About Moser-Trudinger equations with general nonlinearities

"Recent trends in Geometric analysis and applications"

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Joint work with Gabriele Mancini.

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$\Longrightarrow$ $f : x \mapsto \exp(x^2)$ is a "good" candidate for (Q).

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Let us recall where this "4π" comes from

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- If $\int_{\mathbb{D}^2} |\nabla u|^2 dx \le 4\pi$, by denoting $u(x) = u(|x|)$, we have that

$$u(t)^2 \le \left(\int_t^1 u'(\tau)d\tau\right)^2 \le \int_t^1 2\pi\tau u'(\tau)^2 d\tau \int_t^1 \frac{d\tau}{2\pi\tau} \le \frac{4\pi}{2\pi}\log\frac{1}{t}\,,$$

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- and if $\rho < 1$, we get that

$$\begin{aligned}\int_{\mathbb{D}^2} \exp(\rho u^2) dx &\leq \int_0^1 2\pi\tau \exp\left(2\rho \log \frac{1}{\tau} \right) d\tau \\ &\leq 2\pi \int_0^1 \tau^{1-2\rho} d\tau \leq \frac{2\pi}{2-2\rho}.\end{aligned}$$

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Reformulation: If $0 < \beta \leq 4\pi$, the results of Moser and those of Carleson-Chang/Struwe/Flucher give that the supremum

$$\sup_{\{u \in H_0^1, \|u\|^2 = \beta\}} \underbrace{\int_\Omega \exp(u^2)dx}_{:=\Phi_0(u)},$$

is finite and attained by some $u_\beta \geq 0$.

The associated Euler-Lagrange equation

Indeed, u_β satisfies, for some $\lambda \in \mathbb{R}$,

$$\begin{cases} \Delta u = \lambda u \exp(u^2), u > 0 \text{ in } \Omega \ , \\ u = 0 \text{ on } \partial\Omega \ , \\ \beta := \int_\Omega |\nabla u|^2 dx \ , \end{cases} \qquad (MT_0^\beta)$$

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- in annuli, for β slightly larger than $4\pi N$, $N \in \mathbb{N}^\star$: perturbative approach via the finite dimensional reduction method, "Lyapunov-Schmidt" (del Pino-Musso-Ruf (12)).

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  A key point: getting the precise expansion of the energy of blowing-up solutions of (MT_0^β). This energy approaches 4π from above.
- in any non-simply connected domain Ω and for all $\beta > 0$ (with Druet-Malchiodi-Martinazzi).

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- Not without similarity with Chang-Chen-Lin (03) giving a n.s.c. for a mean field type inequality to have an extremal at the first critical level.

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Remark: this parameter λ encodes in some sense less information than the parameter β. For $h \equiv 0$ for instance, $\lambda \to 0$ irrespective of whether u develops 1,2,3..."bubbles".

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\text{c)} & h(\gamma - \frac{t}{\gamma}) = h(\gamma) - \frac{t}{\gamma} h'(\gamma) + O\left(\frac{t^2}{\gamma^3}|h'(\gamma)| + \frac{t_+}{\gamma^2}\right)\,, \\
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General assumptions on h:

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- Given $(\beta_1,\beta_2,\kappa)\in[-2,2]\times\mathbb{R}^2$, an even smooth function h s.t.

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what happens as $\gamma \to +\infty$?

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Assume $L_\infty = +\infty$. *Given any* $\gamma > 0$, *there exists a unique* $(\lambda, u) = (\lambda_\gamma, u_\gamma)$ *solving* (MT_h^{rad}). *Moreover, for all integer* $k \geq 1$, *there exists a family* $(\nu_{k,\gamma})_{\gamma>0}$ *of numbers in* $(0,1)$ *such that, if* $\Lambda_{k,\gamma} > 0$ *is given by* $\Lambda_{k,\gamma} = u_\gamma(\nu_{k,\gamma})$, *we have that* $\lim_{\gamma\to+\infty} \frac{\nu_{k,\gamma}}{\nu_{k+1,\gamma}} = 0$ *and that*

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for all $k \geq 1$, *where* $\Omega_{k,\gamma} = \begin{cases} B_0(\nu_{k+1,\gamma})\backslash B_0(\nu_{k,\gamma}) \text{ if } k\geq 2 \\ B_0(\nu_{k+1,\gamma}) \text{ if } k = 1 \end{cases}$.

In particular, we have that

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- if $L_\infty > 0$, we know that $u_0(x) = L_\infty$ at any blow-up point x (solutions constructed on $\mathbb{D}^2$ in Mancini-T. (18)).

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Let $(\gamma_\alpha)_\alpha$ *be diverging sufficiently slowly to* $+\infty$*, namely* $\log\log\gamma_\alpha = O\left(\log\frac{1}{\alpha}\right)$: *up to a subsequence, assume that* $b_j := \frac{1}{2}\lim_{\alpha\to 0^+} a\gamma_\alpha^{\alpha^j} \in [a/2, +\infty]$ *exists for all* $j \in \mathbb{N}^\star$ *and set*

$$N := \min\left\{j \in \mathbb{N}^\star \ s.t. \ b_j < +\infty\right\} .$$

There exists $(\lambda_\alpha, u_\alpha)_\alpha$ *solving* $(MT^{rad}_{h_\alpha,\gamma_\alpha})$. *Moreover there exists* $u_0 \in C^2_0$ *solving the limiting equation such that* $u_0(0) = b_N \geq a/2$ *and such that the quantization holds true*

Threshold case "$L_\infty \to +\infty$": for all $0 < \alpha \ll 1$, consider

$$\begin{cases} \Delta u_\alpha = \lambda_\alpha u_\alpha \exp(u_\alpha^2 + h_\alpha(u_\alpha)), u_\alpha > 0 \text{ in } \mathbb{D}^2 \ , \\ u_\alpha = 0 \text{ on } \partial\mathbb{D}^2 \ , \\ u_\alpha(0) := \gamma_\alpha \ , \end{cases} \qquad (MT^{rad}_{h_\alpha,\gamma_\alpha})$$

where $h_\alpha = -at^{1+\alpha}, t \geq T$ ($T \gg 1, a > 0$ given) and $\lim_{\alpha\to 0} h_\alpha = h_0$ in $C^1_{loc}(\mathbb{R})$.

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Thank you!