Einstein-type elliptic systems

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Einstein constraint equations

Definition

An $(n+1)$-dimensional **globally hyperbolic space-time** is a Lorentzian manifold $(V \doteq M^n \times \mathbb{R}, \bar{g})$ satisfying the Einstein equations

$$Ric_{\bar{g}} - \frac{1}{2} R_{\bar{g}} \bar{g} = T_{\bar{g},\bar{\psi}}, \tag{1}$$

where $Ric_{\bar{g}}$ and $R_{\bar{g}}$ represent the Ricci tensor and scalar curvature of $\bar{g}$, respectively, and $T_{\bar{g},\bar{\psi}}$ is the energy-momentum tensor depending on $\bar{g}$ and on a collection of matter fields, denoted by $\bar{\psi}$.

Some examples of matter fields

- Scalar field sources: $\bar{\psi} = \bar{\phi}$, a real function on V.

$$T = d\bar{\phi} \otimes d\bar{\phi} - \left(\frac{1}{2} |d\bar{\phi}|^2_{\bar{g}} + U(\bar{\phi}) \right) \bar{g}$$

- Fluid sources (perfect fluids): $\bar{\psi} = (\mu, p, u)$ where μ, p are real functions and u a time-like vector field.

$$T = \mu u^\flat \otimes u^\flat - p \left(\bar{g} + u^\flat \otimes u^\flat \right)$$

- Electromagnetic sources: $\bar{\psi} = \bar{F} \in \Omega^2(V)$

$$T_{\alpha\beta} = \bar{F}_\alpha{}^\lambda \bar{F}_{\beta\lambda} - \frac{1}{4} |\bar{F}|^2_{\bar{g}} \bar{g}_{\alpha\beta}$$

Charged fluid

In realistic situations we would have a combination of these sources as

$$T = T_{\bar{\phi}} + T_{\text{fluid}} + T_{\bar{F}}.$$

We will be interested in the model of a **charged fluid**:

$$T = \mu u^\flat \otimes u^\flat + T_{\bar{F}}. \tag{2}$$

This particular source couple matter equations to the Einstein field equations as we will see in the sequel.

Charged fluid

The coupled system below is known as the Einstein-Maxwell field equations:

$$\begin{aligned} G_{\bar{g}} &= T_{\bar{g},\bar{\psi}}, \\ \mathrm{div}_{\bar{g}}\, T_{\bar{g},\bar{\psi}} &= 0, \end{aligned} \tag{3}$$

where $G_{\bar{g}} \doteq Ric_{\bar{g}} - \frac{1}{2}R_{\bar{g}}\bar{g}$ is the Einstein tensor and the second equation is a consequence of the contracted Bianchi identities.

This last equation typically imposes conservation equations on the fluids sources. More explicitly

$$\begin{aligned} G_{\bar{g}} &= \mu u^{\flat} \otimes u^{\flat} + T_{\bar{F}} \\ \mathrm{div}_{\bar{g}}\, \bar{F} &= -q u^{\flat} \\ d\bar{F} &= 0 \\ \mathrm{div}_{\bar{g}}(\mu u) &= 0 \\ \mu \bar{\nabla}_u u^{\flat} &= q \iota_u \bar{F} \end{aligned} \tag{4}$$

Initial value formulation

The initial value formulation of (3)-(4) depends basically on writing the Einstein-Maxwell field equations as a second order hyperbolic system on the metric in the *normal* form

$$\partial_{tt}\bar{g} = \Box\bar{g} + \ldots$$

with initial conditions $\bar{g}|_{t=0}, \partial_t\bar{g}|_{t=0}$ plus the initial data for the physical fields involved $(\bar{\phi}, \bar{F}, \mu, u, \ldots)$.

The initial data $\bar{g}_{t=0}$ is exactly the induced metric on $M = M \times \{0\}$ whereas $\partial_t\bar{g}|_{t=0}$ may be expressed in terms of the second fundamental form of M as follows:

$$K = -\frac{1}{2N}\left(\partial_t g_t - \pounds_\beta g_t\right)|_{t=0},$$

Here, g_t is the Riemannian induced metric on the slice $M \times \{t\}$. The function N on M is the *lapse* and the vector field β on M is the *shift* corresponding to the coordinate frame we are considering in V.

Initial value formulation

The Codazzi-Gauss equations impose constraints on g and K, then on the initial data $g_{t=0}$ and $\partial_t g_t|_{t=0}$. The resulting constraint equations are actually the following:

$$\begin{aligned} R_g - |K|_g^2 + (\operatorname{tr}_g K)^2 &= 2\epsilon \\ \operatorname{div}_g K - d \operatorname{tr}_g K &= J, \end{aligned} \tag{5}$$

where

$$\epsilon \doteq T(n,n)|_{t=0} \quad \text{and} \quad J \doteq -T(n,\cdot)|_{t=0}.$$

Thus, in terms of well-posedness of the initial value problem, the system (5) becomes a **necessary** condition for the initial data (g, K).

It is a remarkable fact that, for many sources of interest (scalar fields, perfect fluids, Einstein-Maxwell among others), the constraint equations (5) are also a **sufficient** condition to guarantee short-time existence.

Initial value formulation

The space-time equations associated with matter fields can impose further constraints.

This is the case for the electromagnetic field equations (and more generally, for Yang-Mills fields coupled to the Einstein equations.) In the case of a charged fluid, the full system of constraint equations reads as follows:

$$\begin{aligned} R_g - |K|_g^2 + (\operatorname{tr}_g K)^2 &= 2\epsilon \\ \operatorname{div}_g K - d \operatorname{tr}_g K &= J, \\ \operatorname{div}_g E &= \widetilde{q}, \end{aligned} \tag{6}$$

where E is a vector field on M constructed from $\bar{F}$.

Conformal method

The constraint equations appear as a highly underdetermined system. It is therefore natural to split the initial data into arbitrarily prescribed data and unknowns.

The proposal of the conformal method is to consider the following *gauges*

$$
\begin{aligned}
g &= \phi^{\frac{4}{n-2}}\gamma \\
K &= \phi^{-2}\widetilde{K} + \frac{\tau}{n}g, \\
\widetilde{K} &= \pounds_{\gamma,conf}X + U,
\end{aligned}
\tag{7}
$$

where we have denoted $\tau \doteq \mathrm{tr}_g K$ and defined the conformal Killing operator

$$\pounds_{\gamma,conf}X \doteq \pounds_X\gamma - \frac{2}{n}\,\mathrm{div}_\gamma X\,\gamma.$$

Furthermore, in the above decomposition, the symmetric traceless $(0,2)$-tensor field U is supposed to be *transverse*, *i.e*, $\mathrm{div}_\gamma U = 0$.

Conformal method

Using the conformal splitting above, we rewrite the Gauss-Codazzi constraints as follows:

$$\begin{aligned} &\Delta_\gamma \phi - c_n R_\gamma \phi + c_n |\widetilde{K}|^2_\gamma \, \phi^{-\frac{3n-2}{n-2}} + c_n \left(\frac{1-n}{n} \tau^2 + 2\epsilon \right) \phi^{\frac{n+2}{n-2}} = 0, \\ &\Delta_{\gamma, conf} X - \left(\frac{n-1}{n} D\tau + J \right) \phi^{\frac{2n}{n-2}} = 0, \end{aligned} \tag{8}$$

where

$$\Delta_{\gamma, conf} X \doteq \mathrm{div}_\gamma (\pounds_{\gamma, conf} X)$$

is the conformal Killing Laplacian.

Notice that in the above formulation, the sources ϵ and J have been treated as **known** quantities.

Typically, the specific form of the energy-momentum tensor imposes some conformal scaling for these sources. In particular, we say that J is **York-scaled** if

$$J = \phi^{-\frac{2n}{n-2}} \widetilde{J},$$

where $\widetilde{J}$ is constructed with freely **prescribed** data.

Conformal method

In this case, under a CMC assumption, the constraints read as follows.

$$\begin{aligned} &\Delta_\gamma \phi - c_n R_\gamma \phi + c_n |\widetilde{K}|^2_\gamma \, \phi^{-\frac{3n-2}{n-2}} + c_n \left(\frac{1-n}{n} \tau^2 + 2\epsilon \right) \phi^{\frac{n+2}{n-2}} = 0, \\ &\Delta_{\gamma, conf} X = \widetilde{J}. \end{aligned} \tag{9}$$

Notice that we are treating these equations as equations for (ϕ, X) with free parameters

$$(\gamma, \tau, U, \epsilon, \widetilde{J})$$

and the only coupling between the two equations in (9) is through $\widetilde{K}(X)$.

Thus, if the momentum constraint is solvable for some prescribed $\tilde{J}$, then the equations **decouple** and we are left with the study of the Lichnerowicz equation.

Analysis in the CMC case

We will first review the CMC case with York scaled momentum. Here, we suppose M is a **closed** n-dimensional manifold, with $n \geq 3$. The following standard linear result is useful in this context.

Theorem (Conformal Laplacian elliptic properties)

Let (M, γ) be an n-dimensional closed manifold, with $\gamma \in W_2^p$, with $p > \frac{n}{2}$. Then $\Delta_{\gamma,conf} : W_2^p \to L^p$ is Fredholm of index zero. Furthermore, its Kernel coincides with the space of conformal Killing fields (CKF) of γ.

Once we have the **no-CKF** condition the momentum constraint is solvable for any $\widetilde{J} \in L^p$.

Thus, plugging this solution into the Hamiltonian constraint, we get that solving the full system reduces to finding positive solutions for the Lichnerowicz equation.

Analysis in the CMC case

More generally, we can examine the solutions of a non-linear PDE of the form

$$\Delta_\gamma \phi = f(\cdot, \phi) \doteq \sum_I a_I \phi^I, \tag{10}$$

where I denotes the different non-linearities appearing in the appropriate Hamiltonian constraint and $a_I \in L^p$. In this context, we have the following result.

Theorem (Existence for the Lichnerowicz equation)

Consider a closed n-dimensional Riemannian manifold (M, γ), with $n \geq 3$ and $\gamma \in W_2^p$ with $p > \frac{n}{2}$. Consider the equation (10) with the coefficients $a_I \in L^p$. Furthermore, assume that (10) admits a pair of sub and supersolution ϕ_- and ϕ_+, both in W_2^p, satisfying $0 < \ell \leq \phi_- \leq \phi_+ \leq m$, for some positive numbers ℓ, m. Then, equation (10) admits a positive solution.

Analysis in the CMC case

Idea of the proof.

The idea is to construct the solution iteratively by solving a sequence of linear problems. For doing that, we consider the shifted operator

$$\Delta_\gamma - a : W_2^p \to L^p,$$

which is invertible for any choice of $a \in L^p$ and $a > 0$ a.e. Then, given

$$\phi_0 \doteq \phi_-,$$

the sequence $\{\phi_n\}_{n=0}^{\infty} \subset W_2^p$ generated by solving the equations

$$\Delta_\gamma \phi_{n+1} - a\phi_{n+1} = f(\phi_n) - a\phi_n \doteq f_a(\phi_n)$$

is well-defined.

Analysis in the CMC case

Then, we proceed as follows:

- A combination of the maximum principle with an appropriate choice of the function a tailored to guarantee that f_a is a decreasing function on ϕ, for $\ell \leq \phi \leq m$, implies that the sequence is trapped between the barriers, that is, $\phi_- \leq \phi_n \leq \phi_+$.
- Then, standard *a priori* estimates are used to guarantee that the sequence is bounded in W_2^p.
- The compact embedding $W_2^p \hookrightarrow C^0$ guarantees that we can extract a C^0-convergent subsequence, that is, we get $\phi_n \xrightarrow{C^0} \phi$ up to restriction to a subsequence
- Finally, elliptic estimates and the C^0-convergence imply that $\{\phi_n\}$ is actually Cauchy W_2^p, which finishes the proof.

$\square$

Analysis in the CMC case

With this theorem, we conclude that the construction of barriers is enough to solve the Lichnerowicz equation. The existence of barriers is linked to Yamabe invariant according to the following CMC-vacuum classification (Isenberg, 1995):

	$\tau = 0$ $U = 0$	$\tau = 0$ $U \neq 0$	$\tau \neq 0$ $U = 0$	$\tau \neq 0$ $U \neq 0$
$\mathcal{Y}_\gamma > 0$	No	Yes	No	Yes
$\mathcal{Y}_\gamma = 0$	Yes	No	No	Yes
$\mathcal{Y}_\gamma < 0$	No	No	Yes	Yes

Similar classifications can be achieved for the CMC case with some types of energy sources with appropriate scalings (Y. Choquet-Bruhat, J. Isenberg and D. Pollack, 2007). Further results have been obtained for compact manifolds with boundary which allow partial classifications (M. Holst and G. Tsogtgerel, 2013).

Analysis for coupled systems

Concerning the full coupled system some near CMC results were established by means of fixed point and implicit function arguments (J. Isenberg and V. Moncrief, 1996).

The first results that consider far from CMC solutions was established by M. Holst *et al.* in 2008,[1] which was followed by the work of D. Maxwell in 2009. [2]

[1] Holst, M., Nagy, G. and Tsogtgerel G., Commun. Math. Phys., 288: 547 (2009)
[2] D. Maxwell, Math. Res. Lett., 16, no. 4, 627 - 645 (2009).

Analysis in the non-CMC case

The main idea for constructing far from CMC solutions in the above results is to consider York-scaled sources and notice that solutions of the constraint equations in these conditions can be obtained from fixed points of the map

$$\mathcal{F} : W_2^p \mapsto W_2^p,$$
$$\bar{\phi} \mapsto \phi$$

defined by $\mathcal{F} = \mathcal{H} \circ \mathcal{M}$, where the maps $\mathcal{M}$ and $\mathcal{H}$ are the solution maps associated with the momentum constraint for fixed ϕ and for the Hamiltonian constraint for fixed X, respectively. That is,

$$\bar{\phi} \in W_2^p \xrightarrow{\mathcal{M}} X_{\bar{\phi}} \in W_2^p \xrightarrow{\mathcal{H}} \phi_{\bar{\phi}} \in W_2^p$$

Analysis in the non-CMC case

- Assuming no-CKF condition implies that the map $\mathcal{M}$ is well-defined and has nice mapping properties.

- On the other hand, the construction and properties of the map $\mathcal{H}$ are not that immediate. In the papers referred to above, this construction is subtlety different in each case.

- The work of D. Maxwell (2005) on the construction of barriers relies on much more refined versions of the classification theorems we briefly commented above.

Analysis in the non-CMC case

In order to illustrate these points, we quote the following result

Theorem (D. Maxwell)

Let $\gamma \in W_2^p$ with $p > 3$ be a Yamabe-positive metric on a smooth compact 3-manifold. Suppose γ has no conformal Killing fields, $U \in W_1^p$ is a TT tensor, and $\tau \in W_1^p$. If $U \not\equiv 0$ and if $||U||_{C^0}$ is sufficiently small, then there exists a solution $(\phi, X) \in W_2^p$ for the conformal problem.

Analysis in the non-CMC case

- Similar results have been established combining different hypotheses on $\mathcal{Y}_\gamma$, U and τ. Basically, the restrictions appear when constructing the global barriers.

- Similar results have been extended by Dilts *et al.* (2014) and Holst-Meier (2015) to the asymptotically euclidean case.

- Finally, some natural questions arise regarding the method described above. Namely, is it possible to apply these ideas to less restrictive physical situations, for instance to sources which are **not York-scaled**? Or is possible to treat more general systems such as **charged fluids** or Einstein-YM?

- In what follows we will give some contributions to these problems in the context of AE manifolds.

Non-compact case: AE manifolds

Definition (Euclidean at infinity manifolds)

An n-dimensional smooth Riemannian manifold (M, e) is called Euclidean at infinity if there is a compact set K such that $M \backslash K$ is the disjoint union of a finite number of open sets U_i, such that each (U_i, e) is isometric to the exterior of an open ball in Euclidean space.

In what follows, we denote by $W^p_{s,\delta}$ the **weighted** Sobolev spaces of tensor fields on the manifold (M, e) Euclidean at infinity for which

$$(1 + d^2)^{\frac{1}{2}(m+\delta)} D^m u \in L^p, \quad \text{for} \quad 0 \leq m \leq s,$$

in the metric e, where D represents the e-covariant derivative and $d = d(p, \cdot)$ the distance in e to a fixed point p.

Definition (AE manifolds)

Let (M, e) be a manifold euclidean at infinity and g be a Riemannian metric on M. Then we will say that (M, g) is $W^p_{s,\delta}$-AE if $g - e \in W^p_{s,\delta}$ for some $\delta > -\frac{n}{p}$.

Einstein-Maxwell: conformal method

In this context, the Einstein-Maxwell constraint equations are rewritten (after suitable scaling of the sources of the model) as

$$
\begin{aligned}
&\Delta_\gamma \phi = c_n R_\gamma \phi - c_n |\widetilde{K}|^2_\gamma \, \phi^{-\frac{3n-2}{n-2}} - c_n \left(2\epsilon_1 - r_n \tau^2\right) \phi^{\frac{n+2}{n-2}} - 2 c_n \epsilon_2 \, \phi^{-3} \\
&- 2 c_n \epsilon_3 \, \phi^{\frac{n-6}{n-2}}, \\
&\Delta_{\gamma, conf} X = r_n D\tau \, \phi^{\frac{2n}{n-2}} + \omega_1 \, \phi^{2 \frac{n+1}{n-2}} - \omega_2 \\
&\Delta_\gamma f = \widetilde{q} \, \phi^{\frac{2n}{n-2}},
\end{aligned}
\tag{11}
$$

where

$$
\epsilon_1 = \mu \left(1 + |\widetilde{u}|^2_\gamma\right)^{\frac{1}{2}}, \quad 2\epsilon_2 = |df|^2_\gamma + |\vartheta|^2_\gamma, \quad \epsilon_3 = \frac{1}{4} |\widetilde{F}|^2_\gamma,
$$

$$
\omega_1 = \mu \left(1 + |\widetilde{u}|^2_\gamma\right)^{\frac{1}{2}} \widetilde{u}^\flat, \quad \omega_2 = \widetilde{F}(df^\sharp + \vartheta^\sharp, \cdot), \quad \widetilde{q} = q(1 + |\widetilde{u}|^2_\gamma)
$$

Here, f and ϑ come from a Helmholtz decomposition $E^\flat = df + \vartheta$ with $\mathrm{div}_\gamma \, \vartheta = 0$.

Some analytical tools

Lemma (Weak maximum principle)

Consider a Euclidean at infinity manifold M and let ψ be a $W^p_{2,loc}$-solution to the problem

$$\Delta_\gamma \psi - a\psi \leq 0 \quad on \quad M, \tag{12}$$

where $\gamma \in W^p_{2,\delta}$, $p > \frac{n}{2}$, $\delta > -\frac{n}{p}$, $a \in L^p_{\delta+2}$ and $a \geq 0$ a.e. If ψ tends to constants $A_j > 0$ on each end E_j, then $\psi \geq 0$ on M.

Lemma (Strong maximum principle)

Suppose that (M, γ) is a $W^p_{2,\rho}$-AE manifold with $p > \frac{n}{2}$ and $\rho > -\frac{n}{p}$ and let $a \in L^p_{\rho+2}$ with $a \geq 0$. Suppose that $u \in W^p_{2,loc}$ is non-negative and satisfies

$$\Delta_\gamma u - au \leq 0 \tag{13}$$

If $u(x) = 0$ for some $x \in M$, then $u \equiv 0$.

Some analytical tools

Proposition

Let (M,γ) be a $W^p_{2,\rho}$-AE manifold with $p > n$ and $\rho > -\frac{n}{p}$. Consider the operators

$$\begin{aligned} \mathcal{P}_1 : W^p_{2,\delta}(M,\mathbb{R}) &\to L^p_{\delta+2}(M,\mathbb{R}), \\ u &\mapsto \Delta_\gamma u - au, \\ \mathcal{P}_2 : W^p_{2,\delta}(M,TM) &\to L^p_{\delta+2}(M,T^*M), \\ X &\mapsto \Delta_{\gamma,conf} X \end{aligned} \tag{14}$$

where $a \in L^p_{\delta+2}$ is a non-negative function and $\delta > -\frac{n}{p}$. Then, if

$$-\frac{n}{p} < \delta < n - 2 - \frac{n}{p},$$

both operators define isomorphisms between the respective spaces.

Some analytical tools

The next auxiliary result will be used to prove existence of solutions for the Lichnerowicz equation which have some a prescribed asymptotic behaviour on the ends of M.

Lemma

Let (M, g) be a $W^p_{2,\rho}$ asymptotically euclidean manifold, with $\rho > -\frac{n}{p}$ with ends $\{E_j\}_{j=1}^N$ and let $\{A_j\}_{j=1}^N$ be bounded functions on M with $\Delta_\gamma A_j \in L^p_{\delta+2}$ for some $\delta > -\frac{n}{p}$. Then, there is a unique solution to the equation

$$\Delta_\gamma \omega = 0, \tag{15}$$

on M such that ω tends to A_j in E_j as $x \to \infty$ along E_j. Furthermore,

$$\min_j \inf_M A_j \leq \omega \leq \max_j \sup_M A_j. \tag{16}$$

A general result

With the above results at hand, we prescribe the behaviour of the conformal factor at infinity by choosing a set of constants $\{A_j\}_{j=1}^N$ and a harmonic function ω asymptotic to A_j on each end E_j and writing

$$\phi = \omega + \varphi, \quad \text{with} \quad \varphi \in W^p_{2,\delta}.$$

Now, consider the linear operator

$$\begin{aligned} &\mathcal{P} : W^p_{2,\delta} \to L^p_{\delta+2}, \\ &(\varphi, f, X) \mapsto (\Delta_\gamma \varphi, \Delta_\gamma f, \Delta_{\gamma,conf} X) \end{aligned} \tag{17}$$

and denote by $\mathbf{F}$ the map taking $(\phi, f, X) \mapsto \mathbf{F}(\phi, f, X)$, where $\mathbf{F}(\phi, f, X)$ stands for the *polynomial* appearing in the right hand side of (11).

In this setting, we rewrite the system (11) more compactly as

$$\mathcal{P}(\psi) = \mathbf{F}(\psi), \tag{18}$$

where $\psi = (\varphi, f, X) \in W^p_{2,\delta}$.

A fixed point argument

At this point, the idea is to solve the above problem by solving a sequence of linear problems. In particular, given $\psi_0 \in W^p_{2,\delta}$, if we get a unique solution for

$$\mathcal{P}(\psi) = \mathbf{F}(\psi_0),$$

which is given by $\psi_1 = \mathcal{P}^{-1}F(\psi_0)$, we can begin an iteration scheme, where we could now use ψ_1 as a source and solve the linear problem for this source and begin an iteration procedure.

A fixed point in this iterative process solves the problem (18).

A fixed point argument

In this procedure, we will need the following stronger notion of barriers.

Definition

We say that ϕ_- is a **strong global subsolution** for the Hamiltonian constraint associated with a charged fluid, if there are numbers $M_f, M_X > 0$ such that

$$\Delta_\gamma \phi_- \geq \sum_I a_I(f, X)\phi_-^I \quad \text{for all} \quad f \in B_{M_f}, \quad X \in B_{M_X},$$

where $B_{M_f}, B_{M_X} \subset W_{2,\delta}^p$ stand for the closed balls of radii M_f and M_X respectively. Strong global supersolutions are defined analogously.

Now, assuming the same kind of hypotheses as Holst *et al.* and Maxwell and Dilts *et al.*, we can establish the following result concerning the charged fluid Einstein constraint equations.

Main result

Theorem

Let (M,γ) be a $W^p_{2,\delta}$-AE manifold, with $p > n$ and $\delta > -\frac{n}{p}$, and consider the constraint equations for a charged fluid given by (11) on M. Assume that the Hamiltonian constraint admits a compatible pair of strong global sub and super-solutions given by ϕ_- and ϕ_+, which are, respectively, asymptotic to harmonic functions $\omega_\pm$ tending to positive constants $\{A^\pm_j\}_{j=1}^N$ on each end $\{E_j\}_{j=1}^N$. Fix a harmonic function ω asymptotic to constants $\{A_j\}_{j=1}^N$ on each end satisfying $0 < A^-_j \leq A_j \leq A^+_j$ and suppose that the solution map

$$\mathcal{F} : W^p_{2,\delta}(M;E) \mapsto W^p_{2,\delta}(M;E),$$
$$\psi = (\varphi, f, Y) \mapsto \mathcal{F}(\psi) \doteq \mathcal{P}^{-1} \circ \boldsymbol{F}(\psi).$$

preserves the balls $B_{M_f}, B_{M_X} \subset W^p_{2,\delta}$ whenever $\phi_- \leq \omega + \varphi \leq \varphi_+$. Then, the system admits a solution $(\phi = \omega + \varphi, f, X)$ with $(\varphi, f, X) \in W^p_{2,\delta}$ and $\phi > 0$.

Idea of the proof .

- First introduce a shift in the Hamiltonian constraint (as in the Isenberg's proof), which is chosen so as to make the modified right hand side a decreasing function on φ.
- Define $\varphi_0 \doteq \phi_- - \omega$, take $f_0 \in B_{M_f}$ and $X_0 \in B_{M_X}$ and use the linear properties of the maps $\mathcal{P}_1$ and $\mathcal{P}_2$ to produce the sequence $\{\phi_n = \omega + \varphi_n, f_n, X_n\}$.
- Use the maximum principle, the choice of shift function and the existence of strong global barriers to show that $\phi_- \leq \phi_n \leq \phi_+$.
- Use elliptic estimates to prove that the sequence $\{\varphi_n\} \subset W^p_{2,\delta}$ is bounded.
- Use the compact embedding $W^p_{2,\delta} \hookrightarrow W^p_{1,\delta'}$, with $\delta > \delta'$ to extract a convergent subsequence $\{\phi_n = \varphi_n + \omega, f_n, X_n\}$.
- Use elliptic estimates to improve convergence.

Searching for barriers

This general result shows that the problem of finding solutions of the charged fluid constraint equations becomes the problem of finding strong global barriers for this system.

Following ideas of Holst and Meier and Dilts *et al.*, we can produce such barriers under smallness assumptions on the free parameters.

We then get the following results in two distinct scenarios.

Some results

Theorem (Far from CMC - $\mathcal{Y}_g > 0$)

Let (M, γ) be a $W^p_{2,\delta}$-Yamabe positive AE manifold, with $p > n$ and $\delta > -\frac{n}{p}$. Consider $\tau \in W^p_{1,\delta+1}(M, \mathbb{R})$, $U \in W^p_{1,\delta+1}(M, T^0_2 M)$, $\widetilde{F} \in W^p_{1,\delta+1}(M, \Lambda^2 TM)$, $\mu \in L^p_{\delta+2}(M, \mathbb{R})$, $\vartheta \in W^p_{1,\delta+1}(M, T^*M)$ and $\widetilde{q} \in L^p_{\delta+2}(M, \mathbb{R})$. If $U, \widetilde{F}, \mu, \vartheta$ and $\widetilde{q}$ are sufficiently small, there is a $W^p_{2,\delta}$-solution to the conformal problem (11).

Theorem (Near CMC - $\mathcal{Y}_g \leq 0$)

Let (M, γ) be a $W^p_{2,\delta}$-AE manifold, with $p > n$ and $\delta > -\frac{n}{p}$. Consider $\tau \in W^p_{1,\delta+1}(M, \mathbb{R})$, $U \in W^p_{1,\delta+1}(M, T^0_2 M)$, $\widetilde{F} \in W^p_{1,\delta+1}(M, \Lambda^2 TM)$, $\mu \in L^p_{\delta+2}(M, \mathbb{R})$, $\vartheta \in W^p_{1,\delta+1}(M, T^*M)$ and $\widetilde{q} \in L^p_{\delta+2}(M, \mathbb{R})$. If $U, \widetilde{F}, \mu, \vartheta, \widetilde{q}, ||D\tau||$ are sufficiently small and $c_n R_\gamma + b_n \tau^2 \geq 0$ on the subset where $R_\gamma < 0$, there is a $W^p_{2,\delta}$-solution to the conformal problem (11).

Boundary conditions

The above results can be extended to AE manifolds with suitable boundary conditions, for instance modelling (marginally) trapped surfaces. The conformal formulation of such conditions is given by

$$\hat{\nu}(\phi) = -a_n H\phi + (d_n\tau + a_n\theta_-)\phi^{\frac{n}{n-2}} + a_n\left(\frac{1}{2}|\theta_-| - r_n\tau\right) v^{\frac{2n}{n-2}}\phi^{-\frac{n}{n-2}},$$

$$\pounds_{\gamma,conf} X(\hat{\nu},\cdot) = -\left(\left(\frac{1}{2}|\theta_-| - r_n\tau\right) v^{\frac{2n}{n-2}} + U(\hat{\nu},\hat{\nu})\right)\hat{\nu},$$

$$\langle \widetilde{E}, \hat{\nu}\rangle_\gamma = E_{\hat{\nu}},$$

where $\hat{\nu}$ is the outward pointing γ-unit normal to ∂M; $H = \mathrm{div}_\gamma \hat{\nu}$ is the mean curvature of ∂M as an embedded hypersurface of (M,γ), taken with respect to $-\hat{\nu}$; θ_- denotes one of the *expansion coefficients* on ∂M and the data v, θ_- and τ is supposed to satisfy the following condition along the boundary

$$\frac{1}{2}|\theta_-| - r_n\tau \geq 0,$$

$$v \geq \phi.$$

Final comments

- All of the above is not limited to the constraint equations for charged fluids. The main existence theorem encompasses a large class of semi-linear elliptic systems. Clearly, the construction of the barriers has to be adapted to each particular case.

- These results are obtained as a test case for applying the method for a system of fourth-order equations that we are considering as a version of the field equations to quadratic functionals of the curvature.

Final comments

Indeed, we are currently considering functionals as

$$\mathcal{A}[\bar{g}] = \int \alpha R_{\bar{g}}^2 + \beta |Ric_{\bar{g}}|_{\bar{g}}^2$$

whose Euler-Lagrange equations has the form

$$A_{\bar{g}} \doteq -\beta \, \Box Ric_{\bar{g}} - \left(2\alpha + \frac{1}{2}\beta\right) \Box R_{\bar{g}} \bar{g} + \ldots = 0$$

The very first attempt has been to consider it in terms of a (necessarily) coupled system of second order equations.

Thank you for your attention!