{4\pi\lambda} + \ell + h$ and integrating on $\mathbb{S}^2$, we see

$$\int_{\mathbb{S}^2}\left(-a\Delta u + \frac{1}{4\pi}\right)\left(\frac{1}{4\pi\lambda} + \ell + h\right) d\mu = \int_{\mathbb{S}^2} e^{2u}\left(\frac{1}{4\pi\lambda} + \ell + h\right)^2 d\mu.$$

Using the fact $u \in \mathcal{S}_2$ it becomes

$$\begin{aligned} & a\int_{\mathbb{S}^2} u\,(2\ell + 6h)\, d\mu \\ = & \int_{\mathbb{S}^2} e^{2u}\,(\ell+h)^2\, d\mu \\ = & \int_{\mathbb{S}^2}\left(e^{2u}-1\right)(\ell+h)^2\, d\mu + \int_{\mathbb{S}^2}\ell^2 d\mu + \int_{\mathbb{S}^2} h^2 d\mu. \end{aligned}$$

It follows that

$$o\left(\|\ell\| + \|h\|\right) = \int_{\mathbb{S}^2}\ell^2 d\mu + \int_{\mathbb{S}^2} h^2 d\mu + o\left(\|\ell\|^2 + \|h\|^2\right).$$

Hence

$$\|\ell\|^2 + \|h\|^2 = o\left(\|\ell\| + \|h\|\right).$$

We get $\|\ell\| + \|h\| = o(1)$.

Now we claim that $\|u_a\|_{L^\infty} = o(1)$. Indeed since

$$\begin{aligned} & \left\| e^{2u}\left(\frac{1}{4\pi\lambda} + \ell + h\right) - \frac{1}{4\pi}\right\|_{L^2} \\ \leq & \left\| e^{2u}\left(\frac{1}{4\pi\lambda} - \frac{1}{4\pi}\right)\right\|_{L^2} + \frac{1}{4\pi}\left\|e^{2u} - 1\right\|_{L^2} + \left\|e^{2u}\left(\ell + h\right)\right\|_{L^2} \\ = & \ o(1), \end{aligned}$$

it follows from (5.6) and standard elliptic theory that $\|u_a\|_{W^{2,2}} = o(1)$. Sobolev embedding theorem tells us $\|u_a\|_{L^\infty} = o(1)$.

At last we observe that $e^{2u} - \lambda$ is perpendicular to $\mathbb{R}$, $\mathcal{H}_1$ and $\mathcal{H}_2$, hence

$$\begin{aligned} & 12\int_{\mathbb{S}^2}\left(e^{2u} - \lambda\right)^2 d\mu \\ \leq & \int_{\mathbb{S}^2}\left|\nabla e^{2u}\right|^2 d\mu \\ = & \ 4\int_{\mathbb{S}^2} e^{4u}\left|\nabla u\right|^2 d\mu \\ = & \int_{\mathbb{S}^2}\nabla u \cdot \nabla e^{4u} d\mu \\ = & \int_{\mathbb{S}^2}\left(-\Delta u\right) e^{4u} d\mu \\ = & \int_{\mathbb{S}^2}\left(-\Delta u\right)\left(e^{4u} - \lambda^2\right) d\mu \\ = & \ \frac{1}{a}\int_{\mathbb{S}^2}\left[e^{2u}\left(\frac{1}{4\pi\lambda} + \ell + h\right) - \frac{1}{4\pi}\right]\left(e^{4u} - \lambda^2\right) d\mu \\ = & \ \frac{1 + o(1)}{2\pi a}\int_{\mathbb{S}^2}\left(e^{2u} - \lambda\right)^2 d\mu + \frac{1}{a}\int_{\mathbb{S}^2} e^{2u}\left(\ell + h\right)\left(e^{4u} - \lambda^2\right) d\mu. \end{aligned}$$

On the other hand,

$$\begin{aligned} & \int_{\mathbb{S}^2} e^{2u}\left(\ell + h\right)\left(e^{4u} - \lambda^2\right) d\mu \\ = & \int_{\mathbb{S}^2}\left(e^{2u} - \lambda\right)\left(\ell + h\right)\left(e^{4u} - \lambda^2\right) d\mu + \lambda\int_{\mathbb{S}^2}\left(\ell + h\right)\left(e^{4u} - \lambda^2\right) d\mu \\ = & \ o(1)\int_{\mathbb{S}^2}\left(e^{2u} - \lambda\right)^2 d\mu + \lambda\int_{\mathbb{S}^2}\left(\ell + h\right)\left(e^{4u} - 2\lambda e^{2u} + \lambda^2\right) d\mu \\ = & \ o(1)\int_{\mathbb{S}^2}\left(e^{2u} - \lambda\right)^2 d\mu + \lambda\int_{\mathbb{S}^2}\left(\ell + h\right)\left(e^{2u} - \lambda\right)^2 d\mu \\ = & \ o(1)\int_{\mathbb{S}^2}\left(e^{2u} - \lambda\right)^2 d\mu. \end{aligned}$$

Here we have used the fact $u \in \mathcal{S}_2$. Plug this equality back we see

$$\left(12 - \frac{1}{2\pi a} + o(1)\right)\int_{\mathbb{S}^2}\left(e^{2u} - \lambda\right)^2 d\mu \leq 0.$$

Since a is close to $\frac{1}{8\pi}$, we get $\int_{\mathbb{S}^2}\left(e^{2u}-\lambda\right)^2 d\mu=0$. Hence u must be constant function. In view of the fact $\overline{u}=0$, we get $u=0$. This finishes the proof of Theorem 5.1.

6. A revisit of Lebedev-Milin type inequalities on $\mathbb{S}^1$

In this section we will show the above method on $\mathbb{S}^2$ provides a variational approach for a sequence of Lebedev-Milin type inequalities on $\mathbb{S}^1$. Let D be the unit disk in the plane and $\mathbb{S}^1=\partial D$ be the unit circle. We use θ as the usual angle variable and identify $\mathbb{R}^2$ as $\mathbb{C}$.

Theorem 6.1. *For $m\in\mathbb{N}$, $u\in H^1(D)$ with $\int_{\mathbb{S}^1} ud\theta=0$ and $\int_{\mathbb{S}^1} e^u e^{ik\theta}d\theta=0$ for $k=1,\cdots,m$, we have*

$$\log\left(\frac{1}{2\pi}\int_{\mathbb{S}^1} e^u d\theta\right)\le\frac{1}{4\pi(m+1)}\|\nabla u\|^2_{L^2(D)}. \tag{6.1}$$

Moreover equality holds if and only if $u(z)=\log\frac{1}{|1-\xi z^{m+1}|^2}$ for some $\xi\in\mathbb{C}$ with $|\xi|<1$.

For $m=1$, (6.1) is proved in [OsPS] by variational method. As observed in [Wi], (6.1) follows from the work of Grenander-Szego [GrS] on Toeplitz determinants.

On $\mathbb{S}^1$, the Moser-Trudinger inequality (1.2) is replaced by the Beurling-Chang-Marshall inequality (see [ChM, corollary 2]): for $u\in H^1(D)\setminus\{0\}$ with $\int_{\mathbb{S}^1} ud\theta=0$, we have

$$\int_{\mathbb{S}^1} e^{\pi\frac{u^2}{\|\nabla u\|^2_{L^2(D)}}}d\theta\le c. \tag{6.2}$$

Similar to (1.9)–(1.12), for any nonnegative integer k, we write

$$\begin{aligned}
\mathcal{P}_k &= \left\{\text{real polynomials on } \mathbb{R}^2 \text{ with degree at most } k\right\}; && (6.3)\\
\mathring{\mathcal{P}}_k &= \left\{p\in\mathcal{P}_k:\int_{\mathbb{S}^1} pd\theta=0\right\}; && (6.4)\\
H_k &= \left\{\text{degree } k \text{ homogeneous real polynomials on } \mathbb{R}^2\right\}; && (6.5)\\
\mathcal{H}_k &= \{h\in H_k:\Delta_{\mathbb{R}^2}h=0\}=span_{\mathbb{R}}\left\{\operatorname{Re}\left(z^k\right),\operatorname{Im}\left(z^k\right)\right\}. && (6.6)
\end{aligned}$$

Note that

$$\left.\mathcal{H}_k\right|_{\mathbb{S}^1}=span_{\mathbb{R}}\{\cos k\theta,\sin k\theta\} \tag{6.7}$$

and

$$\left.\mathring{\mathcal{P}}_k\right|_{\mathbb{S}^1}=span_{\mathbb{R}}\{\cos j\theta,\sin j\theta: j\in\mathbb{N}, j\le k\}. \tag{6.8}$$

Corresponds to Definition 1.1, we have for $m\in\mathbb{N}$,

$$\begin{aligned}
&\mathcal{N}_m\left(\mathbb{S}^1\right) && (6.9)\\
=\ &\Big\{N\in\mathbb{N}:\exists z_1,\cdots,z_N\in\mathbb{S}^1 \text{ and } \nu_1,\cdots,\nu_N\in[0,\infty) \text{ s.t. for any } p\in\mathcal{P}_m,\\
&\nu_1 p(z_1)+\cdots+\nu_N p(z_N)=\frac{1}{2\pi}\int_{\mathbb{S}^1} pd\theta.\Big\}
\end{aligned}$$

and $N_m\left(\mathbb{S}^1\right)=\min\mathcal{N}_m\left(\mathbb{S}^1\right)$. Unlike the case on $\mathbb{S}^2$, it is known that

$$N_m\left(\mathbb{S}^1\right)=m+1. \tag{6.10}$$

Indeed if $N \in \mathcal{N}_m\left(\mathbb{S}^1\right)$, we must have $N \geq m+1$. Otherwise, for the $z_1, \cdots, z_N \in \mathbb{S}^1$ in (6.9), we let $f(z) = (z - z_1) \cdots (z - z_N)$, then $\operatorname{Re} f, \operatorname{Im} f \in \mathcal{P}_m$. It follows that

$$\frac{1}{2\pi} \int_{\mathbb{S}^1} f d\theta = \nu_1 f(z_1) + \cdots + \nu_N f(z_N) = 0.$$

On the other hand, we clearly have

$$\frac{1}{2\pi} \int_{\mathbb{S}^1} f d\theta = (-1)^N z_1 \cdots z_N \neq 0.$$

This gives us a contradiction. Hence $N_m\left(\mathbb{S}^1\right) \geq m+1$. On the other hand, for $1 \leq k \leq m+1$, we let $\nu_k = \frac{1}{m+1}$ and $z_k = e^{\frac{2k\pi}{m+1} i}$. It follows that $m+1 \in \mathcal{N}_m\left(\mathbb{S}^1\right)$. Hence $N_m\left(\mathbb{S}^1\right) = m+1$.

Now we are ready to state the analogue of Theorem 1.1 on $\mathbb{S}^1$.

Lemma 6.1. *Assume $m \in \mathbb{N}$, $u \in H^1(D)$ such that $\int_{\mathbb{S}^1} u d\theta = 0$ and $\int_{\mathbb{S}^1} e^u e^{ik\theta} d\theta = 0$ for $1 \leq k \leq m$, then for any $\varepsilon > 0$ we have*

$$\begin{aligned} \log \int_{\mathbb{S}^1} e^u d\theta &\leq \left(\frac{1}{4\pi N_m\left(\mathbb{S}^1\right)} + \varepsilon\right) \|\nabla u\|_{L^2(D)}^2 + c_\varepsilon \\ &= \left(\frac{1}{4\pi(m+1)} + \varepsilon\right) \|\nabla u\|_{L^2(D)}^2 + c_\varepsilon. \end{aligned} \tag{6.11}$$

Note that for $m = 1$, Lemma 6.1 is treated in [OsPS, lemma 2.5]. We can prove Lemma 6.1 by replacing (1.2) with (6.2) and following the approach in Section 2 and Section 3. The detail is left to interested readers.

To continue we denote

$$\mathcal{S}_m = \left\{u \in H^1(D) : \int_{\mathbb{S}^1} u d\theta = 0, \int_{\mathbb{S}^1} e^u e^{ik\theta} d\theta = 0 \text{ for } k = 1, \cdots, m\right\}. \tag{6.12}$$

Let $a \in \left(\frac{1}{4\pi(m+1)}, \frac{1}{4\pi m}\right)$, then it follows from Lemma 6.1 that

$$\inf_{u \in \mathcal{S}_m} \left[a \|\nabla u\|_{L^2(D)}^2 - \log\left(\frac{1}{2\pi} \int_{\mathbb{S}^1} e^u d\theta\right)\right] \tag{6.13}$$

is achieved.

Let u be a minimizer for (6.13), then u is smooth and for some real numbers β_k and γ_k,

$$\begin{aligned} -\Delta u &= 0 \text{ in } D; \\ 2a \frac{\partial u}{\partial \nu} - \frac{e^u}{\int_{\mathbb{S}^1} e^u d\theta} &= -\frac{1}{2\pi} + \sum_{k=1}^{m} (\beta_k \cos k\theta + \gamma_k \sin k\theta) e^u. \end{aligned}$$

Here ν is the unit outer normal direction of $\mathbb{S}^1$. Let

$$v = u - \log\left(2a \int_{\mathbb{S}^1} e^u d\theta\right), \tag{6.14}$$

then v is smooth and

$$\begin{aligned}
-\Delta v &= 0 \text{ in } D; \\
\frac{\partial v}{\partial \nu} + \frac{1}{4\pi a} &= e^v + \sum_{k=1}^{m} \left(c_k e^{ik\theta} + \overline{c_k} e^{-ik\theta} \right) e^v; \\
\int_{\mathbb{S}^1} e^v e^{ik\theta} d\theta &= 0 \text{ for } k = 1, \cdots, m.
\end{aligned}$$

Here $c_1, \cdots, c_m$ are complex constants. Next we claim $c_k = 0$ for all k. For the case $m = 1$, this is proved in [OsPS, lemma 2.6].

Lemma 6.2. *Let $m \in \mathbb{N}$, $\alpha > 0$, $v \in C^\infty\left(\overline{D}\right)$ such that $\int_{\mathbb{S}^1} e^v e^{ik\theta} d\theta = 0$ for $k = 1, \cdots, m$ and*

$$-\Delta v = 0 \text{ in } D; \tag{6.15}$$

$$\frac{\partial v}{\partial \nu} + \alpha = e^v + \sum_{k=1}^{m} \left(c_k e^{ik\theta} + \overline{c_k} e^{-ik\theta} \right) e^v; \tag{6.16}$$

here ν is the unit outer normal direction of $\mathbb{S}^1$ and $c_1, \cdots, c_m$ are complex constants, then $c_k = 0$ for $1 \le k \le m$.

Proof. We write

$$\begin{aligned}
v|_{\mathbb{S}^1} &= \sum_{k=-\infty}^{\infty} a_k e^{ik\theta}, \quad a_k \in \mathbb{C}, \overline{a_k} = a_{-k}; \\
e^v|_{\mathbb{S}^1} &= \sum_{k=-\infty}^{\infty} b_k e^{ik\theta}, \quad b_k \in \mathbb{C}, \overline{b_k} = b_{-k}.
\end{aligned}$$

It follows from the assumption that

$$b_k = 0 \text{ for } 1 \le |k| \le m. \tag{6.17}$$

Using (6.15) and (6.16) we see

$$\sum_{k=-\infty}^{\infty} |k| \, a_k e^{ik\theta} + \alpha = \left(1 + \sum_{j=1}^{m} c_j e^{ij\theta} + \sum_{j=1}^{m} \overline{c_j} e^{-ij\theta} \right) \sum_{k=-\infty}^{\infty} b_k e^{ik\theta}.$$

Compare the constant term on both sides and using (6.17) we get $b_0 = \alpha$. On the other hand, for $k \neq 0$, we have

$$|k| \, a_k = b_k + \sum_{j=1}^{m} c_j b_{k-j} + \sum_{j=1}^{m} \overline{c_j} b_{k+j}. \tag{6.18}$$

Next we observe that

$$\partial_\theta \left(e^v \right) = e^v \partial_\theta v,$$

hence

$$\sum_{k=-\infty}^{\infty} k b_k e^{ik\theta} = \left(\sum_{j=-\infty}^{\infty} j a_j e^{ij\theta} \right) \left(\sum_{k=-\infty}^{\infty} b_k e^{ik\theta} \right).$$

It follows that

$$k b_k = \sum_{j=-\infty}^{\infty} j a_j b_{k-j}. \tag{6.19}$$

Plug (6.18) into (6.19), we get

$$kb_k=\sum_{j=-\infty}^{\infty}\operatorname{sgn}(j)\left[b_j+\sum_{s=1}^{m}c_sb_{j-s}+\sum_{s=1}^{m}\overline{c_s}b_{j+s}\right]b_{k-j}.$$

In particular, for $1\leq k\leq m$, it becomes

$$\begin{aligned} kb_k &= \sum_{j=1}^{k}b_jb_{k-j}+\sum_{s=1}^{m}c_s\sum_{j=1}^{k+s}b_{j-s}b_{k-j}+\sum_{s=1}^{k}\overline{c_s}\sum_{j=1}^{k-s}b_{j+s}b_{k-j}\\ &\quad+\sum_{s=k+1}^{m}\overline{c_s}\sum_{j=k-s+1}^{0}b_{j+s}b_{k-j}. \end{aligned}$$

Using (6.17) we get $\alpha^2c_k=0$, hence $c_k=0$. ∎

It follows from Lemma 6.2 that the function v defined in (6.14) satisfies

$$\begin{aligned} -\Delta v &= 0 \text{ in } D;\\ \frac{\partial v}{\partial\nu}+\frac{1}{4\pi a} &= e^v \text{ on } \mathbb{S}^1. \end{aligned}$$

Since $\frac{1}{4\pi a}\in(m,m+1)$, it follows from [OsPS, lemma 2.3] that v is a constant function. Hence any minimizer of (6.13) must be 0. In another word, for any $u\in\mathcal{S}_m$,

$$\log\left(\frac{1}{2\pi}\int_{\mathbb{S}^1}e^ud\theta\right)\leq a\left\|\nabla u\right\|_{L^2(D)}^2.$$

Let $a\to\frac{1}{4\pi(m+1)}$, we get (6.1).

If $u\in\mathcal{S}_m$ such that

$$\log\left(\frac{1}{2\pi}\int_{\mathbb{S}^1}e^ud\theta\right)=\frac{\left\|\nabla u\right\|_{L^2(D)}^2}{4\pi(m+1)},$$

then u is smooth and for some real numbers β_k and γ_k,

$$\begin{aligned} -\Delta u &= 0 \text{ in } D;\\ \frac{1}{2\pi(m+1)}\frac{\partial u}{\partial\nu}-\frac{e^u}{\int_{\mathbb{S}^1}e^ud\theta} &= -\frac{1}{2\pi}+\sum_{k=1}^{m}\left(\beta_k\cos k\theta+\gamma_k\sin k\theta\right)e^u. \end{aligned}$$

Let

$$v=u-\log\frac{\int_{\mathbb{S}^1}e^ud\theta}{2\pi(m+1)},$$

it follows from Lemma 6.2 that

$$\begin{aligned} -\Delta v &= 0 \text{ in } D;\\ \frac{\partial v}{\partial\nu}+m+1 &= e^v \text{ on } \mathbb{S}^1. \end{aligned}$$

By [Wa, theorem 7], we can find $\xi\in\mathbb{C}$ with $|\xi|<1$ such that

$$v(z)=\log\frac{(m+1)\left(1-|\xi|^2\right)}{\left|1-\xi z^{m+1}\right|^2}.$$

Using the fact $\int_{\mathbb{S}^1}ud\theta=0$, we see $u(z)=\log\frac{1}{|1-\xi z^{m+1}|^2}$.

At last calculation shows for any $\xi \in \mathbb{C}$ with $|\xi| < 1$, if we write $u_\xi(z) = \log \frac{1}{|1-\xi z^{m+1}|^2}$, then $u_\xi \in \mathcal{S}_m$ and

$$\log\left(\frac{1}{2\pi}\int_{\mathbb{S}^1} e^{u_\xi} d\theta\right) = \log\frac{1}{1-|\xi|^2} = \frac{1}{4\pi(m+1)}\|\nabla u_\xi\|^2_{L^2(D)}.$$

Theorem 6.1 follows.

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