On the Sobolev quotient in sub-Riemannian geometry

Joint work with J.H.Cheng and P.Yang

Andrea Malchiodi (SNS)

Pisa, Feb. 10, 2020

The Yamabe problem

The Yamabe problem

On compact Riemannian manifolds of dimension $n \geq 3$, Yamabe in 1960 posed the problem of finding conformal metrics with constant scalar curvature.

The Yamabe problem

On compact Riemannian manifolds of dimension $n \geq 3$, Yamabe in 1960 posed the problem of finding conformal metrics with constant scalar curvature.

This is a conformal generalization of the Uniformization problem, and Yamabe intended to use it as a step to solve Poincaré's conjecture.

The Yamabe problem

On compact Riemannian manifolds of dimension $n \geq 3$, Yamabe in 1960 posed the problem of finding conformal metrics with constant scalar curvature.

This is a conformal generalization of the Uniformization problem, and Yamabe intended to use it as a step to solve Poincaré's conjecture.

If R_g is the scalar curvature, setting $\tilde{g}(x) = \lambda(x)g(x) = u(x)^{\frac{4}{n-2}}g(x)$, $u(x)$ one has to find on M a positive solution of

$$(Y) \qquad -c_n \Delta u + R_g u = \overline{R}\, u^{\frac{n+2}{n-2}}; \qquad\qquad c_n = 4\frac{n-1}{n-2}, \quad \overline{R} \in \mathbb{R}.$$

The Sobolev-Yamabe quotient

The Sobolev-Yamabe quotient

Recall the equation

$$(Y) \qquad -c_n \Delta u + R_g u = \overline{R}\, u^{\frac{n+2}{n-2}}; \qquad\qquad c_n = 4\frac{n-1}{n-2}, \quad \overline{R} \in \mathbb{R}.$$

The Sobolev-Yamabe quotient

Recall the equation

$$(Y) \qquad -c_n \Delta u + R_g u = \overline{R}\, u^{\frac{n+2}{n-2}}; \qquad c_n = 4\frac{n-1}{n-2}, \quad \overline{R} \in \mathbb{R}.$$

Considering $\overline{R}$ as a Lagrange multiplier, one can try to find solutions by minimizing the *Sobolev-Yamabe quotient*

$$Q_{SY}(u) = \frac{\int_M \left(c_n |\nabla u|^2 + R_g u^2\right) dV}{\left(\int_M |u|^{2^*} dV\right)^{\frac{2}{2^*}}}; \qquad 2^* = \frac{2n}{n-2}.$$

The Sobolev-Yamabe quotient

Recall the equation

$$(Y) \qquad -c_n \Delta u + R_g u = \overline{R}\, u^{\frac{n+2}{n-2}}; \qquad\qquad c_n = 4\frac{n-1}{n-2}, \quad \overline{R} \in \mathbb{R}.$$

Considering $\overline{R}$ as a Lagrange multiplier, one can try to find solutions by minimizing the *Sobolev-Yamabe quotient*

$$Q_{SY}(u) = \frac{\int_M \left(c_n |\nabla u|^2 + R_g u^2 \right) dV}{\left(\int_M |u|^{2^*} dV \right)^{\frac{2}{2^*}}}; \qquad\qquad 2^* = \frac{2n}{n-2}.$$

The *Sobolev-Yamabe constant* is defined as

$$Y(M, [g]) = \inf_{u \not\equiv 0} Q_{SY}(u).$$

The Sobolev-Yamabe quotient

Recall the equation

$$(Y) \qquad -c_n \Delta u + R_g u = \overline{R}\, u^{\frac{n+2}{n-2}}; \qquad c_n = 4\frac{n-1}{n-2}, \quad \overline{R} \in \mathbb{R}.$$

Considering $\overline{R}$ as a Lagrange multiplier, one can try to find solutions by minimizing the *Sobolev-Yamabe quotient*

$$Q_{SY}(u) = \frac{\int_M \left(c_n |\nabla u|^2 + R_g u^2\right) dV}{\left(\int_M |u|^{2^*} dV\right)^{\frac{2}{2^*}}}; \qquad 2^* = \frac{2n}{n-2}.$$

The *Sobolev-Yamabe constant* is defined as

$$Y(M,[g]) = \inf_{u \not\equiv 0} Q_{SY}(u).$$

The number $Y(M,[g])$ depends only on the conformal class $[g]$ of g.

The Sobolev-Yamabe quotient

Recall the equation

$$(Y) \qquad -c_n \Delta u + R_g u = \overline{R}\, u^{\frac{n+2}{n-2}}; \qquad\qquad c_n = 4\frac{n-1}{n-2}, \quad \overline{R} \in \mathbb{R}.$$

Considering $\overline{R}$ as a Lagrange multiplier, one can try to find solutions by minimizing the *Sobolev-Yamabe quotient*

$$Q_{SY}(u) = \frac{\int_M \left(c_n |\nabla u|^2 + R_g u^2\right) dV}{\left(\int_M |u|^{2^*} dV\right)^{\frac{2}{2^*}}}; \qquad\qquad 2^* = \frac{2n}{n-2}.$$

The *Sobolev-Yamabe constant* is defined as

$$Y(M, [g]) = \inf_{u \not\equiv 0} Q_{SY}(u).$$

The number $Y(M, [g])$ depends only on the conformal class $[g]$ of g.

$(M, [g])$ is said to be of *negative, zero or positive Yamabe class* when $Y(M, [g])$ is negative, zero or positive.

The Sobolev quotient in $\mathbb{R}^n$ $(n \geq 3)$

The Sobolev quotient in $\mathbb{R}^n$ $(n \geq 3)$

In $\mathbb{R}^n$ one has the *Sobolev-Gagliardo-Nirenberg* inequality

$$\|u\|^2_{L^{2^*}(\mathbb{R}^n)} \leq B_n \int_{\mathbb{R}^n} |\nabla u|^2 dx; \qquad\qquad u \in C^\infty_c(\mathbb{R}^n).$$

The Sobolev quotient in $\mathbb{R}^n$ $(n \geq 3)$

In $\mathbb{R}^n$ one has the *Sobolev-Gagliardo-Nirenberg* inequality

$$\|u\|^2_{L^{2^*}(\mathbb{R}^n)} \leq B_n \int_{\mathbb{R}^n} |\nabla u|^2 dx; \qquad u \in C^\infty_c(\mathbb{R}^n).$$

As for $Y(M,[g])$, define the *Sobolev quotient* $S_n = \inf_u \frac{\int_{\mathbb{R}^n} c_n |\nabla u|^2 dx}{\|u\|^2_{L^{2^*}(\mathbb{R}^n)}}$.

The Sobolev quotient in $\mathbb{R}^n$ $(n \geq 3)$

In $\mathbb{R}^n$ one has the *Sobolev-Gagliardo-Nirenberg* inequality

$$\|u\|^2_{L^{2^*}(\mathbb{R}^n)} \leq B_n \int_{\mathbb{R}^n} |\nabla u|^2 dx; \qquad u \in C^\infty_c(\mathbb{R}^n).$$

As for $Y(M,[g])$, define the *Sobolev quotient* $S_n = \inf_u \frac{\int_{\mathbb{R}^n} c_n |\nabla u|^2 dx}{\|u\|^2_{L^{2^*}(\mathbb{R}^n)}}$.

([Aubin, '76], [Talenti, '76]) Completing $C^\infty_c(\mathbb{R}^n)$, S_n is attained by

$$U_{p,\lambda}(x) := \frac{\lambda^{\frac{n-2}{2}}}{(1+\lambda^2|x-p|^2)^{\frac{n-2}{2}}}; \qquad p \in \mathbb{R}^n, \lambda > 0.$$

The Sobolev quotient in $\mathbb{R}^n$ $(n \geq 3)$

In $\mathbb{R}^n$ one has the *Sobolev-Gagliardo-Nirenberg* inequality

$$\|u\|^2_{L^{2^*}(\mathbb{R}^n)} \leq B_n \int_{\mathbb{R}^n} |\nabla u|^2 dx; \qquad u \in C^\infty_c(\mathbb{R}^n).$$

As for $Y(M,[g])$, define the *Sobolev quotient* $S_n = \inf_u \frac{\int_{\mathbb{R}^n} c_n |\nabla u|^2 dx}{\|u\|^2_{L^{2^*}(\mathbb{R}^n)}}$.

([Aubin, '76], [Talenti, '76]) Completing $C^\infty_c(\mathbb{R}^n)$, S_n is attained by

$$U_{p,\lambda}(x) := \frac{\lambda^{\frac{n-2}{2}}}{(1+\lambda^2|x-p|^2)^{\frac{n-2}{2}}}; \qquad p \in \mathbb{R}^n, \lambda > 0.$$

• Since S^n is conformal to $\mathbb{R}^n$, one has that $Y(S^n,[g_{S^n}]) = S_n$.

The Sobolev quotient of domains of $\mathbb{R}^n$

The Sobolev quotient of domains of $\mathbb{R}^n$

Also for a (say, bounded smooth) domain $\Omega \subseteq \mathbb{R}^n$ one can consider the Sobolev quotient for functions supported in Ω

$$\inf_{u \in C_c^\infty(\Omega)} \frac{\int_{\mathbb{R}^n} c_n |\nabla u|^2 dx}{\|u\|^2_{L^{2^*}(\mathbb{R}^n)}}.$$

The Sobolev quotient of domains of $\mathbb{R}^n$

Also for a (say, bounded smooth) domain $\Omega \subseteq \mathbb{R}^n$ one can consider the Sobolev quotient for functions supported in Ω

$$\inf_{u \in C_c^\infty(\Omega)} \frac{\int_{\mathbb{R}^n} c_n |\nabla u|^2 dx}{\|u\|^2_{L^{2^*}(\mathbb{R}^n)}}.$$

In this case the infimum coincides with S_n, but it is never attained because of the lack of compactness of the embedding.

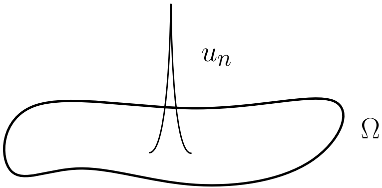
The Sobolev quotient of domains of $\mathbb{R}^n$

Also for a (say, bounded smooth) domain $\Omega \subseteq \mathbb{R}^n$ one can consider the Sobolev quotient for functions supported in Ω

$$\inf_{u \in C_c^\infty(\Omega)} \frac{\int_{\mathbb{R}^n} c_n |\nabla u|^2 dx}{\|u\|^2_{L^{2^*}(\mathbb{R}^n)}}.$$

In this case the infimum coincides with S_n, but it is never attained because of the lack of compactness of the embedding.

Minimizing sequences u_n tend to concentrate indefinitely inside Ω.



Brief history on the Yamabe problem

Brief history on the Yamabe problem

- In 1960 Yamabe attempted to solve (Y) by subcritical approximation.

Brief history on the Yamabe problem

- In 1960 Yamabe attempted to solve (Y) by subcritical approximation.

- In 1968 Trudinger proved that (Y) is solvable provided $Y(M,[g]) \leq \varepsilon_n$ for some $\varepsilon_n > 0$.

Brief history on the Yamabe problem

- In 1960 Yamabe attempted to solve (Y) by subcritical approximation.

- In 1968 Trudinger proved that (Y) is solvable provided $Y(M,[g]) \leq \varepsilon_n$ for some $\varepsilon_n > 0$. In particular for negative and zero Yamabe class.

- In 1960 Yamabe attempted to solve (Y) by subcritical approximation.

- In 1968 Trudinger proved that (Y) is solvable provided $Y(M,[g]) \leq \varepsilon_n$ for some $\varepsilon_n > 0$. In particular for negative and zero Yamabe class.

- In 1976 Aubin proved that (Y) is solvable provided $Y(M,[g]) < S_n$.

Brief history on the Yamabe problem

- In 1960 Yamabe attempted to solve (Y) by subcritical approximation.

- In 1968 Trudinger proved that (Y) is solvable provided $Y(M,[g]) \leq \varepsilon_n$ for some $\varepsilon_n > 0$. In particular for negative and zero Yamabe class.

- In 1976 Aubin proved that (Y) is solvable provided $Y(M,[g]) < S_n$. He also verified this inequality when $n \geq 6$ and (M,g) is not *locally conformally flat*, unless $(M,g) \simeq (S^n, g_{S^n})$.

Brief history on the Yamabe problem

- In 1960 Yamabe attempted to solve (Y) by subcritical approximation.

- In 1968 Trudinger proved that (Y) is solvable provided $Y(M,[g]) \leq \varepsilon_n$ for some $\varepsilon_n > 0$. In particular for negative and zero Yamabe class.

- In 1976 Aubin proved that (Y) is solvable provided $Y(M,[g]) < S_n$. He also verified this inequality when $n \geq 6$ and (M,g) is not *locally conformally flat*, unless $(M,g) \simeq (S^n, g_{S^n})$.

- In 1984 Schoen proved that $Y(M,[g]) < S_n$ in all other cases, i.e. $n \leq 5$ or (M,g) locally conformally flat, unless $(M,g) \simeq (S^n, g_{S^n})$.

On the inequality $Y(M,[g]) < S_n$

On the inequality $Y(M, [g]) < S_n$

The inequality is proved using Aubin-Talenti's functions.

On the inequality $Y(M,[g]) < S_n$

The inequality is proved using Aubin-Talenti's functions. Given $p \in M$, consider a conformal metric $\tilde{g} \simeq U_{p,\lambda}^{\frac{4}{n-2}} g$ with λ large.

On the inequality $Y(M,[g]) < S_n$

The inequality is proved using Aubin-Talenti's functions. Given $p \in M$, consider a conformal metric $\tilde{g} \simeq U_{p,\lambda}^{\frac{4}{n-2}} g$ with λ large. Since locally $(M,g) \simeq \mathbb{R}^n$ and since $U_{p,\lambda}$ is highly concentrated, $Q_{SY}(U_{p,\lambda}) \simeq S_n$, with small correction terms due to the geometry of M.

On the inequality $Y(M,[g]) < S_n$

The inequality is proved using Aubin-Talenti's functions. Given $p \in M$, consider a conformal metric $\tilde{g} \simeq U_{p,\lambda}^{\frac{4}{n-2}} g$ with λ large. Since locally $(M,g) \simeq \mathbb{R}^n$ and since $U_{p,\lambda}$ is highly concentrated, $Q_{SY}(U_{p,\lambda}) \simeq S_n$, with small correction terms due to the geometry of M.

Since $U_{p,\lambda}$ decays like $\frac{1}{|x|^{n-2}}$ at infinity, it is more *localized* in large dimension.

On the inequality $Y(M,[g]) < S_n$

The inequality is proved using Aubin-Talenti's functions. Given $p \in M$, consider a conformal metric $\tilde{g} \simeq U_{p,\lambda}^{\frac{4}{n-2}} g$ with λ large. Since locally $(M,g) \simeq \mathbb{R}^n$ and since $U_{p,\lambda}$ is highly concentrated, $Q_{SY}(U_{p,\lambda}) \simeq S_n$, with small correction terms due to the geometry of M.

Since $U_{p,\lambda}$ decays like $\frac{1}{|x|^{n-2}}$ at infinity, it is more *localized* in large dimension. Aubin proved that for $n \geq 6$ the corrections are given by $-\frac{|W_g|^2(p)}{\lambda^4}$, a local quantity depending on the Weyl tensor.

On the inequality $Y(M,[g]) < S_n$

The inequality is proved using Aubin-Talenti's functions. Given $p \in M$, consider a conformal metric $\tilde{g} \simeq U_{p,\lambda}^{\frac{4}{n-2}} g$ with λ large. Since locally $(M,g) \simeq \mathbb{R}^n$ and since $U_{p,\lambda}$ is highly concentrated, $Q_{SY}(U_{p,\lambda}) \simeq S_n$, with small correction terms due to the geometry of M.

Since $U_{p,\lambda}$ decays like $\frac{1}{|x|^{n-2}}$ at infinity, it is more *localized* in large dimension. Aubin proved that for $n \geq 6$ the corrections are given by $-\frac{|W_g|^2(p)}{\lambda^4}$, a local quantity depending on the Weyl tensor.

For $n \leq 5$ the correction is of *global nature.*

On the inequality $Y(M,[g]) < S_n$

The inequality is proved using Aubin-Talenti's functions. Given $p \in M$, consider a conformal metric $\tilde{g} \simeq U_{p,\lambda}^{\frac{4}{n-2}} g$ with λ large. Since locally $(M,g) \simeq \mathbb{R}^n$ and since $U_{p,\lambda}$ is highly concentrated, $Q_{SY}(U_{p,\lambda}) \simeq S_n$, with small correction terms due to the geometry of M.

Since $U_{p,\lambda}$ decays like $\frac{1}{|x|^{n-2}}$ at infinity, it is more *localized* in large dimension. Aubin proved that for $n \geq 6$ the corrections are given by $-\frac{|W_g|^2(p)}{\lambda^4}$, a local quantity depending on the Weyl tensor.

For $n \leq 5$ the correction is of *global nature.* Heuristics:

On the inequality $Y(M,[g]) < S_n$

The inequality is proved using Aubin-Talenti's functions. Given $p \in M$, consider a conformal metric $\tilde{g} \simeq U_{p,\lambda}^{\frac{4}{n-2}} g$ with λ large. Since locally $(M,g) \simeq \mathbb{R}^n$ and since $U_{p,\lambda}$ is highly concentrated, $Q_{SY}(U_{p,\lambda}) \simeq S_n$, with small correction terms due to the geometry of M.

Since $U_{p,\lambda}$ decays like $\frac{1}{|x|^{n-2}}$ at infinity, it is more *localized* in large dimension. Aubin proved that for $n \geq 6$ the corrections are given by $-\frac{|W_g|^2(p)}{\lambda^4}$, a local quantity depending on the Weyl tensor.

For $n \leq 5$ the correction is of *global nature*. Heuristics: if $u \simeq U_{p,\lambda}$ then

$$L_g u := -c_n \Delta u + R_g u \simeq U_{p,\lambda}^{\frac{n+2}{n-2}} \simeq \frac{1}{\lambda} \delta_p.$$

On the inequality $Y(M,[g]) < S_n$

The inequality is proved using Aubin-Talenti's functions. Given $p \in M$, consider a conformal metric $\tilde{g} \simeq U_{p,\lambda}^{\frac{4}{n-2}} g$ with λ large. Since locally $(M,g) \simeq \mathbb{R}^n$ and since $U_{p,\lambda}$ is highly concentrated, $Q_{SY}(U_{p,\lambda}) \simeq S_n$, with small correction terms due to the geometry of M.

Since $U_{p,\lambda}$ decays like $\frac{1}{|x|^{n-2}}$ at infinity, it is more *localized* in large dimension. Aubin proved that for $n \geq 6$ the corrections are given by $-\frac{|W_g|^2(p)}{\lambda^4}$, a local quantity depending on the Weyl tensor.

For $n \leq 5$ the correction is of *global nature.* Heuristics: if $u \simeq U_{p,\lambda}$ then

$$L_g u := -c_n \Delta u + R_g u \simeq U_{p,\lambda}^{\frac{n+2}{n-2}} \simeq \frac{1}{\lambda} \delta_p.$$

At large scales an approximate solution looks like the Green's function G_p of the operator L_g.

On the inequality $Y(M,[g]) < S_n$

The inequality is proved using Aubin-Talenti's functions. Given $p \in M$, consider a conformal metric $\tilde{g} \simeq U_{p,\lambda}^{\frac{4}{n-2}} g$ with λ large. Since locally $(M,g) \simeq \mathbb{R}^n$ and since $U_{p,\lambda}$ is highly concentrated, $Q_{SY}(U_{p,\lambda}) \simeq S_n$, with small correction terms due to the geometry of M.

Since $U_{p,\lambda}$ decays like $\frac{1}{|x|^{n-2}}$ at infinity, it is more *localized* in large dimension. Aubin proved that for $n \geq 6$ the corrections are given by $-\frac{|W_g|^2(p)}{\lambda^4}$, a local quantity depending on the Weyl tensor.

For $n \leq 5$ the correction is of *global nature*. Heuristics: if $u \simeq U_{p,\lambda}$ then

$$L_g u := -c_n \Delta u + R_g u \simeq U_{p,\lambda}^{\frac{n+2}{n-2}} \simeq \frac{1}{\lambda}\delta_p.$$

At large scales an approximate solution looks like the Green's function G_p of the operator L_g. If $G_p \simeq \frac{1}{|x|^{n-2}} + A$ at p, the correction is $-A/\lambda^{n-2}$.

A brief excursion in general relativity

A brief excursion in general relativity

To understand the value of A, general relativity comes into play.

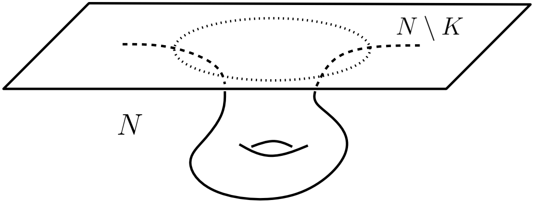
To understand the value of A, general relativity comes into play.

A manifold $(N^3, \tilde{g})$ is *asymptotically flat* if it is a union of a compact set K (possibly with topology), and such that $N \setminus K$ (called *end*) is diffeomorphic to $\mathbb{R}^3 \setminus B_1(0)$.

To understand the value of A, general relativity comes into play.

A manifold $(N^3, \tilde{g})$ is *asymptotically flat* if it is a union of a compact set K (possibly with topology), and such that $N \setminus K$ (called *end*) is diffeomorphic to $\mathbb{R}^3 \setminus B_1(0)$. It is required that the metric satisfies

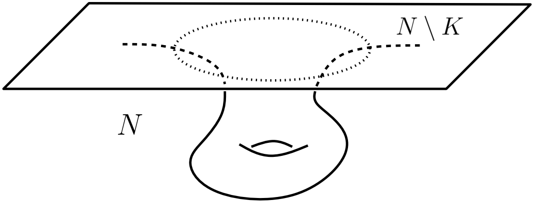
$$\tilde{g}_{ij} \to \delta_{ij} \qquad \text{at infinity} \quad \text{(with some rate).}$$



To understand the value of A, general relativity comes into play.

A manifold $(N^3, \tilde{g})$ is *asymptotically flat* if it is a union of a compact set K (possibly with topology), and such that $N \setminus K$ (called *end*) is diffeomorphic to $\mathbb{R}^3 \setminus B_1(0)$. It is required that the metric satisfies

$$\tilde{g}_{ij} \to \delta_{ij} \qquad \text{at infinity} \quad \text{(with some rate).}$$



Such manifolds describe *initial data sets* for isolated gravitational systems, and a similar definition holds for multiple *ends*.

Some examples

Example 1: Schwartzschild metric (two ends).

Example 1: Schwartzschild metric (two ends). It describes a static black hole of total mass m.

Some examples

Example 1: Schwartzschild metric (two ends). It describes a static black hole of total mass m. In polar coordinates (r, ξ) the expression is

$$\tilde{g}_{Schw} = \left(1 + \frac{m}{2r}\right)^4 \left(dr^2 + r^2 d\xi^2\right).$$

Some examples

Example 1: Schwartzschild metric (two ends). It describes a static black hole of total mass m. In polar coordinates (r, ξ) the expression is

$$\tilde{g}_{Schw} = \left(1 + \frac{m}{2r}\right)^4 \left(dr^2 + r^2 d\xi^2\right).$$

At $r = \frac{m}{2}$ there is a minimal surface, representing the *event horizon*.

Some examples

Example 1: Schwartzschild metric (two ends). It describes a static black hole of total mass m. In polar coordinates (r, ξ) the expression is

$$\tilde{g}_{Schw} = \left(1 + \frac{m}{2r}\right)^4 \left(dr^2 + r^2 d\xi^2\right).$$

At $r = \frac{m}{2}$ there is a minimal surface, representing the *event horizon*.

Example 2: Conformal blow-ups.

Example 1: Schwartzschild metric (two ends). It describes a static black hole of total mass m. In polar coordinates (r, ξ) the expression is

$$\tilde{g}_{Schw} = \left(1 + \frac{m}{2r}\right)^4 \left(dr^2 + r^2 d\xi^2\right).$$

At $r = \frac{m}{2}$ there is a minimal surface, representing the *event horizon*.

Example 2: Conformal blow-ups. Given a compact Riemannian three-manifold (M, g) and $p \in M$, one can consider a conformal metric on $\tilde{g}$ on $M \setminus \{p\}$ of the following form

$$\tilde{g} = f(x)\, g; \qquad f(x) \simeq \frac{1}{d(x,p)^4}.$$

Some examples

Example 1: Schwartzschild metric (two ends). It describes a static black hole of total mass m. In polar coordinates (r, ξ) the expression is

$$\tilde{g}_{Schw} = \left(1 + \frac{m}{2r}\right)^4 \left(dr^2 + r^2 d\xi^2\right).$$

At $r = \frac{m}{2}$ there is a minimal surface, representing the *event horizon*.

Example 2: Conformal blow-ups. Given a compact Riemannian three-manifold (M, g) and $p \in M$, one can consider a conformal metric on $\tilde{g}$ on $M \setminus \{p\}$ of the following form

$$\tilde{g} = f(x)\, g; \qquad f(x) \simeq \frac{1}{d(x,p)^4}.$$

Then, in normal coordinates x at p, setting $y = \frac{x}{|x|^2}$ (Kelvin inversion) one has an asymptotically flat manifold in y-coordinates

$$\tilde{g}(x) \simeq \frac{dx^2}{|x|^4} \simeq dy^2, \qquad (y \text{ large}).$$

Einstein's equation in vacuum

Einstein's equation in vacuum

It governs the structure of space-time according to general relativity

Einstein's equation in vacuum

It governs the structure of space-time according to general relativity

$$\text{(Einstein tensor)} \qquad E_{ij} := R_{ij} - \frac{1}{2} R_g \, g_{ij} = 0.$$

Einstein's equation in vacuum

It governs the structure of space-time according to general relativity

$$\text{(Einstein tensor)} \qquad E_{ij} := R_{ij} - \frac{1}{2} R_g \, g_{ij} = 0.$$

Here R_{ij} is the Ricci tensor, and R_g the scalar curvature.

Einstein's equation in vacuum

It governs the structure of space-time according to general relativity

$$\text{(Einstein tensor)} \qquad E_{ij} := R_{ij} - \frac{1}{2} R_g \, g_{ij} = 0.$$

Here R_{ij} is the Ricci tensor, and R_g the scalar curvature.

This equation is variational, with Euler-Lagrange functional given by

$$\mathcal{A}(g) := \int_M R_g \, dV_g \qquad \underline{\text{Einstein-Hilbert functional}}.$$

Einstein's equation in vacuum

It governs the structure of space-time according to general relativity

$$\text{(Einstein tensor)} \qquad E_{ij} := R_{ij} - \frac{1}{2} R_g \, g_{ij} = 0.$$

Here R_{ij} is the Ricci tensor, and R_g the scalar curvature.

This equation is variational, with Euler-Lagrange functional given by

$$\mathcal{A}(g) := \int_M R_g \, dV_g \qquad \underline{\text{Einstein-Hilbert functional}}.$$

In fact, one has

$$\frac{d}{dg} \left(R_g \, dV_g \right) [h] = - \left(h^{ij} E_{ij} + \text{div } X \right) dV_g,$$

where X is some vector field.

The mass of an asymptotically flat manifold

The mass of an asymptotically flat manifold

If we consider variations that preserve asymptotic flatness, then the divergence term has a role

The mass of an asymptotically flat manifold

If we consider variations that preserve asymptotic flatness, then the divergence term has a role (flux at infinity)

The mass of an asymptotically flat manifold

If we consider variations that preserve asymptotic flatness, then the divergence term has a role (flux at infinity), and

$$\frac{d}{dg}(\mathcal{A}(g) + m(g))[h] = \int_M h^{ij} E_{ij}\, dV.$$

The mass of an asymptotically flat manifold

If we consider variations that preserve asymptotic flatness, then the divergence term has a role (flux at infinity), and

$$\frac{d}{dg}(\mathcal{A}(g) + m(g))[h] = \int_M h^{ij} E_{ij}\, dV.$$

The quantity $m(g)$, called *ADM mass* ([ADM, '60]), is defined as

$$m(g) := \lim_{r \to \infty} \oint_{S_r} (\partial_k\, g_{jk} - \partial_j\, g_{kk})\, \nu^j d\sigma.$$

The mass of an asymptotically flat manifold

If we consider variations that preserve asymptotic flatness, then the divergence term has a role (flux at infinity), and

$$\frac{d}{dg}(\mathcal{A}(g) + m(g))[h] = \int_M h^{ij} E_{ij}\, dV.$$

The quantity $m(g)$, called *ADM mass* ([ADM, '60]), is defined as

$$m(g) := \lim_{r \to \infty} \oint_{S_r} (\partial_k\, g_{jk} - \partial_j\, g_{kk})\, \nu^j d\sigma.$$

Example 1: Schwartzschild. m_{ADM} = black-hole mass.

If we consider variations that preserve asymptotic flatness, then the divergence term has a role (flux at infinity), and

$$\frac{d}{dg}(\mathcal{A}(g) + m(g))[h] = \int_M h^{ij} E_{ij}\, dV.$$

The quantity $m(g)$, called *ADM mass* ([ADM, '60]), is defined as

$$m(g) := \lim_{r \to \infty} \oint_{S_r} (\partial_k\, g_{jk} - \partial_j\, g_{kk})\, \nu^j d\sigma.$$

Example 1: Schwartzschild. $m_{ADM} =$ black-hole mass.

Example 2: Conformal blow-ups. If G_p is the Green's function of an elliptic operator on $\hat{M}$ with pole at p, then $G_p(x) \simeq d(x,p)^{-1}$.

The mass of an asymptotically flat manifold

If we consider variations that preserve asymptotic flatness, then the divergence term has a role (flux at infinity), and

$$\frac{d}{dg}(\mathcal{A}(g) + m(g))[h] = \int_M h^{ij} E_{ij}\, dV.$$

The quantity $m(g)$, called *ADM mass* ([ADM, '60]), is defined as

$$m(g) := \lim_{r\to\infty} \oint_{S_r} (\partial_k\, g_{jk} - \partial_j\, g_{kk})\, \nu^j d\sigma.$$

Example 1: Schwartzschild. m_{ADM} = black-hole mass.

Example 2: Conformal blow-ups. If G_p is the Green's function of an elliptic operator on $\hat{M}$ with pole at p, then $G_p(x) \simeq d(x,p)^{-1}$.
If $f(x) = G_p^4 \simeq d(x,p)^{-4}$ and $\tilde{g}(x) = f(x)g(x)$, then

$$m_{ADM} = \lim_{x\to p}\left(G_p(x) - \frac{1}{d(x,p)}\right) = A.$$

The Positive Mass Theorem

The Positive Mass Theorem

Theorem ([Schoen-Yau, '79 ('81, '17)])

The Positive Mass Theorem

Theorem ([Schoen-Yau, '79 ('81, '17)])

If $R_g \geq 0$ then $m(g) \geq 0$.

Theorem ([Schoen-Yau, '79 ('81, '17)])

If $R_g \geq 0$ then $m(g) \geq 0$. In case $m(g) = 0$, then (M, g) is isometric to the flat Euclidean space $(\mathbb{R}^3, dx^2)$.

The Positive Mass Theorem

Theorem ([Schoen-Yau, '79 ('81, '17)])

If $R_g \geq 0$ then $m(g) \geq 0$. In case $m(g) = 0$, then (M, g) is isometric to the flat Euclidean space $(\mathbb{R}^3, dx^2)$.

Physically, this means that a positive local energy density implies a positive global energy for the system.

Theorem ([Schoen-Yau, '79 ('81, '17)])

If $R_g \geq 0$ then $m(g) \geq 0$. In case $m(g) = 0$, then (M, g) is isometric to the flat Euclidean space $(\mathbb{R}^3, dx^2)$.

Physically, this means that a positive local energy density implies a positive global energy for the system. (But one cannot just integrate!)

Theorem ([Schoen-Yau, '79 ('81, '17)])

If $R_g \geq 0$ then $m(g) \geq 0$. In case $m(g) = 0$, then (M, g) is <u>isometric</u> to the flat Euclidean space $(\mathbb{R}^3, dx^2)$.

Physically, this means that a positive <u>local</u> energy density implies a positive <u>global</u> energy for the system. (But one cannot just integrate!)

The proof used the construction of <u>stable</u> asymptotically planar minimal surfaces assuming $m < 0$, obtaining then a contradiction from the second variation formula using $R_g \geq 0$.

CR manifolds

CR manifolds

We deal with three-dimensional manifolds with a non-integrable two-dimensional distribution (contact structure) ξ.

CR manifolds

We deal with three-dimensional manifolds with a non-integrable two-dimensional distribution (contact structure) ξ.

We also have a CR structure (complex rotation) $J : \xi \to \xi$ s.t. $J^2 = -1$.

CR manifolds

We deal with three-dimensional manifolds with a non-integrable two-dimensional distribution (contact structure) ξ.

We also have a CR structure (complex rotation) $J : \xi \to \xi$ s.t. $J^2 = -1$.

Given J as above, we have locally a vector field Z_1 such that

$$JZ_1 = iZ_1; \qquad JZ_{\overline{1}} = -iZ_{\overline{1}} \qquad \text{where} \quad Z_{\overline{1}} = \overline{(Z_1)}.$$

CR manifolds

We deal with three-dimensional manifolds with a non-integrable two-dimensional distribution (contact structure) ξ.

We also have a CR structure (complex rotation) $J : \xi \to \xi$ s.t. $J^2 = -1$.

Given J as above, we have locally a vector field Z_1 such that

$$JZ_1 = iZ_1; \qquad JZ_{\overline{1}} = -iZ_{\overline{1}} \qquad \text{where} \quad Z_{\overline{1}} = \overline{(Z_1)}.$$

A contact form θ is a 1-form annihilating ξ

CR manifolds

We deal with three-dimensional manifolds with a non-integrable two-dimensional distribution (contact structure) ξ.

We also have a CR structure (complex rotation) $J : \xi \to \xi$ s.t. $J^2 = -1$.

Given J as above, we have locally a vector field Z_1 such that

$$J Z_1 = i Z_1; \qquad J Z_{\overline{1}} = -i Z_{\overline{1}} \qquad \text{where} \quad Z_{\overline{1}} = \overline{(Z_1)}.$$

A contact form θ is a 1-form annihilating ξ: we assume that $\theta \wedge d\theta \neq 0$ everywhere on M (pseudoconvexity).

CR manifolds

We deal with three-dimensional manifolds with a non-integrable two-dimensional distribution (contact structure) ξ.

We also have a CR structure (complex rotation) $J : \xi \to \xi$ s.t. $J^2 = -1$.

Given J as above, we have locally a vector field Z_1 such that

$$JZ_1 = iZ_1; \qquad JZ_{\overline{1}} = -iZ_{\overline{1}} \qquad \text{where} \quad Z_{\overline{1}} = \overline{(Z_1)}.$$

A contact form θ is a 1-form annihilating ξ: we assume that $\theta \wedge d\theta \neq 0$ everywhere on M (pseudoconvexity).

This condition is quite important for the study of biholomorphic mappings and the $\overline{\partial}$-Neumann problem ([Beals-Fefferman-Grossman, '83]).

Examples

The *Heisenberg group* (flat model) $\mathbb{H}^1 = \{(z,t) \in \mathbb{C} \times \mathbb{R}\}$.

Examples

The *Heisenberg group* (flat model) $\mathbb{H}^1 = \{(z,t) \in \mathbb{C} \times \mathbb{R}\}$. Setting

$$\overset{\circ}{Z}_1 = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial z} + i\overline{z}\frac{\partial}{\partial t}\right); \qquad \overset{\circ}{Z}_{\overline{1}} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial \overline{z}} - iz\frac{\partial}{\partial t}\right),$$

ξ_0 is spanned by real and imaginary parts of $\overset{\circ}{Z}_1$.

The *Heisenberg group* (flat model) $\mathbb{H}^1 = \{(z,t) \in \mathbb{C} \times \mathbb{R}\}$. Setting

$$\overset{\circ}{Z}_1 = \frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial z} + i\overline{z} \frac{\partial}{\partial t} \right); \qquad \overset{\circ}{Z}_{\overline{1}} = \frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial \overline{z}} - iz \frac{\partial}{\partial t} \right),$$

ξ_0 is spanned by real and imaginary parts of $\overset{\circ}{Z}_1$. The standard CR structure $J_0 : \xi_0 \to \xi_0$ verifies $J_0 \overset{\circ}{Z}_1 = i \overset{\circ}{Z}_1$.

Examples

The *Heisenberg group* (flat model) $\mathbb{H}^1 = \{(z,t) \in \mathbb{C} \times \mathbb{R}\}$. Setting

$$\overset{\circ}{Z}_1 = \frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial z} + i\overline{z} \frac{\partial}{\partial t} \right); \qquad \overset{\circ}{Z}_{\overline{1}} = \frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial \overline{z}} - iz \frac{\partial}{\partial t} \right),$$

ξ_0 is spanned by real and imaginary parts of $\overset{\circ}{Z}_1$. The standard CR structure $J_0 : \xi_0 \to \xi_0$ verifies $J_0 \overset{\circ}{Z}_1 = i \overset{\circ}{Z}_1$. $\overset{\circ}{\theta} = dt + iz d\overline{z} - i\overline{z} dz.$

The *Heisenberg group* (flat model) $\mathbb{H}^1 = \{(z,t) \in \mathbb{C} \times \mathbb{R}\}$. Setting

$$\overset{\circ}{Z}_1 = \frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial z} + i\overline{z} \frac{\partial}{\partial t} \right); \qquad \overset{\circ}{Z}_{\overline{1}} = \frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial \overline{z}} - iz \frac{\partial}{\partial t} \right),$$

ξ_0 is spanned by real and imaginary parts of $\overset{\circ}{Z}_1$. The standard CR structure $J_0 : \xi_0 \to \xi_0$ verifies $J_0 \overset{\circ}{Z}_1 = i \overset{\circ}{Z}_1$. $\overset{\circ}{\theta} = dt + izd\overline{z} - i\overline{z}dz$.

Boundaries of complex domains.

The *Heisenberg group* (flat model) $\mathbb{H}^1 = \{(z,t) \in \mathbb{C} \times \mathbb{R}\}$. Setting

$$\overset{\circ}{Z}_1 = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial z} + i\overline{z}\frac{\partial}{\partial t}\right); \qquad \overset{\circ}{Z}_{\overline{1}} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial \overline{z}} - iz\frac{\partial}{\partial t}\right),$$

ξ_0 is spanned by real and imaginary parts of $\overset{\circ}{Z}_1$. The standard CR structure $J_0 : \xi_0 \to \xi_0$ verifies $J_0 \overset{\circ}{Z}_1 = i \overset{\circ}{Z}_1$. $\quad \overset{\circ}{\theta} = dt + iz d\overline{z} - i\overline{z} dz$.

Boundaries of complex domains. Consider $\Omega \subset \mathbb{C}^2$ and J_2 the standard complex rotation in $\mathbb{C}^2$.

Examples

The *Heisenberg group* (flat model) $\mathbb{H}^1 = \{(z,t) \in \mathbb{C} \times \mathbb{R}\}$. Setting

$$\overset{\circ}{Z}_1 = \frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial z} + i\overline{z} \frac{\partial}{\partial t} \right); \qquad \overset{\circ}{Z}_{\overline{1}} = \frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial \overline{z}} - iz \frac{\partial}{\partial t} \right),$$

ξ_0 is spanned by real and imaginary parts of $\overset{\circ}{Z}_1$. The standard CR structure $J_0 : \xi_0 \to \xi_0$ verifies $J_0 \overset{\circ}{Z}_1 = i \overset{\circ}{Z}_1$. $\overset{\circ}{\theta} = dt + iz d\overline{z} - i\overline{z} dz$.

Boundaries of complex domains. Consider $\Omega \subset \mathbb{C}^2$ and J_2 the standard complex rotation in $\mathbb{C}^2$. Given $p \in \partial\Omega$ one can consider the subset ξ_p of $T_p\partial\Omega$ which is invariant by J_2.

The *Heisenberg group* (flat model) $\mathbb{H}^1 = \{(z,t) \in \mathbb{C} \times \mathbb{R}\}$. Setting

$$\overset{\circ}{Z}_1 = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial z} + i\overline{z}\frac{\partial}{\partial t}\right); \qquad \overset{\circ}{Z}_{\overline{1}} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial \overline{z}} - iz\frac{\partial}{\partial t}\right),$$

ξ_0 is spanned by real and imaginary parts of $\overset{\circ}{Z}_1$. The standard CR structure $J_0 : \xi_0 \to \xi_0$ verifies $J_0 \overset{\circ}{Z}_1 = i \overset{\circ}{Z}_1$. $\overset{\circ}{\theta} = dt + iz d\overline{z} - i\overline{z} dz$.

Boundaries of complex domains. Consider $\Omega \subset \mathbb{C}^2$ and J_2 the standard complex rotation in $\mathbb{C}^2$. Given $p \in \partial\Omega$ one can consider the subset ξ_p of $T_p\partial\Omega$ which is invariant by J_2. We take ξ_p as contact distribution, and $J|_{\xi_p}$ as the CR structure J.

The Webster curvature of a CR three-manifold

The Webster curvature of a CR three-manifold

In 1983 Webster introduced *scalar curvature* W, to study the biholomorphy problem, which behaves conformally like the scalar curvature.

The Webster curvature of a CR three-manifold

In 1983 Webster introduced *scalar curvature* W, to study the biholomorphy problem, which behaves conformally like the scalar curvature.

Changing conformally the contact form, if $\hat{\theta} = u^2\theta$, then $W_{\hat{\theta}}$ is given by

$$-4\Delta_b u + W_\theta u = W_{\hat{\theta}} u^3.$$

The Webster curvature of a CR three-manifold

In 1983 Webster introduced *scalar curvature* W, to study the biholomorphy problem, which behaves conformally like the scalar curvature.

Changing conformally the contact form, if $\hat{\theta} = u^2\theta$, then $W_{\hat{\theta}}$ is given by

$$-4\Delta_b u + W_\theta u = W_{\hat{\theta}} u^3.$$

Here Δ_b is the *sub-laplacian* on M: roughly, the *laplacian in the contact directions*

The Webster curvature of a CR three-manifold

In 1983 Webster introduced *scalar curvature* W, to study the biholomorphy problem, which behaves conformally like the scalar curvature.

Changing conformally the contact form, if $\hat{\theta} = u^2\theta$, then $W_{\hat{\theta}}$ is given by

$$-4\Delta_b u + W_\theta u = W_{\hat{\theta}} u^3.$$

Here Δ_b is the *sub-laplacian* on M: roughly, the *laplacian in the contact directions* (use Hörmander's theory (commutators) to recover regularity).

In 1983 Webster introduced *scalar curvature* W, to study the biholomorphy problem, which behaves conformally like the scalar curvature.

Changing conformally the contact form, if $\hat{\theta} = u^2\theta$, then $W_{\hat{\theta}}$ is given by

$$-4\Delta_b u + W_\theta u = W_{\hat{\theta}} u^3.$$

Here Δ_b is the *sub-laplacian* on M: roughly, the *laplacian in the contact directions* (use Hörmander's theory (commutators) to recover regularity).

As before, we can define a *Sobolev-Webster quotient*, a *Webster class*, and try to uniformize W as we did for the scalar curvature.

In 1983 Webster introduced *scalar curvature* W, to study the biholomorphy problem, which behaves conformally like the scalar curvature.

Changing conformally the contact form, if $\hat{\theta} = u^2\theta$, then $W_{\hat{\theta}}$ is given by

$$-4\Delta_b u + W_\theta u = W_{\hat{\theta}} u^3.$$

Here Δ_b is the *sub-laplacian* on M: roughly, the *laplacian in the contact directions* (use Hörmander's theory (commutators) to recover regularity).

As before, we can define a *Sobolev-Webster quotient*, a *Webster class*, and try to uniformize W as we did for the scalar curvature. In real dimension $n \geq 5$ Jerison and Lee (1989) proved the counterparts of Trudinger and Aubin's results.

The Webster curvature of a CR three-manifold

In 1983 Webster introduced *scalar curvature* W, to study the biholomorphy problem, which behaves conformally like the scalar curvature.

Changing conformally the contact form, if $\hat{\theta} = u^2\theta$, then $W_{\hat{\theta}}$ is given by

$$-4\Delta_b u + W_\theta u = W_{\hat{\theta}} u^3.$$

Here Δ_b is the *sub-laplacian* on M: roughly, the *laplacian in the contact directions* (use Hörmander's theory (commutators) to recover regularity).

As before, we can define a *Sobolev-Webster quotient*, a *Webster class*, and try to uniformize W as we did for the scalar curvature. In real dimension $n \geq 5$ Jerison and Lee (1989) proved the counterparts of Trudinger and Aubin's results. In real dimension $n = 3$ non-minimal solutions were found in [Gamara (et al.), '01].

Green's function and mass in three dimensions (CR)

Green's function and mass in three dimensions (CR)

In 3D the Green's function still appears.

Green's function and mass in three dimensions (CR)

In 3D the Green's function still appears. In suitable coordinates at $p \in M$

$$G_p \simeq 1/\rho^2 + A,$$

where $\rho^4(z,t) = |z|^4 + t^2$, $(z,t) \in \mathbb{H}^1$ is the *homogeneous distance*.

Green's function and mass in three dimensions (CR)

In 3D the Green's function still appears. In suitable coordinates at $p \in M$

$$G_p \simeq 1/\rho^2 + A,$$

where $\rho^4(z,t) = |z|^4 + t^2$, $(z,t) \in \mathbb{H}^1$ is the *homogeneous distance.* Blowing-up the contact form θ using G_p, we obtain an asymptotically (Heisenberg) flat manifold and define its mass, proportional to A.

Green's function and mass in three dimensions (CR)

In 3D the Green's function still appears. In suitable coordinates at $p \in M$

$$G_p \simeq 1/\rho^2 + A,$$

where $\rho^4(z,t) = |z|^4 + t^2$, $(z,t) \in \mathbb{H}^1$ is the *homogeneous distance*. Blowing-up the contact form θ using G_p, we obtain an asymptotically (Heisenberg) flat manifold and define its <u>mass</u>, proportional to A.

However, one crucial difference between dimension three and higher is the *embeddability* of abstract CR manifolds ([Chen-Shaw, '01]).

Green's function and mass in three dimensions (CR)

In 3D the Green's function still appears. In suitable coordinates at $p \in M$

$$G_p \simeq 1/\rho^2 + A,$$

where $\rho^4(z,t) = |z|^4 + t^2$, $(z,t) \in \mathbb{H}^1$ is the *homogeneous distance.* Blowing-up the contact form θ using G_p, we obtain an asymptotically (Heisenberg) flat manifold and define its mass, proportional to A.

However, one crucial difference between dimension three and higher is the *embeddability* of abstract CR manifolds ([Chen-Shaw, '01]). There is a fourth-order (Paneitz) operator $P = \Delta_b^2 + l.o.t.$ which plays a role here.

Green's function and mass in three dimensions (CR)

In 3D the Green's function still appears. In suitable coordinates at $p \in M$

$$G_p \simeq 1/\rho^2 + A,$$

where $\rho^4(z,t) = |z|^4 + t^2$, $(z,t) \in \mathbb{H}^1$ is the *homogeneous distance*. Blowing-up the contact form θ using G_p, we obtain an asymptotically (Heisenberg) flat manifold and define its mass, proportional to A.

However, one crucial difference between dimension three and higher is the *embeddability* of abstract CR manifolds ([Chen-Shaw, '01]). There is a fourth-order (Paneitz) operator $P = \Delta_b^2 + l.o.t.$ which plays a role here.

Theorem ([Chanillo-Chiu-Yang, '12])

Green's function and mass in three dimensions (CR)

In 3D the Green's function still appears. In suitable coordinates at $p \in M$

$$G_p \simeq 1/\rho^2 + A,$$

where $\rho^4(z,t) = |z|^4 + t^2$, $(z,t) \in \mathbb{H}^1$ is the *homogeneous distance.* Blowing-up the contact form θ using G_p, we obtain an asymptotically (Heisenberg) flat manifold and define its mass, proportional to A.

However, one crucial difference between dimension three and higher is the *embeddability* of abstract CR manifolds ([Chen-Shaw, '01]). There is a fourth-order (Paneitz) operator $P = \Delta_b^2 + l.o.t.$ which plays a role here.

Theorem ([Chanillo-Chiu-Yang, '12]) Let M^3 be a compact CR manifold.

Green's function and mass in three dimensions (CR)

In 3D the Green's function still appears. In suitable coordinates at $p \in M$

$$G_p \simeq 1/\rho^2 + A,$$

where $\rho^4(z,t) = |z|^4 + t^2$, $(z,t) \in \mathbb{H}^1$ is the *homogeneous distance*. Blowing-up the contact form θ using G_p, we obtain an asymptotically (Heisenberg) flat manifold and define its mass, proportional to A.

However, one crucial difference between dimension three and higher is the *embeddability* of abstract CR manifolds ([Chen-Shaw, '01]). There is a fourth-order (Paneitz) operator $P = \Delta_b^2 + l.o.t.$ which plays a role here.

Theorem ([Chanillo-Chiu-Yang, '12]) Let M^3 be a compact CR manifold. If $P \geq 0$ and $W > 0$, then M embeds into some $\mathbb{C}^N$.

Green's function and mass in three dimensions (CR)

In 3D the Green's function still appears. In suitable coordinates at $p \in M$

$$G_p \simeq 1/\rho^2 + A,$$

where $\rho^4(z,t) = |z|^4 + t^2$, $(z,t) \in \mathbb{H}^1$ is the *homogeneous distance*. Blowing-up the contact form θ using G_p, we obtain an asymptotically (Heisenberg) flat manifold and define its mass, proportional to A.

However, one crucial difference between dimension three and higher is the *embeddability* of abstract CR manifolds ([Chen-Shaw, '01]). There is a fourth-order (Paneitz) operator $P = \Delta_b^2 + l.o.t.$ which plays a role here.

Theorem ([Chanillo-Chiu-Yang, '12]) Let M^3 be a compact CR manifold. If $P \geq 0$ and $W > 0$, then M embeds into some $\mathbb{C}^N$.

More relations between P and embeddability properties of CR manifolds in [Chanillo-Case-Yang, '16], [Takeuchi, '19].

A positive mass theorem in CR geometry

Theorem 1 ([Cheng-M.-Yang, '17])

Theorem 1 ([Cheng-M.-Yang, '17])

Let (M^3, J, θ) be a compact CR manifold.

A positive mass theorem in CR geometry

Theorem 1 ([Cheng-M.-Yang, '17])

Let (M^3, J, θ) be a compact CR manifold. Suppose the Webster class is positive, and that the Paneitz operator P is non-negative.

A positive mass theorem in CR geometry

Theorem 1 ([Cheng-M.-Yang, '17])

Let (M^3, J, θ) be a compact CR manifold. Suppose the Webster class is positive, and that the Paneitz operator P is non-negative. Let $p \in M$ and let $\tilde{\theta}$ be a blown-up of contact form at p.

A positive mass theorem in CR geometry

Theorem 1 ([Cheng-M.-Yang, '17])

Let (M^3, J, θ) be a compact CR manifold. Suppose the Webster class is positive, and that the Paneitz operator P is non-negative. Let $p \in M$ and let $\tilde{\theta}$ be a blown-up of contact form at p. Then

(a) the CR mass m of $(M, J, \tilde{\theta})$ is non negative;

Theorem 1 ([Cheng-M.-Yang, '17])

Let (M^3, J, θ) be a compact CR manifold. Suppose the Webster class is positive, and that the Paneitz operator P is non-negative. Let $p \in M$ and let $\tilde{\theta}$ be a blown-up of contact form at p. Then

(a) the CR mass m of $(M, J, \tilde{\theta})$ is non negative;

(b) if $m = 0$, (M, J, θ) is conformally equivalent to a standard $S^3 (\simeq \mathbb{H}^1)$.

A positive mass theorem in CR geometry

Theorem 1 ([Cheng-M.-Yang, '17])

Let (M^3, J, θ) be a compact CR manifold. Suppose the Webster class is positive, and that the Paneitz operator P is non-negative. Let $p \in M$ and let $\tilde{\theta}$ be a blown-up of contact form at p. Then

(a) the CR mass m of $(M, J, \tilde{\theta})$ is non negative;

(b) if $m = 0$, (M, J, θ) is conformally equivalent to a standard $S^3 (\simeq \mathbb{H}^1)$.

- The proof uses a tricky integration by parts

Theorem 1 ([Cheng-M.-Yang, '17])

Let (M^3, J, θ) be a compact CR manifold. Suppose the Webster class is positive, and that the Paneitz operator P is non-negative. Let $p \in M$ and let $\tilde{\theta}$ be a blown-up of contact form at p. Then

(a) the CR mass m of $(M, J, \tilde{\theta})$ is non negative;

(b) if $m = 0$, (M, J, θ) is conformally equivalent to a standard $S^3 (\simeq \mathbb{H}^1)$.

• The proof uses a tricky integration by parts: the main idea was to bring-in the Paneitz operator to write the mass as sum of squares.

Theorem 1 ([Cheng-M.-Yang, '17])

Let (M^3, J, θ) be a compact CR manifold. Suppose the Webster class is positive, and that the Paneitz operator P is non-negative. Let $p \in M$ and let $\tilde{\theta}$ be a blown-up of contact form at p. Then

(a) the CR mass m of $(M, J, \tilde{\theta})$ is non negative;

(b) if $m = 0$, (M, J, θ) is conformally equivalent to a standard $S^3(\simeq \mathbb{H}^1)$.

• The proof uses a tricky integration by parts: the main idea was to bring-in the Paneitz operator to write the mass as sum of squares.

• Positivity of the mass implies that the Sobolev-Webster quotient of the manifold is lower than that of the sphere, and minimizers exist.

On the positivity condition for the Paneitz operator

On the positivity condition for the Paneitz operator

Consider S^3 in $\mathbb{C}^2$. Its *standard CR structure* $J_{(0)}$ is given by

$$J_{(0)} Z_1^{S^3} = i Z_1^{S^3}; \qquad\qquad Z_1^{S^3} = \bar{z}^2 \frac{\partial}{\partial z^1} - \bar{z}^1 \frac{\partial}{\partial z^2}.$$

On the positivity condition for the Paneitz operator

Consider S^3 in $\mathbb{C}^2$. Its *standard CR structure* $J_{(0)}$ is given by

$$J_{(0)} Z_1^{S^3} = i Z_1^{S^3}; \qquad\qquad Z_1^{S^3} = \bar{z}^2 \frac{\partial}{\partial z^1} - \bar{z}^1 \frac{\partial}{\partial z^2}.$$

It turns out that most perturbations of the standard structure are non embeddable ([Burns-Epstein, '90]).

On the positivity condition for the Paneitz operator

Consider S^3 in $\mathbb{C}^2$. Its *standard CR structure* $J_{(0)}$ is given by

$$J_{(0)} Z_1^{S^3} = i Z_1^{S^3}; \qquad Z_1^{S^3} = \bar{z}^2 \frac{\partial}{\partial z^1} - \bar{z}^1 \frac{\partial}{\partial z^2}.$$

It turns out that most perturbations of the standard structure are non embeddable ([Burns-Epstein, '90]).

Interesting case are *Rossi spheres* S^3_s, from [H.Rossi, '65]

On the positivity condition for the Paneitz operator

Consider S^3 in $\mathbb{C}^2$. Its *standard CR structure* $J_{(0)}$ is given by

$$J_{(0)} Z_1^{S^3} = i Z_1^{S^3}; \qquad\qquad Z_1^{S^3} = \bar{z}^2 \frac{\partial}{\partial z^1} - \bar{z}^1 \frac{\partial}{\partial z^2}.$$

It turns out that most perturbations of the standard structure are non embeddable ([Burns-Epstein, '90]).

Interesting case are *Rossi spheres* S^3_s, from [H.Rossi, '65]: these are homogeneous, of positive Webster class, have the same contact structure as the standard S^3 but a *distorted complex rotation* $J_{(s)}$ for $s \in (-\varepsilon, \varepsilon)$

$$J_{(s)}(Z_1^{S^3} + s\bar{Z}_1^{S^3}) = i\left(Z_1^{S^3} + s\bar{Z}_1^{S^3}\right).$$

On the positivity condition for the Paneitz operator

Consider S^3 in $\mathbb{C}^2$. Its *standard CR structure* $J_{(0)}$ is given by

$$J_{(0)} Z_1^{S^3} = i Z_1^{S^3}; \qquad Z_1^{S^3} = \bar{z}^2 \frac{\partial}{\partial z^1} - \bar{z}^1 \frac{\partial}{\partial z^2}.$$

It turns out that most perturbations of the standard structure are non embeddable ([Burns-Epstein, '90]).

Interesting case are *Rossi spheres* S^3_s, from [H.Rossi, '65]: these are homogeneous, of positive Webster class, have the same contact structure as the standard S^3 but a *distorted complex rotation* $J_{(s)}$ for $s \in (-\varepsilon, \varepsilon)$

$$J_{(s)}(Z_1^{S^3} + s\bar{Z}_1^{S^3}) = i \left(Z_1^{S^3} + s\bar{Z}_1^{S^3} \right).$$

In these cases the Paneitz operator cannot be positive-definite.

On the positivity condition for the Paneitz operator

Consider S^3 in $\mathbb{C}^2$. Its *standard CR structure* $J_{(0)}$ is given by

$$J_{(0)} Z_1^{S^3} = i Z_1^{S^3}; \qquad Z_1^{S^3} = \bar{z}^2 \frac{\partial}{\partial z^1} - \bar{z}^1 \frac{\partial}{\partial z^2}.$$

It turns out that most perturbations of the standard structure are non embeddable ([Burns-Epstein, '90]).

Interesting case are *Rossi spheres* S^3_s, from [H.Rossi, '65]: these are homogeneous, of positive Webster class, have the same contact structure as the standard S^3 but a *distorted complex rotation* $J_{(s)}$ for $s \in (-\varepsilon, \varepsilon)$

$$J_{(s)}(Z_1^{S^3} + s\bar{Z}_1^{S^3}) = i \left(Z_1^{S^3} + s\bar{Z}_1^{S^3} \right).$$

In these cases the Paneitz operator cannot be positive-definite.

Theorem 2 ([Cheng-M.-Yang, '19])

On the positivity condition for the Paneitz operator

Consider S^3 in $\mathbb{C}^2$. Its *standard CR structure* $J_{(0)}$ is given by

$$J_{(0)} Z_1^{S^3} = i Z_1^{S^3}; \qquad Z_1^{S^3} = \bar{z}^2 \frac{\partial}{\partial z^1} - \bar{z}^1 \frac{\partial}{\partial z^2}.$$

It turns out that most perturbations of the standard structure are non embeddable ([Burns-Epstein, '90]).

Interesting case are *Rossi spheres* S^3_s, from [H.Rossi, '65]: these are homogeneous, of positive Webster class, have the same contact structure as the standard S^3 but a *distorted complex rotation* $J_{(s)}$ for $s \in (-\varepsilon, \varepsilon)$

$$J_{(s)}(Z_1^{S^3} + s\bar{Z}_1^{S^3}) = i\left(Z_1^{S^3} + s\bar{Z}_1^{S^3}\right).$$

In these cases the Paneitz operator cannot be positive-definite.

Theorem 2 ([Cheng-M.-Yang, '19])

For small $s \neq 0$, the CR mass of S^3_s is negative ($m_s \simeq -18\pi s^2$).

Some ideas of the proof

Some ideas of the proof

As we saw, the mass is related to the next-order term in the expansion of the Green's function (Robin's function).

Some ideas of the proof

As we saw, the mass is related to the next-order term in the expansion of the Green's function (Robin's function). Determining it is in general a hard problem, since it is a global object.

Some ideas of the proof

As we saw, the mass is related to the next-order term in the expansion of the Green's function (Robin's function). Determining it is in general a hard problem, since it is a global object.

Fixing a *pole* $p \in S^3$, we find suitable s-coordinates (near p) to expand the Green's function as $G_{p,(s)} \simeq \frac{1}{\rho_{(s)}^2} + A_{(s)}$, with $A_{(s)}$ unknown.

Some ideas of the proof

As we saw, the mass is related to the next-order term in the expansion of the Green's function (Robin's function). Determining it is in general a hard problem, since it is a global object.

Fixing a *pole* $p \in S^3$, we find suitable s-coordinates (near p) to expand the Green's function as $G_{p,(s)} \simeq \frac{1}{\rho^2_{(s)}} + A_{(s)}$, with $A_{(s)}$ unknown.

On the other hand, it is possible to Taylor-expand in s the equation

$$-4\Delta_b^{(s)} G_{(s)} + W_{(s)} G_{(s)} = \delta_p$$

away from p, in the standard coordinates of $\mathbb{C}^2$.

Some ideas of the proof

As we saw, the mass is related to the next-order term in the expansion of the Green's function (Robin's function). Determining it is in general a hard problem, since it is a global object.

Fixing a *pole* $p \in S^3$, we find suitable s-coordinates (near p) to expand the Green's function as $G_{p,(s)} \simeq \frac{1}{\rho^2_{(s)}} + A_{(s)}$, with $A_{(s)}$ unknown.

On the other hand, it is possible to Taylor-expand in s the equation

$$-4\Delta_b^{(s)} G_{(s)} + W_{(s)} G_{(s)} = \delta_p$$

away from p, in the standard coordinates of $\mathbb{C}^2$.

One then needs to verify that the two expansions match, obtaining then the asymptotic behaviour for $s \to 0$ of $A_{(s)}$, proportional to the mass. $\square$

CR Sobolev quotient of Rossi spheres

Theorem 3 ([Cheng-M.-Yang, '19])

CR Sobolev quotient of Rossi spheres

Theorem 3 ([Cheng-M.-Yang, '19])

For small $s \neq 0$ the infimum of the Sobolev-Webster quotient of Rossi spheres is not attained (and is equal to that of the standard S^3).

Theorem 3 ([Cheng-M.-Yang, '19])

For small $s \neq 0$ the infimum of the Sobolev-Webster quotient of Rossi spheres is not attained (and is equal to that of the standard S^3).

Sketch of the proof.

- If a function has low Sobolev-Webster quotient on a Rossi sphere S^3_s it has low Sobolev-Webster quotient also on the standard $S^3 = S^3_0$.

Theorem 3 ([Cheng-M.-Yang, '19])

For small $s \neq 0$ the infimum of the Sobolev-Webster quotient of Rossi spheres is not attained (and is equal to that of the standard S^3).

Sketch of the proof.

- If a function has low Sobolev-Webster quotient on a Rossi sphere S^3_s it has low Sobolev-Webster quotient also on the standard $S^3 = S^3_0$.

- Minima of the quotient on S^3_0 were classifed in [Jerison-Lee, '88] as (CR counterparts of) Aubin-Talenti functions: call them U^{CR}_λ ($\lambda > 0$).

CR Sobolev quotient of Rossi spheres

Theorem 3 ([Cheng-M.-Yang, '19])

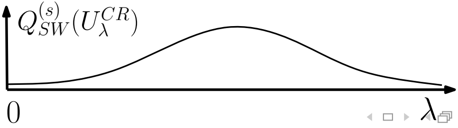
For small $s \neq 0$ the infimum of the Sobolev-Webster quotient of Rossi spheres is not attained (and is equal to that of the standard S^3).

Sketch of the proof.

- If a function has low Sobolev-Webster quotient on a Rossi sphere S^3_s it has low Sobolev-Webster quotient also on the standard $S^3 = S^3_0$.

- Minima of the quotient on S^3_0 were classifed in [Jerison-Lee, '88] as (CR counterparts of) Aubin-Talenti functions: call them U^{CR}_λ ($\lambda > 0$).

- For $|s| \neq 0$ small, the Webster quotient of the functions U^{CR}_λ has a profile of this kind, for λ in a <u>fixed compact set</u> of $(0, \infty)$



CR Sobolev quotient of Rossi spheres

CR Sobolev quotient of Rossi spheres

It remains to understand the case in which minimizers were close in the Sobolev sense to functions U_λ^{CR} with λ large (λ small is analogous).

CR Sobolev quotient of Rossi spheres

It remains to understand the case in which minimizers were close in the Sobolev sense to functions U_λ^{CR} with λ large (λ small is analogous).

For s fixed and λ large (depending on s) it is possible to show that $Q_{SW}^{(s)}(U_\lambda^{CR}) \simeq Q_{SW}^{(0)}(S^3) - \frac{m_{(s)}}{\lambda^2} + O(\lambda^{-3})$, which is larger than $Q_{SW}^{(0)}(S^3)$ since the mass $m_{(s)}$ is negative.

CR Sobolev quotient of Rossi spheres

It remains to understand the case in which minimizers were close in the Sobolev sense to functions U_λ^{CR} with λ large (λ small is analogous).

For s fixed and λ large (depending on s) it is possible to show that $Q_{SW}^{(s)}(U_\lambda^{CR}) \simeq Q_{SW}^{(0)}(S^3) - \frac{m_{(s)}}{\lambda^2} + O(\lambda^{-3})$, which is larger than $Q_{SW}^{(0)}(S^3)$ since the mass $m_{(s)}$ is negative.

However in this way we cannot guarantee high energy for all values of λ: some *intermediate range* is missing.

CR Sobolev quotient of Rossi spheres

It remains to understand the case in which minimizers were close in the Sobolev sense to functions U_λ^{CR} with λ large (λ small is analogous).

For s fixed and λ large (depending on s) it is possible to show that $Q_{SW}^{(s)}(U_\lambda^{CR}) \simeq Q_{SW}^{(0)}(S^3) - \frac{m_{(s)}}{\lambda^2} + O(\lambda^{-3})$, which is larger than $Q_{SW}^{(0)}(S^3)$ since the mass $m_{(s)}$ is negative.

However in this way we cannot guarantee high energy for all values of λ: some *intermediate range* is missing. To cover that too, we exploit an *isomorphism* between S^3_{+s} and S^3_{-s}.

CR Sobolev quotient of Rossi spheres

It remains to understand the case in which minimizers were close in the Sobolev sense to functions U_λ^{CR} with λ large (λ small is analogous).

For s fixed and λ large (depending on s) it is possible to show that $Q_{SW}^{(s)}(U_\lambda^{CR}) \simeq Q_{SW}^{(0)}(S^3) - \frac{m_{(s)}}{\lambda^2} + O(\lambda^{-3})$, which is larger than $Q_{SW}^{(0)}(S^3)$ since the mass $m_{(s)}$ is negative.

However in this way we cannot guarantee high energy for all values of λ: some *intermediate range* is missing. To cover that too, we exploit an *isomorphism* between S^3_{+s} and S^3_{-s}. By evenness in the parameter s, this implies that indeed

$$Q_{SW}^{(s)}(U_\lambda^{CR}) \simeq Q_{SW}^{(0)}(S^3) - \frac{m_{(s)}}{\lambda^2} + O(s^2\lambda^{-3}),$$

proving a strict inequality for all λ's. $\square$

Comments

Comments

We have seen that the CR Sobolev quotient of S^3_s, a closed manifold, behaves like that of a domain in $\mathbb{R}^n$.

Comments

We have seen that the CR Sobolev quotient of S^3_s, a closed manifold, behaves like that of a domain in $\mathbb{R}^n$. This fact seems to be tightly related to the non-embeddability of Rossi spheres.

Comments

We have seen that the CR Sobolev quotient of S^3_s, a <u>closed</u> manifold, behaves like that of a domain in $\mathbb{R}^n$. This fact seems to be tightly related to the non-embeddability of Rossi spheres.

To understand the phenomenon more in general, recall that the standard metric of S^n is a saddle point of the Einstein-Hilbert functional

$$g \quad \longrightarrow \quad \int_{S^n} R_g dV_g.$$

Comments

We have seen that the CR Sobolev quotient of S^3_s, a closed manifold, behaves like that of a domain in $\mathbb{R}^n$. This fact seems to be tightly related to the non-embeddability of Rossi spheres.

To understand the phenomenon more in general, recall that the standard metric of S^n is a saddle point of the Einstein-Hilbert functional

$$g \quad \longrightarrow \quad \int_{S^n} R_g dV_g.$$

It is a minimum in the conformal class and a maximum outside of it.

We have seen that the CR Sobolev quotient of S^3_s, a closed manifold, behaves like that of a domain in $\mathbb{R}^n$. This fact seems to be tightly related to the non-embeddability of Rossi spheres.

To understand the phenomenon more in general, recall that the standard metric of S^n is a saddle point of the Einstein-Hilbert functional

$$g \quad \longrightarrow \quad \int_{S^n} R_g dV_g.$$

It is a minimum in the conformal class and a maximum outside of it.

In the (3D) CR case one has positive second variations also for special non-embeddable directions ([Bland, '94]).

Comments

We have seen that the CR Sobolev quotient of S^3_s, a closed manifold, behaves like that of a domain in $\mathbb{R}^n$. This fact seems to be tightly related to the non-embeddability of Rossi spheres.

To understand the phenomenon more in general, recall that the standard metric of S^n is a saddle point of the Einstein-Hilbert functional

$$g \quad \longrightarrow \quad \int_{S^n} R_g dV_g.$$

It is a minimum in the conformal class and a maximum outside of it.

In the (3D) CR case one has positive second variations also for special non-embeddable directions ([Bland, '94]). It would be interesting to observe this change of sign also the mass and the Sobolev quotient.

Some open problems

Another problem recently settled is compactness of solutions to Yamabe's equation ([Druet, '04], [Li-Zhang, '05-'06], [Brendle-Marques, '08], [Khuri-Marques-Schoen, '09]).

Some open problems

Another problem recently settled is compactness of solutions to Yamabe's equation ([Druet, '04], [Li-Zhang, '05-'06], [Brendle-Marques, '08], [Khuri-Marques-Schoen, '09]). Compactness holds if and only if $n \leq 24$.

Some open problems

Another problem recently settled is compactness of solutions to Yamabe's equation ([Druet, '04], [Li-Zhang, '05-'06], [Brendle-Marques, '08], [Khuri-Marques-Schoen, '09]). Compactness holds if and only if $n \leq 24$.

Compactness for the CR case is entirely open.

Some open problems

Another problem recently settled is compactness of solutions to Yamabe's equation ([Druet, '04], [Li-Zhang, '05-'06], [Brendle-Marques, '08], [Khuri-Marques-Schoen, '09]). Compactness holds if and only if $n \leq 24$.

Compactness for the CR case is entirely open. One reason is that profiles of blow-ups are not classified.

Some open problems

Another problem recently settled is compactness of solutions to Yamabe's equation ([Druet, '04], [Li-Zhang, '05-'06], [Brendle-Marques, '08], [Khuri-Marques-Schoen, '09]). Compactness holds if and only if $n \leq 24$.

Compactness for the CR case is entirely open. One reason is that profiles of blow-ups are not classified. This concerns entire positive solutions to

$$-\Delta_b u = u^{\frac{Q+2}{Q-2}} \quad \text{in } \mathbb{H}^n; \qquad Q = 2n+2.$$

Some open problems

Another problem recently settled is compactness of solutions to Yamabe's equation ([Druet, '04], [Li-Zhang, '05-'06], [Brendle-Marques, '08], [Khuri-Marques-Schoen, '09]). Compactness holds if and only if $n \leq 24$.

Compactness for the CR case is entirely open. One reason is that profiles of blow-ups are not classified. This concerns entire positive solutions to

$$-\Delta_b u = u^{\frac{Q+2}{Q-2}} \quad \text{in } \mathbb{H}^n; \qquad Q = 2n + 2.$$

Assuming finite volume, it is done in [Jerison-Lee, '88].

Some open problems

Another problem recently settled is compactness of solutions to Yamabe's equation ([Druet, '04], [Li-Zhang, '05-'06], [Brendle-Marques, '08], [Khuri-Marques-Schoen, '09]). Compactness holds if and only if $n \leq 24$.

Compactness for the CR case is entirely open. One reason is that profiles of blow-ups are not classified. This concerns entire positive solutions to

$$-\Delta_b u = u^{\frac{Q+2}{Q-2}} \quad \text{in } \mathbb{H}^n; \qquad Q = 2n + 2.$$

Assuming finite volume, it is done in [Jerison-Lee, '88]. Howeverwe may not have this assumption, and moving planes do not work.

Some open problems

Another problem recently settled is compactness of solutions to Yamabe's equation ([Druet, '04], [Li-Zhang, '05-'06], [Brendle-Marques, '08], [Khuri-Marques-Schoen, '09]). Compactness holds if and only if $n \leq 24$.

Compactness for the CR case is entirely open. One reason is that profiles of blow-ups are not classified. This concerns entire positive solutions to

$$-\Delta_b u = u^{\frac{Q+2}{Q-2}} \quad \text{in } \mathbb{H}^n; \qquad Q = 2n + 2.$$

Assuming finite volume, it is done in [Jerison-Lee, '88]. Howeverwe may not have this assumption, and moving planes do not work.

A related problem concerns the classification of

$$-\Delta_b u = u^p \quad \text{in } \mathbb{H}^n; \qquad p < \frac{Q+2}{Q-2}.$$

Some open problems

Another problem recently settled is compactness of solutions to Yamabe's equation ([Druet, '04], [Li-Zhang, '05-'06], [Brendle-Marques, '08], [Khuri-Marques-Schoen, '09]). Compactness holds if and only if $n \leq 24$.

Compactness for the CR case is entirely open. One reason is that profiles of blow-ups are not classified. This concerns entire positive solutions to

$$-\Delta_b u = u^{\frac{Q+2}{Q-2}} \quad \text{in } \mathbb{H}^n; \qquad Q = 2n + 2.$$

Assuming finite volume, it is done in [Jerison-Lee, '88]. Howeverwe may not have this assumption, and moving planes do not work.

A related problem concerns the classification of

$$-\Delta_b u = u^p \quad \text{in } \mathbb{H}^n; \qquad p < \frac{Q+2}{Q-2}.$$

In $\mathbb{R}^n$ it was shown in [Gidas-Spruck, '81] that $u \equiv 0$.

Some open problems

Another problem recently settled is compactness of solutions to Yamabe's equation ([Druet, '04], [Li-Zhang, '05-'06], [Brendle-Marques, '08], [Khuri-Marques-Schoen, '09]). Compactness holds if and only if $n \leq 24$.

Compactness for the CR case is entirely open. One reason is that profiles of blow-ups are not classified. This concerns entire positive solutions to

$$-\Delta_b u = u^{\frac{Q+2}{Q-2}} \quad \text{in } \mathbb{H}^n; \qquad Q = 2n+2.$$

Assuming finite volume, it is done in [Jerison-Lee, '88]. Howeverwe may not have this assumption, and moving planes do not work.

A related problem concerns the classification of

$$-\Delta_b u = u^p \quad \text{in } \mathbb{H}^n; \qquad p < \frac{Q+2}{Q-2}.$$

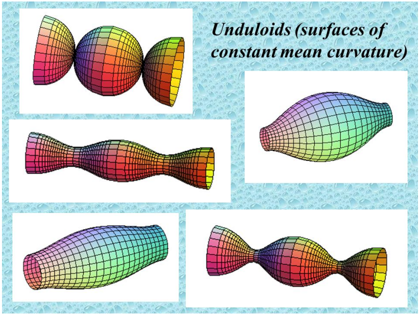
In $\mathbb{R}^n$ it was shown in [Gidas-Spruck, '81] that $u \equiv 0$. In $\mathbb{H}^n$, there are partial results in [Birindelli-Capuzzo Dolcetta-Cutrì, 97], for $p < \frac{Q}{Q-2}$.

A similar problem holds for *singular solutions* on $\mathbb{H}^n \setminus \{0\}$.

A similar problem holds for *singular solutions* on $\mathbb{H}^n \setminus \{0\}$. In $\mathbb{R}^n \setminus \{0\}$ such solutions were classified in [Caffarelli-Gidas-Spruck, '89] and were shown to be radial.

A similar problem holds for *singular solutions* on $\mathbb{H}^n \setminus \{0\}$. In $\mathbb{R}^n \setminus \{0\}$ such solutions were classified in [Caffarelli-Gidas-Spruck, '89] and were shown to be radial.

For the Heisenberg group there is a recent construction in [Afeltra, '19], where solutions similar to *Delaunay's unduloids* were produced.



Thanks for your attention