Strichartz estimates and Fourier restriction theorems in the Heisenberg group

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Dispersive and sub-elliptic PDEs
Pisa, Italy

February 10-12, 2020

Joint work with

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→ Main references:

BBG-19 H.Bahouri, DB, I.Gallagher,
Strichartz estimates and Fourier restriction theorems in the Heisenberg group, Preprint ArXiv, 2019

Outline

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The Schrödinger equation on $\mathbb{R}^n$

The Schrödinger equation on $\mathbb{R}^n$

$$\begin{cases} i\partial_t u - \Delta u = 0 \\ u_{|t=0} = u_0 \,, \end{cases}$$

From the explicit expression of the solution, using Fourier analysis:

$$u(t,\cdot) = \frac{\mathrm{e}^{i\frac{|\cdot|^2}{4t}}}{(4\pi i t)^{\frac{n}{2}}} \star u_0 \,.$$

one obtains the basic dispersive estimate (for $t \neq 0$)

$$\|u(t,\cdot)\|_{L^\infty(\mathbb{R}^n)} \leq \frac{1}{(4\pi|t|)^{\frac{n}{2}}} \|u_0\|_{L^1(\mathbb{R}^n)} \tag{1}$$

Strichartz estimates

For initial data $u_0 \in L^2(\mathbb{R}^n)$ we have the following Strichartz estimate

$$\|u\|_{L^q(\mathbb{R},L^p(\mathbb{R}^n))} \leq C_{p,q}\|u_0\|_{L^2(\mathbb{R}^n)}\,, \tag{2}$$

where (p,q) satisfies the scaling admissibility condition

$$\frac{2}{q}+\frac{n}{p}=\frac{n}{2}, \qquad q \geq 2,\ (n,q,p) \neq (2,2,\infty)$$

The dispersive inequality also yields the following Strichartz inequalities for the inhomogeneous Schrödinger equation $i\partial_t u - \Delta u = f$

$$\|u\|_{L^q(\mathbb{R},L^p(\mathbb{R}^n))} \leq C_{p,q,p_1,q_1}\left(\|u_0\|_{L^2(\mathbb{R}^n)} + \|f\|_{L^{q_1'}(\mathbb{R},L^{p_1'}(\mathbb{R}^n))}\right), \tag{3}$$

- (p,q) and (p_1,q_1) satisfy the admissibility condition
- a' the dual exponent of any $a \in [1,\infty]$.
- crucial in the study of semilinear and quasilinear Schrödinger equations

Notation in Heisenberg

The d-dimensional Heisenberg group $\mathbb{H}^d = \mathbb{C}^d \times \mathbb{R}$

$$w \cdot w' = (z, s) \cdot (z', s') = \left(z + z', s + s' - 2\Im\left(z\overline{z}'\right)\right),$$

$$\Im\left(z\overline{z}'\right) = \langle x, y'\rangle - \langle x', y\rangle$$

- Convolution product for integrable functions f and g:

$$f \star g(w) = \int_{\mathbb{H}^d} f(w \cdot v^{-1}) g(v)\, dv = \int_{\mathbb{H}^d} f(v) g(v^{-1} \cdot w)\, dv\,.$$

- the anisotropic Lebesgue spaces

$$\|f\|_{L^p_z L^r_s} = \left(\int \left(\int |f(z,s)|^r\, ds\right)^{\frac{p}{r}} dz\right)^{\frac{1}{p}}.$$

- the family of dilation operators $(\delta_a)_{a>0}$ (compatible with (**??**)) defined by

$$\delta_a(z, s) = (az, a^2 s)\,.$$

- the homogeneous dimension of $\mathbb{H}^d$ to be $Q = 2d + 2$.

Notation in Heisenberg, II

- the horizontal vector fields X_j and Y_j are defined for $j \in \{1, \dots, d\}$ by

$$X_j = \partial_{x_j} + 2y_j\partial_s, \qquad Y_j = \partial_{y_j} - 2x_j\partial_s \,.$$

- The sublaplacian

$$\Delta_{\mathbb{H}} = \sum_{j=1}^{d} X_j^2 + Y_j^2 \,,$$

- Complex notations $Z_j = X_j + iY_j$

$$\Delta_{\mathbb{H}} u = \sum_{j=1}^{d} Z_j \overline{Z}_j - i4d\partial_s,$$

Remark (on Shrödinger equation in $\mathbb{H}$)

$$i\partial_t u - \Delta_{\mathbb{H}} u = 0 \quad \Leftrightarrow \quad i(\partial_t + 4d\partial_s)u = \sum_{j=1}^{d} Z_j \overline{Z}_j u$$

No dispersion in Heisenberg

The linear Schrödinger equations on $\mathbb{H}^d$ associated with the sublaplacian

$$(S_{\mathbb{H}}) \quad \begin{cases} i\partial_t u - \Delta_{\mathbb{H}} u = f \\ u_{|t=0} = u_0 \,, \end{cases}$$

$(S_{\mathbb{H}})$ is a model for totally non-dispersive evolution equations.

Theorem

There exists a function u_0 in the Schwartz class $\mathcal{S}(\mathbb{H}^d)$ such that the solution to the free Schrödinger equation $(S_{\mathbb{H}})$ (with $f \equiv 0$) satisfies

$$u(t,z,s) = u_0(z, s + 4td)\,.$$

In particular for all $1 \leq p \leq \infty$

$$\|u(t,\cdot)\|_{L^p(\mathbb{H}^d)} = \|u_0\|_{L^p(\mathbb{H}^d)}$$

$\rightarrow$ One cannot hope for a dispersion phenomenon

- Goal: prove (some) Strichartz estimates in the Heisenberg group

$\rightarrow$ anisotropic norms due to the no-dispersion effect

$\rightarrow$ the original approach of Strichartz, 1977

Vol. 44, No. 3 DUKE MATHEMATICAL JOURNAL© September 1977

RESTRICTIONS OF FOURIER TRANSFORMS TO QUADRATIC SURFACES AND DECAY OF SOLUTIONS OF WAVE EQUATIONS

ROBERT S. STRICHARTZ

§1. Introduction

Let S be a subset of $\mathbb{R}^n$ and $d\mu$ a positive measure supported on S and of temperate growth at infinity. We consider the following two problems:

Problem A. For which values of p, $1 \leq p < 2$, is it true that $f \in L^p(\mathbb{R}^n)$ implies $\hat{f}$ has a well-defined restriction to S in $L^2(d\mu)$ with

$$\left(\int |\hat{f}|^2 d\mu\right)^{1/2} \leq c_p \|f\|_p? \tag{1.1}$$

- The Fourier dual of $\mathbb{R}^n$ is $\mathbb{R}^n$
- The Fourier dual of $\mathbb{H}^d$ is not $\mathbb{H}^d$

Outline

Fourier restriction problem [Stein, Fefferman, Tomas, etc]

Problem: Can we restrict the Fourier transform of an L^p function to a subset ?

- f in $L^1(\mathbb{R}^n)$ implies $\mathcal{F}(f)$ continuous $\to$ OK.
- f in $L^2(\mathbb{R}^n)$ implies $\mathcal{F}(f)$ in $L^2(\widehat{\mathbb{R}}^n)$ $\to$ arbitrary on a zero meas set $\widehat{S}$ of $\widehat{\mathbb{R}}^n$.

$\to$ The Fourier transform of a L^p function, for any $p > 1$, cannot be restricted to hyperplanes.

$$f(x) = \frac{e^{-|x'|^2}}{1+|x_1|} \qquad x = (x_1, x') \in \mathbb{R}^n, \tag{4}$$

belongs to $L^p(\mathbb{R}^n)$, for all $p > 1$, but its Fourier transform does not admit a restriction on the hyperplane $\widehat{S} = \{\xi_1 = 0\}$.

Tomas and Stein

One can restrict the Fourier transform of $L^p(\mathbb{R}^n)$ functions, for $p > 1$ (close to 1), to hypersurfaces $\widehat{S}$ that are "sufficiently curved", (main example: the sphere).

Problem: given a hypersurface $\widehat{S} \subset \widehat{\mathbb{R}}^n$ endowed with a smooth measure $d\sigma$, the restriction problem asks for which pairs (p, q) an inequality of the form

$$\|\mathcal{F}(f)|_{\widehat{S}}\|_{L^q(\widehat{S}, d\sigma)} \leq C\|f\|_{L^p(\mathbb{R}^n)} \tag{5}$$

holds for all f in $\mathcal{S}(\mathbb{R}^n)$.

- The operator R_S is continuous from $L^p(\mathbb{R}^n)$ to $L^q(\widehat{S}, d\sigma)$?

$$R_S f = \mathcal{F}(f)|_{\widehat{S}}$$

$\rightarrow$ not completely settled in its general form (we focus on the case $q = 2$).

- For $q = 2$: the adjoint operator R_S^* is continuous from $L^2(\widehat{S}, d\sigma)$ to $L^{p'}(\mathbb{R}^n)$?

$$R_S^* g = \mathcal{F}^{-1}(g d\sigma)$$

$$\|\mathcal{F}^{-1}(g d\sigma)\|_{L^{p'}(\mathbb{R}^n)} \leq C\|g\|_{L^2(\widehat{S}, d\sigma)} \tag{6}$$

the case $q = 2$

A basic counterexample shows that the range of p for which the estimate holds cannot be the entire interval $1 \leq p \leq 2$;

Example (Knapp)

Let $\widehat{S}$ be the $(n-1)$-dimensional sphere in $\widehat{\mathbb{R}}^n$ endowed with the standard measure $d\mu$. The estimate can hold only if $p \leq \dfrac{2n+2}{n+3} = 2 - \dfrac{4}{n+3}$.

- Consider the equivalent formulation of the estimate

$$\|\widehat{g\sigma}\|_{L^{p'}(\mathbb{R}^n)} \leq C\|g\|_{L^2(S^{n-1})} \tag{7}$$

- Let $\delta > 0$ and let g_δ be the characteristic function on the spherical cap

$$\widehat{C}_\delta = \{x \in \widehat{S} : |x \cdot e_n| < \delta\}.$$

Proof of Knapp, I

- We consider the equivalent formulation of estimate

$$\|\widehat{g\sigma}\|_{L^{p'}(\mathbb{R}^n)} \leq C\|g\|_{L^2(S^{n-1})} \tag{8}$$

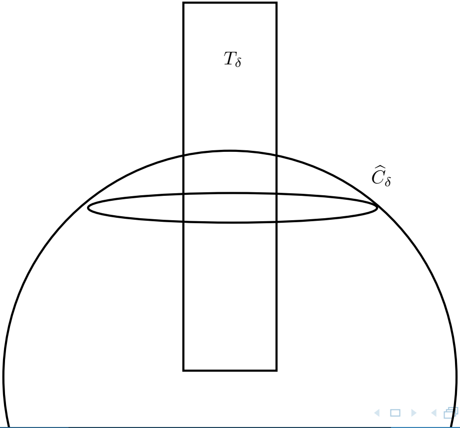
- Let $\delta > 0$ be small and let g_δ be the characteristic function on C_δ.
- $|C_\delta| \sim \delta^{n-1}$. This implies $\|g_\delta\|_{L^2(S^{n-1})} \sim \delta^{(n-1)/2}$.
- If $x \in \mathbb{R}^n$ is orthogonal to the vertical direction

$$|\widehat{g_\delta\sigma}(x)| = \left|\int_{S^{n-1}} e^{ix\cdot\xi} g_\delta(\xi)d\sigma(\xi)\right| = \left|\int_{C_\delta} e^{ix\cdot\xi} d\sigma(\xi)\right| \sim |C_\delta| \sim \delta^{n-1}.$$

$$\|\widehat{g_\delta\sigma}\|_{L^{p'}(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |\widehat{g\sigma}(x)|^{p'} dx\right)^{1/p'}$$

Geometric interpretation

- let T_δ be the tube in the x space oriented orthogonally to the sphere
$$[-\delta^{-1}, \delta^{-1}] \times \ldots \times [-\delta^{-1}, \delta^{-1}] \times [-\delta^{-2}, \delta^{-2}]$$
- $|T_\delta| \sim \delta^{-n-1}$.



Proof of Knapp, II

- For x in T_δ and δ very small the quantity $x \cdot \xi$ is almost zero for $\xi \in C_\delta$.

$$\|\widehat{g_\delta\sigma}\|_{L^{p'}(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |\widehat{g\sigma}(x)|^{p'} dx\right)^{1/p'} \tag{9}$$

$$\geq \left(\int_{T_\delta} |\widehat{g\sigma}(x)|^{p'} dx\right)^{1/p'} \tag{10}$$

$$\sim \left(\int_{T_\delta} \delta^{(n-1)p'} dx\right)^{1/p'} \tag{11}$$

$$\sim \delta^{(n-1)}|T_\delta|^{1/p'} \sim \delta^{(n-1)}\delta^{(-n-1)/p'} \tag{12}$$

The estimate can hence be valid only if (the inequality is $\geq$ since $\delta \to 0$)

$$n - 1 - \frac{n+1}{p'} \geq \frac{n-1}{2}$$

which is the conclusion.

Tomas-Stein

The above range is indeed the correct one in the case of a surface with non vanishing curvature.

Theorem (Tomas-Stein, 1975)

Let $\widehat{S}$ be a smooth compact hypersurface in $\widehat{\mathbb{R}}^n$ with non vanishing Gaussian curvature at every point, and let $d\sigma$ be a smooth measure on $\widehat{S}$. Then

$$\|\mathcal{F}(f)|_{\widehat{S}}\|_{L^2(\widehat{S},d\sigma)} \leq C_p \|f\|_{L^p(\mathbb{R}^n)} .$$

for every $f \in \mathcal{S}(\mathbb{R}^n)$ and every $p \leq (2n+2)/(n+3)$,

- A similar result is possible for surfaces with vanishing Gaussian curvature (that are not flat).
- In this case the range of p is smaller depending on the order of tangency of the surface to its tangent space.
- The assumption about compactness of $\widehat{S}$ can be removed by replacing $d\sigma$ with a compactly supported smooth measure.

Idea of the proof

Equivalent to the continuity from $L^p(\mathbb{R}^n)$ to $L^{p'}(\mathbb{R}^n)$ of the operator

$$R_S^* R_S f = f * \widehat{\sigma} \tag{13}$$

$$\|\mathcal{F}(f)|_{\widehat{S}}\|^2_{L^2(\widehat{S},d\sigma)} = \int (f * \widehat{\sigma}) f dx \leq \|f * \widehat{\sigma}\|_{L^{p'}(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)}$$

Recall that the Fourier transform of the measure $d\sigma$ is a function given by

$$\widehat{\sigma}(\xi) = \int_{\mathbb{R}^n} e^{ix\cdot\xi} d\sigma(x) \tag{14}$$

Let S be a smooth compact hypersurfaces with non-zero Gaussian curvature at every point. Then

$$|\widehat{\sigma}(\xi)| \leq C(1+|\xi|)^{-\frac{n-1}{2}} \tag{15}$$

- If S not compact $\to$ $d\sigma$ is only a positive Radon measure (not finite).
- one can consider its truncation $\mu = \varphi d\sigma$ with φ compact support

From restriction to Strichartz estimates

The classical Schrödinger equation in $\mathbb{R}^n$: taking the inverse Fourier transform

$$u(t,x) = \int_{\widehat{\mathbb{R}}^n} e^{i(x\cdot\xi + t|\xi|^2)} \widehat{u}_0(\xi) d\xi \,. \tag{16}$$

Consider the paraboloid $\widehat{S}$ in the space of frequencies $\widehat{\mathbb{R}}^{n+1} = \widehat{\mathbb{R}} \times \widehat{\mathbb{R}}^n$

$$\widehat{S} = \left\{ (\alpha, \xi) \in \widehat{\mathbb{R}} \times \widehat{\mathbb{R}}^n \mid \alpha = |\xi|^2 \right\}.$$

- Given $\widehat{u}_0 : \widehat{\mathbb{R}}^n \to \mathbb{C}$ define $g : \widehat{S} \to \mathbb{C}$ as $g(|\xi|^2, \xi) = \widehat{u}_0(\xi)$. Then

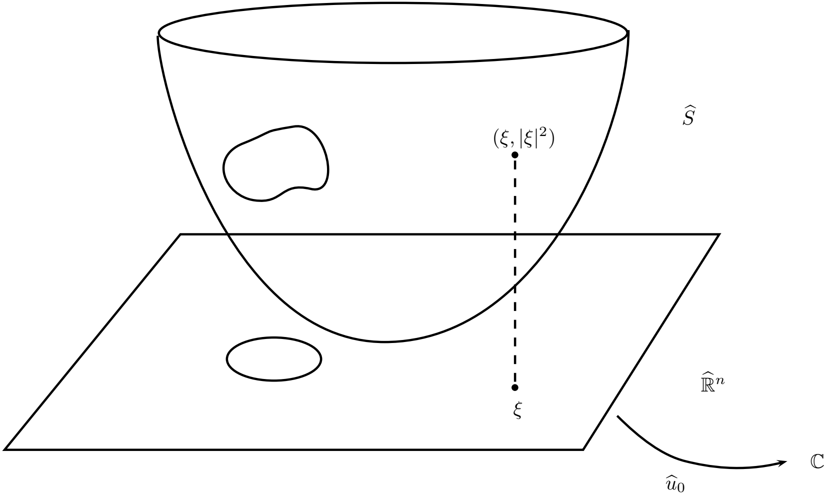
$$u(t,x) = \int_{\mathbb{R}^n} e^{i(x\cdot\xi + t|\xi|^2)} \widehat{u}_0(\xi) d\xi = \int_{\widehat{S}} e^{iy\cdot z} g(z) d\sigma(z)$$

where $y = (t,x)$ and $z = (\alpha, \xi)$.

- Let us endow $\widehat{S}$ with the measure $d\sigma = d\xi$.

$\rightarrow$ $d\sigma$ is not the intrinsic surface measure of $\widehat{S}$, which is $d\mu = \sqrt{1 + 2|\xi|} d\xi$.

Geometric interpretation



The Fourier restriction theorem

$$\|\mathcal{F}^{-1}(gd\sigma)\|_{L^{p'}(\widehat{\mathbb{R}}^{n+1})} \leq C_p \|g\|_{L^2(\widehat{S},d\mu)}, \tag{17}$$

for all $g \in L^2(\widehat{S}, d\mu)$ and all $p' \geq 2(n+2)/n$.

By construction $\|g\|_{L^2(\widehat{S},d\mu)} = \|\widehat{u}_0\|_{L^2(\widehat{\mathbb{R}}^n)} = \|u_0\|_{L^2(\mathbb{R}^n)}$

$\rightarrow$ we stress that we apply the result in dimension $n+1$, i.e., in $\mathbb{R} \times \mathbb{R}^n = \mathbb{R}^{n+1}$

Applying the statement to g related to a initial data u_0 such that $\widehat{u}_0$ is supported on a unit ball

$$\|u\|_{L^{p'}(\mathbb{R}^{n+1})} \leq C\|u_0\|_{L^2(\mathbb{R}^n)}, \tag{18}$$

for all $p' \geq 2(n+2)/n$.

A scaling argument and the density of spectrally localized functions in $L^2(\mathbb{R}^n)$, give the result for $p' = 2 + \frac{4}{n}$. and all $u_0 \in L^2(\mathbb{R}^n)$

Some difficulties

1. Prove a Fourier restriction on the Heisenberg group

- a result of D.Müller $\rightarrow$ specific for the sphere
- what is the sphere? what about paraboloid?

2. We do not exactly need restriction theorems for $\mathbb{H}^d$

- we applied the result to a surface in the space $\mathbb{R}^{n+1} = \mathbb{R} \times \mathbb{R}^n$

$\rightarrow$ the paraboloid for the Schrödinger eq. (the cone for the wave equation).

- when dealing with equations defined on the Heisenberg group $\mathbb{H}^d$, one is naturally lead to consider surfaces in the space $\widehat{\mathbb{R}} \times \widehat{\mathbb{H}}^d$, which is not related to $\mathbb{H}^{d'}$ for some d'.

Outline

The result

A function f on $\mathbb{H}^d$ is said to be *radial* if $f(z,s) = f(|z|,s)$.

Theorem (Bahouri, DB, Gallagher, '19)

Given (p,q) *and* (p_1,q_1) *belonging to the admissible set*

$$\mathcal{A} = \left\{ (p,q) \in [2,\infty]^2 \,/\, q \le p \quad \textit{and} \quad \frac{2}{q} + \frac{2d}{p} = \frac{Q}{2} \right\},$$

the solution to the Schrödinger equation $(S_{\mathbb{H}})$ *with radial data satisfies*

$$\|u\|_{L^\infty_s L^q_t L^p_z} \le C_{p,q,p_1,q_1} \left(\|u_0\|_{L^2(\mathbb{H}^d)} + \|f\|_{L^1_s L^{q_1'}_t L^{p_1'}_z} \right).$$

- a' the dual exponent of any $a \in [1,\infty]$,
- we stress that $L^\infty_s\ L^q_t\ L^p_z \neq L^\infty_t\ L^q_s\ L^p_z$
- for $p = q$ then we get a L^2 estimate (anisotropic)

Switching s and t

- Switching s and t in the proofs one gets the following Strichartz estimates
- a reduced range of indices due to the presence of a velocity coefficient with respect to the variable s

More precisely there holds

$$\|u\|_{L_t^\infty L_s^q L_z^p} \leq C_{p,q,p_1,q_1} \left(\|u_0\|_{L^2(\mathbb{H}^d)} + \|f\|_{L_t^1 L_s^{q_1'} L_z^{p_1'}} \right),$$

for (p,q) and (p_1,q_1) belonging to the admissible set

$$\widetilde{\mathcal{A}} = \left\{ (p,q) \in [2,\infty]^2 \,/\, q \leq p \quad \text{with} \quad \frac{1}{p} + \frac{1}{q} < \frac{1}{2} \quad \text{and} \quad \frac{2}{q} + \frac{2d}{p} = \frac{Q}{2} \right\}.$$

- it is necessary to have at least one L^∞ among t or s
- this inequality is weaker (for instance no $p = q$ case)

Fourier on Heisenberg: a quick view

- The Heisenberg group is noncommutative $\rightarrow$ Fourier transform using irreducible representations of $\mathbb{H}^d$
- the Heisenberg Fourier transform $\mathfrak{F}_{\mathbb{H}} f(\lambda)$ is a family of HS operators on $L^2(\mathbb{R}^d)$

$\rightarrow$ Fourier transform in terms of a frequency set?

- projecting $\mathfrak{F}_{\mathbb{H}}(\lambda)$ onto the o.n. basis of $L^2(\mathbb{R}^d)$ given by Hermite functions.

$$\mathcal{F}_{\mathbb{H}} f(\widehat{w}) = \int_{\mathbb{H}^d} \overline{e^{is\lambda} \mathcal{W}(\widehat{w}, z)}\, f(z,s)\, dz\, ds\,,$$

for any $\widehat{w} = (n, m, \lambda)$ in $\widetilde{\mathbb{H}}^d = \mathbb{N}^{2d} \times \mathbb{R} \setminus \{0\}$, with $\mathcal{W}$ the Wigner transform

$$\mathcal{W}(\widehat{w}, z) = \int_{\mathbb{R}^d} e^{2i\lambda\langle y, x'\rangle} H_{n,\lambda}(x + x') H_{m,\lambda}(-x + x')\, dx'\,.$$

of the (renormalized) Hermite functions $H_{m,\lambda} = |\lambda|^{\frac{d}{4}} H_m(|\lambda|^{\frac{1}{2}} x)$

Some formulas

Inversion and Fourier-Plancherel formulae

$$f(w) = \frac{2^{d-1}}{\pi^{d+1}} \int_{\widetilde{\mathbb{H}}^d} e^{is\lambda} \mathcal{W}(\widehat{w}, Y) \mathcal{F}_{\mathbb{H}} f(\widehat{w}) \, d\widehat{w}$$

and

$$\left(\mathcal{F}_{\mathbb{H}} f | \mathcal{F}_{\mathbb{H}} g\right)_{L^2(\widetilde{\mathbb{H}}^d)} = \frac{\pi^{d+1}}{2^{d-1}} (f|g)_{L^2(\mathbb{H}^d)} ,$$

Action of the Laplacian

$$\mathcal{F}_{\mathbb{H}}(\Delta_{\mathbb{H}} f)(\widehat{w}) = -4|\lambda|(2|m| + d)\mathcal{F}_{\mathbb{H}}(f)(\widehat{w}) .$$

Radial functions $f(z, s) = f(|z|, s)$

$$\mathcal{F}_{\mathbb{H}}(f)(n, m, \lambda) = \mathcal{F}_{\mathbb{H}}(f)(n, m, \lambda)\delta_{n,m} = \mathcal{F}_{\mathbb{H}}(f)(|n|, |n|, \lambda)\delta_{n,m} .$$

Convolution for radial functions

$$\mathcal{F}_{\mathbb{H}}(f \star g)(\ell, \ell, \lambda) = \mathcal{F}_{\mathbb{H}} f(\ell, \ell, \lambda)\mathcal{F}_{\mathbb{H}} g(\ell, \ell, \lambda) .$$

The completion of the frequency set

- The frequency set $\widetilde{\mathbb{H}}^d$ comes with a measure

$$\int_{\widetilde{\mathbb{H}}^d} \theta(\widehat{w})\, d\widehat{w} = \int_{\mathbb{R}} \sum_{(n,m)\in\mathbb{N}^{2d}} \theta(n,m,\lambda)|\lambda|^d\, d\lambda\,.$$

- endowed with a distance

$$d(\widehat{w},\widehat{w}') = \big|\lambda(n+m)-\lambda'(n'+m')\big|_{\ell^1} + \big|(n-m)-(n'-m')|_{\ell^1} + d|\lambda-\lambda'|\,,$$

- $(\widetilde{\mathbb{H}}^d, d)$ it is not complete [$\rightarrow$] build the metric completion, denoted $\widehat{\mathbb{H}}^d$

Some advantages of this metric approach [Bahouri, Chemin, Danchin]

- definition of $\mathcal{S}(\widehat{\mathbb{H}}^d)$,
- $\mathcal{F}_{\mathbb{H}}$ is a bicontinuous isomorphism between $\mathcal{S}(\mathbb{H}^d)$ and $\mathcal{S}(\widehat{\mathbb{H}}^d)$,
- interpretation smoothness $\leftrightarrow$ decay

$\rightarrow$ give a meaning to the unit sphere $\mathbb{S}_{\widehat{\mathbb{H}}^d}$ of $\widehat{\mathbb{H}}^d$.

The result of Müller

- D.Müller [Annals of Math, 1990]: works in terms of spectral decomposition

$$L = \int_0^\infty \lambda dE(\lambda), \qquad \mathcal{P}f = f * G$$

- proves the estimate ("restriction for the sphere"): if $1 \leq p \leq 2$

$$\Big[\sum_{n\in\mathbb{N}^d} \frac{1}{(2|n|+d)^{d+1}} \Big(\sum_{\pm} \Big| \mathcal{F}_{\mathbb{H}}(f)(n,n,\frac{\pm 1}{2|n|+d}) \Big|^2 \Big) \Big]^{\frac{1}{2}} \leq C_p \|f\|_{L^p_z L^1_s}$$

- can be reinterpreted as follows: If $1 \leq p \leq 2$, then for radial f

$$\|\mathcal{F}_{\mathbb{H}}(f)_{|\mathbb{S}_{\widehat{\mathbb{H}}^d}}\|_{L^2(\mathbb{S}_{\widehat{\mathbb{H}}^d})} \leq C_p \|f\|_{L^p_z L^1_s}\,, \tag{19}$$

→ valid on the full interval: for $p \in [1,2]$

→ crucial: the anisotropic norm $L^p_z L^1_s$ ($r = 1$ is necessary in vertical)

- false for $p > 2$

On the surface measure

Recall that for θ Fourier transform of radial function

$$\int_{\widehat{\mathbb{H}}^d} \theta(\widehat{w}) d\widehat{w} = \int_{\mathbb{R}} \sum_{n \in \mathbb{N}^d} \theta(n, n, \lambda) |\lambda|^d \, d\lambda \, .$$

For spherical measures (on sphere of radius R) we want

$$\int_{\widehat{\mathbb{H}}^d} \theta(\widehat{w}) d\widehat{w} = \int_0^\infty \left(\int_{\mathbb{S}_{\widehat{\mathbb{H}}^d}} \theta(\widehat{w}) d\sigma_R(\widehat{w}) \right) dR$$

So we have (change of variable $R^2 = (2|n| + d)|\lambda|$)

$$\int_{\mathbb{S}_{\widehat{\mathbb{H}}^d}} \theta(\widehat{w}) d\sigma_R(\widehat{w}) = \sum_{n \in \mathbb{N}^d} \frac{2R^{2d+1}}{(2|n| + d)^{d+1}} \Big(\sum_{\pm} \theta(n, n, \frac{\pm R^2}{2|n| + d}) \Big)$$

Fourier transform of the surface measure

Up to a measure zero set on $\hat{\mathbb{H}}^d$

$$\mathbb{S}_{\widehat{\mathbb{H}}^d} = \Big\{(n, n, \lambda) \in \widehat{\mathbb{H}}^d \,/\, (2|n| + d)|\lambda| = 1\Big\}$$

By definition, the tempered distribution

$$G = \mathcal{F}_{\mathbb{H}}^{-1}(d\sigma_{\mathbb{S}_{\widehat{\mathbb{H}}^d}})$$

Lemma

G is the bounded function on $\mathbb{H}^d$ defined by

$$G(z, s) = \frac{2^d}{\pi^{d+1}} \sum_{n \in \mathbb{N}^d} \frac{1}{(2|n| + d)^{d+1}} \cos\Big(\frac{s}{2|n| + d}\Big) \mathcal{W}\Big(n, n, 1, \frac{z}{\sqrt{2|n| + d}}\Big) \quad (20)$$

For the sphere of radius $R^{1/2}$ we have the homogeneity property:

$$G_R(z, s) = R^d (G \circ \delta_{\sqrt{R}})(z, s)\,. \quad (21)$$

Strichartz estimate in the Heisenberg group

Let u_0 in $\mathcal{S}(\mathbb{H}^d)$ be radial and consider the Cauchy problem

$$\begin{cases} i\partial_t u - \Delta_{\mathbb{H}} u = 0 \\ u_{|t=0} = u_0 \,. \end{cases}$$

Taking the partial Fourier transform with respect to the variable w we obtain

$$\begin{cases} i\frac{d}{dt}\mathcal{F}_{\mathbb{H}}(u)(t,n,m,\lambda) = -4|\lambda|(2|m|+d)\mathcal{F}_{\mathbb{H}}(u)(t,n,m,\lambda) \\ \mathcal{F}_{\mathbb{H}}(u)_{|t=0} = \mathcal{F}_{\mathbb{H}} u_0 \,. \end{cases}$$

By integration, this leads to

$$\mathcal{F}_{\mathbb{H}}(u)(t,n,m,\lambda) = e^{4it|\lambda|(2|m|+d)}\mathcal{F}_{\mathbb{H}}(u_0)(|n|,|n|,\lambda)\delta_{n,m} \,.$$

$\rightarrow$ Notice that if we set $|m| = 0$ we see the "transport" part

$$\mathcal{F}_{\mathbb{H}}(u)(t,0,0,\lambda) = e^{4it|\lambda|d}\mathcal{F}_{\mathbb{H}}(u_0)(0,0,\lambda) \,.$$

Applying the inverse Fourier formula

$$u(t,z,s) = \frac{2^{d-1}}{\pi^{d+1}} \int_{\widehat{\mathbb{H}}^d} e^{is\lambda}\, \mathcal{W}(\widehat{w}, z)\, e^{4it|\lambda|(2|m|+d)}\, \mathcal{F}_{\mathbb{H}}(u_0)(|n|,|n|,\lambda)\delta_{n,m}\, d\widehat{w}\,.$$

Re-expressed as the inverse Fourier transform in $\widehat{\mathbb{R}} \times \widehat{\mathbb{H}}^d$ of $\mathcal{F}_{\mathbb{H}}(u_0)\, d\Sigma$, where

$$\Sigma = \Big\{(\alpha, \widehat{w}) = \big(\alpha, (n,n,\lambda)\big) \in \widehat{\mathbb{R}} \times \widehat{\mathbb{H}}^d \,/\, \alpha = 4|\lambda|(2|n|+d)\Big\}\,.$$

and we endow Σ with the measure $d\Sigma$ induced by the projection $\widehat{\mathbb{R}} \times \widehat{\mathbb{H}}^d \to \widehat{\mathbb{H}}^d$

$$\int_{\widehat{\mathbb{D}}} \Phi(\alpha, \widehat{w})\, d\Sigma(\alpha, \widehat{w}) = \int_{\widehat{\mathbb{H}}^d} \Phi(4|\lambda|(2|m|+d), \widehat{w})\, d\widehat{w},$$

Then one need a restriction theorem here (recall $\Sigma \to \Sigma_{\mathrm{loc}}$)

Theorem (Bahouri, DB, Gallagher, '19)

If $1 \le q \le p \le 2$, then for f radial

$$\|\mathcal{F}_{\widehat{\mathbb{R}}\times\widehat{\mathbb{H}}^d}(f)|_{\Sigma}\|_{L^2(d\Sigma)} \le C_{p,q} \|f\|_{L^1_s L^q_t L^p_z}\,, \tag{22}$$

Using the dual inequality and assuming that $F_{\mathbb{H}} u_0$ is localized in the unit ball

For any $2 \le p \le q \le \infty$

$$\|u\|_{L^\infty_s L^q_t L^p_z} \le C \|\mathcal{F}_{\mathbb{H}} u_0\|_{L^2(\widehat{\mathbb{H}}^d)} = C \|u_0\|_{L^2(\mathbb{H}^d)} ,$$

- If u_0 is frequency localized in the ball $\mathcal{B}_\Lambda$,

$$u_\Lambda(t,z,s) = u(\Lambda^{-2}t, \Lambda^{-1}z, \Lambda^{-2}s), \qquad u_{0,\Lambda}(z,s) = u_0(\Lambda^{-1}z, \Lambda^{-2}s)$$

- we have

$$\|u_\Lambda\|_{L^\infty_s L^q_t L^p_z} = \Lambda^{\frac{2}{q}+\frac{2d}{p}} \|u\|_{L^\infty_s L^q_t L^p_z}, \qquad \|u_{0,\Lambda}\|_{L^2(\mathbb{H}^d)} = \Lambda^{\frac{Q}{2}} \|u_0\|_{L^2(\mathbb{H}^d)} ,$$

- we infer

$$\|u\|_{L^\infty_s L^q_t L^p_z} \le C \Lambda^{\frac{Q}{2}-\frac{2}{q}-\frac{2d}{p}} \|u_0\|_{L^2(\mathbb{H}^d)} .$$

- It suffices now to choose p and q satisfying ($\rightarrow$ density of spec loc f)

$$\frac{2}{q} + \frac{2d}{p} = \frac{Q}{2}$$

Measure on the paraboloid

Proceeding as for the restriction theorem on the sphere of $\widehat{\mathbb{H}}^d$, let us first compute

$$G_{\Sigma_{\mathrm{loc}}} = \mathcal{F}^{-1}_{\hat{\mathbb{R}} \times \hat{\mathbb{H}}^d}(d\Sigma_{\mathrm{loc}})\,.$$

Lemma

With the above notation, $G_{\Sigma_{\mathrm{loc}}}$ is the bounded function on $\mathbb{R} \times \hat{\mathbb{H}}^d$ defined by

$$G_{\Sigma_{\mathrm{loc}}}(t,w) = 2\pi \int_0^\infty G_\alpha(w)\, e^{-it\,\alpha} \psi(\alpha)\, d\alpha\,, \tag{23}$$

where G_R is the inverse Fourier of the measure of sphere of radius $R^{1/2}$.

This gives for all f in $\mathcal{S}_{\mathrm{rad}}(\mathcal{D})$

$$(R^*_{\Sigma_{\mathrm{loc}}} R_{\Sigma_{\mathrm{loc}}} f)(t,z,s) = \left(\frac{\pi}{2}\right)^d (G_{\Sigma_{\mathrm{loc}}} \star \check{f})(-t,-z,s)\,, \tag{24}$$

Proof for $1 \leq p < 2$ (non endpoint)

Main lemma

$$\|f \star G_{\Sigma_{\text{loc}}}\|_{L^\infty_s L^{q'}_t L^{p'}_z} \lesssim \Big\| \|\mathcal{F}_{\mathbb{R}}(f)(-\alpha,\cdot)\|_{L^p_z L^1_s} \alpha^{d(1-\frac{2}{p'})} \; \psi(\alpha) \Big\|_{L^q_\alpha}$$

- Hölder estimate in α + Hausdorff-Young inequality: for any $a \geq 2$

$$\begin{aligned} \|f \star G_{\Sigma_{\text{loc}}}\|_{L^\infty_s L^{q'}_t L^{p'}_z} &\lesssim \|\mathcal{F}_{\mathbb{R}}(f)\|_{L^a_\alpha L^p_z L^1_s} \|\alpha^{d(1-\frac{2}{p'})} \; \psi(\alpha)\|_{L^b_\alpha} \\ &\lesssim \|f\|_{L^{a'}_t L^p_z L^1_s} \|\alpha^{d(1-\frac{2}{p'})} \; \psi(\alpha)\|_{L^b_\alpha(\mathbb{R})} \, , \end{aligned}$$

  where a' is the conjugate exponent of a and $\frac{1}{a} + \frac{1}{b} = \frac{1}{q}$.
- Finally for $a' = q$ and Minkowski's inequality, we get for $q' \geq p' > 2$

$$\|f \star G_{\Sigma_{\text{loc}}}\|_{L^\infty_s L^{q'}_t L^{p'}_z} \lesssim \|f\|_{L^1_s L^q_t L^p_z}$$

$\rightarrow$ endpoint $p = 2$: ad hoc argument similar to Müller

Outline

Wave equation

The wave equation on $\mathbb{R}^n$

$$(W) \qquad \begin{cases} \partial_t^2 u - \Delta u = 0 \\ (u, \partial_t u)_{|t=0} = (u_0, u_1)\,, \end{cases}$$

The classical dispersive estimate writes (for $t \neq 0$)

$$\|u(t,\cdot)\|_{L^\infty(\mathbb{R}^n)} \leq \frac{C}{|t|^{\frac{n-1}{2}}}\left(\|u_0\|_{L^1(\mathbb{R}^n)} + \|u_1\|_{L^1(\mathbb{R}^n)}\right).$$

$\rightarrow$ oscillatory integrals and stationary phase theorem.

Strichartz estimate for wave equation

$$\|u\|_{L^q(\mathbb{R}, L^p(\mathbb{R}^n))} \leq C(p,q)\Big(\|\nabla u_0\|_{L^2(\mathbb{R}^n)} + \|u_1\|_{L^2(\mathbb{R}^n)}\Big)\,,$$

where (p,q) satisfies the scaling admissibility condition

$$\frac{1}{q} + \frac{n}{p} = \frac{n}{2} - 1, \qquad p, q \geq 2, \quad q < \infty\,.$$

Previous Stricharts estimate for wave on $\mathbb{H}$

On $\mathbb{H}^d$ one can prove a dimension-independent dispersive estimate

$$\|u(t,\cdot)\|_{L^\infty(\mathbb{H}^d)} \leq \frac{C}{|t|^{\frac{1}{2}}}\big(\|u_0\|_{L^1(\mathbb{H}^d)} + \|u_1\|_{L^1(\mathbb{H}^d)}\big),$$

- only the center is involved in the dispersive effect.
- this estimate is optimal.

This dispersive estimate gives rise to a Strichartz estimate

[Bahouri, Gérard, Xu, '00]

$$\|u\|_{L^q_t L^p_{z,s}} \leq C_{p,q,p_1,q_1}\Big(\|\nabla_{\mathbb{H}^d} u_0\|_{L^2(\mathbb{H}^d)} + \|u_1\|_{L^2(\mathbb{H}^d)} + \|f\|_{L^{q_1'}_t L^{p_1'}_{z,s}}\Big)$$

with $\frac{1}{q} + \frac{Q}{p} = \frac{Q}{2} - 1$ and $q \geq 2Q - 1$.

Our result for wave

In the case of the wave equation on $\mathbb{H}$ we obtain the following Strichartz estimate.

Theorem (Bahouri, DB, Gallagher, '19)

With the above notation, given (p,q) and (p_1,q_1) belonging to the admissible set

$$\mathcal{A}^W = \left\{ (p,q) \in [2,\infty]^2 \,/\, q \le p \quad \textit{and} \quad \frac{1}{q} + \frac{2d}{p} = \frac{Q}{2} - 1 \right\},$$

there is a constant C_{p,q,p_1,q_1} such that the solution to the wave equation $(W_{\mathbb{H}})$ associated with radial data satisfies the following Strichartz estimate:

$$\|u\|_{L^\infty_s L^q_t L^p_z} \le C_{p,q,p_1,q_1} \left(\|\nabla_{\mathbb{H}^d} u_0\|_{L^2(\mathbb{H}^d)} + \|u_1\|_{L^2(\mathbb{H}^d)} + \|f\|_{L^1_s L^{q_1'}_t L^{p_1'}_z} \right).$$

- q can be small. In the previous $q \ge 2Q - 1$.
- we pay a price in the s variable

Comments

On the positive side:

- a geometric interpretation of Muller result
- extension to other surfaces in the dual of Heisenberg group
- some new Strichartz estimates for linear Shrödinger/wave equations

Still to do ($\rightarrow$ a lot!):

- remove the radial assumption on the initial data ?
- extend this analysis to more general groups ? 2-step ?
- obtain applications to sub-Riemannian NLS ?

THANKS FOR YOUR ATTENTION !

Appendix : Hermite functions

- $H_{m,\lambda}$ renormalized Hermite function on $\mathbb{R}^d$
- $H_{m,\lambda}(x) = |\lambda|^{\frac{d}{4}} H_m(|\lambda|^{\frac{1}{2}} x)$,
- H_m the Hermite orthonormal basis of $L^2(\mathbb{R}^d)$ given by the eigenfunctions of the harmonic oscillator:

$$-(\Delta - |x|^2)H_m = (2|m| + d)H_m \,,$$

specifically

$$H_m = \left(\frac{1}{2^{|m|} m!} \right)^{\frac{1}{2}} \prod_{j=1}^{d} \big(-\partial_j H_0 + x_j H_0 \big)^{m_j} \,,$$

with

- $H_0(x) = \pi^{-\frac{d}{4}} e^{-\frac{|x|^2}{2}}$
- $m! = m_1! \cdots m_d!$
- $|m| = m_1 + \cdots + m_d$.