Energy estimates for hyperbolic operators with non Lipschitz-continuous coefficients

Daniele Del Santo
(joint work with **F. Colombini** and **F. Fanelli**)

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Strictly hyperbolic equations

Let's consider the operator

$$Lu = \partial_t^2 u - \sum_{j,k=1}^{n} \partial_{x_j}(a_{j,k}(t,x)\partial_{x_k} u)$$

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on the strip $[0, T] \times \mathbb{R}^n$. Suppose that for all $(t,x) \in [0, T] \times \mathbb{R}^n$ and for all $j, k = 1 \dots n$,

$$a_{j,k}(t,x) = a_{k,j}(t,x) \in \mathbb{R}.$$

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$$a_{j,k}(t,x) = a_{k,j}(t,x) \in \mathbb{R}.$$

Suppose that L is **strictly hyperbolic** i.e. there exist $\Lambda_0 \geq \lambda_0 > 0$ such that, for all $(t,x,\xi) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^n$,

$$\lambda_0 |\xi|^2 \leq \sum_{j,k} a_{j,k}(t,x)\xi_j\xi_k \leq \Lambda_0 |\xi|^2.$$

The Cauchy problem for strictly hyperbolic equations

We are interested in the Cauchy problem

$$\begin{cases} Lu = 0 & \text{in } [0,T] \times \mathbb{R}^n, \\ u|_{t=0} = u_0, \quad \partial_t u|_{t=0} = u_1 & \text{in } \mathbb{R}^n. \end{cases} \tag{1}$$

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Is this Cauchy problem well-posed in Sobolev spaces?

(This means that for some $s \in \mathbb{R}$ and for all $u_0 \in H^{s+1}$, $u_1 \in H^s$, there exists a unique $u \in C^0([0,T], H^{s+1}) \cap C^1([0,T], H^s)$ (or possibly $C^0([0,T], H^{s^*+1}) \cap C^1([0,T], H^{s^*})$ with $s^* < s$) in such a way that (1) holds).

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Is it possible to prove an inequality of the type

$$\sup_{0\le t\le T^*}(\|u(t,\cdot)\|_{H^{s^*+1}}+\|\partial_t u(t,\cdot)\|_{H^{s^*}})$$
$$\le C(\|u(0,\cdot)\|_{H^{s+1}}+\|\partial_t u(0\cdot)\|_{H^s}+\int_0^{T^*}\|Lu(\tau,\cdot)\|_{H^{s^*}}\,d\tau), \tag{2}$$

for all $u\in C^2([0,T],H^\infty)$ *(where possibly* $T^*\le T$ *and* $s^*\le s$*)?*

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finite loss of derivatives.

regularity of coefficients vs energy inequality

The focus is on the relations between the

regularity of the coefficients (with respect to time and space)

and the

existence of an energy inequality in Sobolev spaces.

Coefficients depending only on t: Lipschitz and log-Lipschitz case

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then (2) is valid for $s^* = s$ (**no loss**, *classical result*).

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- If the coefficients $a_{j,k}$ are log-Lipschitz-continuous, i.e.

$$\sup_t |a_{j,k}(t+\tau) - a_{j,k}(t)| \leq C|\tau| \log(\frac{1}{|\tau|} + 1),$$

  then (2) for $s^* < s$ (**finite loss**, *Colombini, De Giorgi and Spagnolo '79*).

Coefficients depending only on t: Zygmund and log-Zygmund case

- If the coefficients $a_{j,k}$ are Zygmund-continuous, i.e.

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- If the coefficients $a_{j,k}$ are log-Lipschitz-continuous, i.e.

$$\sup_{t,x} |a_{j,k}(t+\tau, x+y) - a_{j,k}(t,x)| \leq C(|\tau| + |y|) \log(\frac{1}{|\tau| + |y|} + 1),$$

  then (2) for $s^* < s \in]-1, 0[$ (**finite loss**, *Colombini and Lerner '95*).

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then (2) is valid for $s^* = s = -1/2$ (**no loss, but only for**

$$u_0 \in H^{1/2}, \ u_1 \in H^{-1/2},$$

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- If the coefficients $a_{j,k}$ are log-Zygmund-continuous in t and log-Lipschitz-continuous in x, i.e.

  $$\sup_{t,x} |a_{jk}(t+\tau, x) + a_{jk}(t-\tau, x) - 2a_{jk}(t,x)| \leq C_0 |\tau| \log(\frac{1}{|\tau|} + 1),$$

  $$\sup_{t,x} |a_{jk}(t, x+y) - a_{jk}(t,x)| \leq C_1 |y| \log(\frac{1}{|y|} + 1).$$

  then (2) for $s^* < s \in]-1, 0[$ (**finite loss**, *Colombini, DS, Fanelli and Métivier, Comm. PDE '13*).

The problem

It is not clear what happens if the coefficients are depending on t and x, they are Zygmund-continuous and s is different from $-1/2$.

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Conjecture

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Here we present a partial answer, for coefficients which are

Zygmund-continuous in t and Lipschitz-continuous in x.

Statement of the result

Theorem (Colombini, DS and Fanelli)

Suppose that there exist constants C_0, $C_1 > 0$ such that, for all j, $k = 1, \ldots, n$ and for all $\tau \in \mathbb{R}$, $y \in \mathbb{R}^n$,

$$\sup_{t,x} |a_{jk}(t+\tau, x) + a_{jk}(t-\tau, x) - 2a_{jk}(t,x)| \leq C_0|\tau|,$$

$$\sup_{t,x} |a_{jk}(t, x+y) - a_{jk}(t,x)| \leq C_1|y|.$$

Then, for all fixed $s \in]-1, 0]$, there exists a constant $C > 0$, depending only on s and T, such that

$$\sup_{0 \leq t \leq T} (\|u(t,\cdot)\|_{H^{s+1}} + \|\partial_t u(t,\cdot)\|_{H^s})$$
$$\leq C(\|u(0,\cdot)\|_{H^{s+1}} + \|\partial_t u(0,\cdot)\|_{H^s} + \int_0^T \|Lu(\tau,\cdot)\|_{H^s}\, d\tau),$$

for all $u \in C^2([0, T], H^\infty)$.

Colombini-De Giorgi-Spagnolo's proof/1

- case $n = 1$, i.e.

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- Introduce $a_\varepsilon = \varrho_\varepsilon * a$, then, since a is Log-Lipschitz, we have
$$\sup_t |a(t) - a_\varepsilon(t)| \leq C\varepsilon \log(\tfrac{1}{\varepsilon} + 1),$$
$$\sup_t |a_\varepsilon'(t)| \leq C \log(\tfrac{1}{\varepsilon} + 1).$$

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- We have, uniformly in ε,
$$\int (1 + |\xi|^2)^s E_\varepsilon(t,\xi)d\xi \sim \|u(t,\cdot)\|^2_{H^{s+1}} + \|\partial_t u(t,\cdot)\|^2_{H^s}.$$

Colombini-De Giorgi-Spagnolo's proof/2

- Differentiating the approximate energy and using the equation

$$\partial_t E_\varepsilon(t,\xi) = 2(a_\varepsilon(t) - a(t))|\xi|^2 v v' + a'_\varepsilon(t)|\xi|^2|v|^2 + 2vv'$$

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so that, using Grönwall lemma,

$$\begin{aligned} E_\varepsilon(t,\xi) &\le E_\varepsilon(0,\xi)\exp\left[C\left(\int_0^T |a'_\varepsilon|\,dt + |\xi|\int_0^T |a-a_\varepsilon|\,dt + \int_0^T 1\,dt\right)\right] \\ &\le E_\varepsilon(0,\xi)\exp\left[C((\log\tfrac{1}{\varepsilon}+1) + |\xi|\varepsilon(\log\tfrac{1}{\varepsilon}+1)+1)\right] \end{aligned}$$

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- **Key point**: choose $\varepsilon = |\xi|^{-1}$: the approximation rate of the coefficients depend on the variable ξ, i.e. on the point of the phase space. We obtain

$$\begin{aligned} E_{|\xi|^{-1}}(t,\xi) &\le E_{|\xi|^{-1}}(0,\xi)\exp(C(\log(|\xi|+1)+1)) \\ &\le C' E_{|\xi|^{-1}}(0,\xi)(1+|\xi|)^C. \end{aligned}$$

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- Consider the **Tarama's approximate energy**

$$\tilde{E}_\varepsilon(t,\xi) = \frac{1}{\sqrt{a_\varepsilon}} |v'(t) + \frac{a'_\varepsilon}{4a_\varepsilon} v(t)|^2 + \sqrt{a_\varepsilon}|\xi|^2 |v(t)|^2,$$

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Tarama's proof/2

- Differentiating the approximate energy and using the equation $\partial_t \tilde{E}_\varepsilon(t,\xi)$ is

$$\frac{2}{\sqrt{a_\varepsilon}}\Big(v'(t)+\frac{a'_\varepsilon}{4a_\varepsilon}v(t)\Big)\Big(\big(\frac{a'_\varepsilon}{4a_\varepsilon}\big)'-\big(\frac{a'_\varepsilon}{4a_\varepsilon}\big)^2+(a_\varepsilon(t)-a(t))|\xi|^2\Big)v$$

so that, using Grönwall lemma,

$$\begin{aligned}\tilde{E}_\varepsilon(t,\xi) &\le \tilde{E}_\varepsilon(0,\xi)\exp\Big[C\big(\big(\tfrac{1}{|\xi|}\int_0^T |a''_\varepsilon|+|a'_\varepsilon|^2\,dt\big)+\big(|\xi|\int_0^T|a-a_\varepsilon|\,dt\big)\big)\Big]\\ &\le \tilde{E}_\varepsilon(0,\xi)\exp\Big[C\big(\tfrac{1}{|\xi|\varepsilon}+|\xi|\varepsilon\big)\Big].\end{aligned}$$

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- Choosing also in this case $\varepsilon=|\xi|^{-1}$ we have

$$\tilde{E}_\varepsilon(t,\xi)\leq C\tilde{E}_\varepsilon(0,\xi)$$

and the energy estimate follows **without loss of derivatives**.

Tools: Littlewood-Paley decomposition/1

Let $\psi \in C^\infty([0,+\infty[,\mathbb{R})$ such that ψ is non-increasing and

$$\psi(t) = 1 \quad \text{for} \quad 0 \le t \le \frac{11}{10}, \qquad \psi(t) = 0 \quad \text{for} \quad t \ge \frac{19}{10}.$$

We set, for $\xi \in \mathbb{R}^d$,

$$\chi(\xi) = \psi(|\xi|), \qquad \varphi(\xi) = \chi(\xi) - \chi(2\xi).$$

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Given a tempered distribution u, the dyadic blocks are defined by

$$\Delta_0 u = \chi(D)u = \mathcal{F}^{-1}(\chi(\xi)\hat{u}(\xi)),$$

$$\Delta_j u = \varphi(2^{-j}D)u = \mathcal{F}^{-1}(\varphi(2^{-j}\xi)\hat{u}(\xi)) \quad \text{if} \quad j \geq 1,$$

where we have denoted by $\mathcal{F}^{-1}$ the inverse of the Fourier transform. We introduce also the operator

$$S_k u = \sum_{j=0}^{k} \Delta_j u = \mathcal{F}^{-1}(\chi(2^{-k}\xi)\hat{u}(\xi)).$$

Tools: Littlewood-Paley decomposition/2

It is well known the characterization of classical Sobolev spaces via Littlewood-Paley decomposition: for any $s \in \mathbb{R}$, $u \in \mathcal{S}'$,

$$u \in H^s$$

if and only if

$$\forall j,\ \Delta_j u \in L^2 \quad \text{and} \quad \sum 2^{2js} \|\Delta_j u\|_{L^2}^2 < +\infty$$

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Moreover, in such a case, there exists a constant $C_s > 1$ such that

$$\frac{1}{C_s} \sum_{j=0}^{+\infty} 2^{2js} \|\Delta_j u\|_{L^2}^2 \leq \|u\|_{H^s}^2 \leq C_s \sum_{j=0}^{+\infty} 2^{2js} \|\Delta_j u\|_{L^2}^2 .$$

Tools: Littlewood-Paley decomposition/3

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Proposition

Let $u \in L^\infty(\mathbb{R}^d)$. We have the following:

$$u \in Lip(\mathbb{R}^d) \quad \text{if and only if} \quad \sup_j \|\nabla S_j u\|_{L^\infty} < +\infty,$$

$$u \in Zyg(\mathbb{R}^d) \quad \text{if and only if} \quad \sup_j 2^j \|\Delta_j u\|_{L^\infty} < +\infty,$$

$$u \in LogLip(\mathbb{R}^d) \quad \text{if and only if} \quad \sup_j \frac{\|\nabla S_j u\|_{L^\infty}}{j} < +\infty,$$

$$u \in LogZyg(\mathbb{R}^d) \quad \text{if and only if} \quad \sup_j \frac{2^j \|\Delta_j u\|_{L^\infty}}{j} < +\infty.$$

Tools: paradifferential calculus with parameters/1

Let $\gamma \geq 1$ and consider $\psi_\gamma \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ with the following properties

- there exist $\varepsilon_1 < \varepsilon_2 < 1$ such that

$$\psi_\gamma(\eta, \xi) = \begin{cases} 1 & \text{for} \quad |\eta| \leq \varepsilon_1(\gamma + |\xi|), \\ 0 & \text{for} \quad |\eta| \geq \varepsilon_2(\gamma + |\xi|); \end{cases}$$

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- for all $(\beta, \alpha) \in \mathbb{N}^d \times \mathbb{N}^d$, there exists $C_{\beta,\alpha} \geq 0$ such that

$$|\partial_\eta^\beta \partial_\xi^\alpha \psi_\gamma(\eta, \xi)| \leq C_{\beta,\alpha}(\gamma + |\xi|)^{-|\alpha|-|\beta|}.$$

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Define now

$$G^{\psi_\gamma}(x, \xi) = (\mathcal{F}_\eta^{-1}\psi_\gamma)(x, \xi),$$

where $\mathcal{F}_\eta^{-1}\psi_\gamma$ is the inverse of the Fourier transform of ψ_γ with respect to the η variable.

Tools: paradifferential calculus with parameters/2

Let $a \in L^\infty$. We associate to a the classical pseudodifferential symbol

$$\sigma_{a,\gamma}(x,\xi) = (\psi_\gamma(D_x,\xi)a)(x,\xi) = (G^{\psi_\gamma}(\cdot,\xi) * a)(x),$$

and we define the **paradifferential operator associate to a** as the classical pseudodifferential operator associated to $\sigma_{a,\gamma}$, i.e.

$$T_a^\gamma u(x) = \sigma_a(D_x)u(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d_\xi} \sigma_a(x,\xi)\hat{u}(\xi)\,d\xi.$$

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It is possible to choose ψ_γ in such a way that T_a^1 is the usual Bony's paraproduct operator

$$T_a^1 u = \sum_{k=0}^{+\infty} S_k a \Delta_{k+3} u,$$

while, in the general case,

$$T_a^\gamma u = S_{\mu-1} a S_{\mu+2} u + \sum_{k=\mu}^{+\infty} S_k a \Delta_{k+3} u, \quad \text{with} \quad \mu = [\log_2 \gamma].$$

Tools: low regularity symbols and calculus/1

We deal with paradifferential operators having symbols with limited regularity in time and space.

Tools: low regularity symbols and calculus/1

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Definition

A symbol of order m is a function $a(t, x, \xi, \gamma)$ which is locally bounded on $[0,\ T] \times \mathbb{R}^n \times \mathbb{R}^n \times [1, +\infty[$, of class C^∞ with respect to ξ such that, for all $\alpha \in \mathbb{N}^n$, there exists $C_\alpha > 0$ such that, for all (t, x, ξ, γ),

$$|\partial_\xi^\alpha a(t, x, \xi, \gamma)| \leq C_\alpha (\gamma + |\xi|)^{m-|\alpha|}.$$

We take now a symbol a of order $m \geq 0$, Zygmund-continuous with respect to t, uniformly with respect to x and Lipschitz-continuous with respect to x, uniformly with respect to t. We smooth out a with respect to time via a convolution with a mollifier, and call a_ε the smoothed symbol. We consider the classical symbol σ_{a_ε} obtained from a_ε via convolution with G^{ψ_γ}.

Tools: low regularity symbols and calculus/2

Proposition

Under the previous hypotheses, one has:

$$
\begin{aligned}
|\partial_\xi^\alpha \sigma_{a_\varepsilon}(t,x,\xi,\gamma)| &\leq C_\alpha(\gamma+|\xi|)^{m-|\alpha|},\\
|\partial_x^\beta\partial_\xi^\alpha \sigma_{a_\varepsilon}(t,x,\xi,\gamma)| &\leq C_{\beta,\alpha}(\gamma+|\xi|)^{m-|\alpha|+|\beta|-1},\\
|\partial_\xi^\alpha \sigma_{\partial_t a_\varepsilon}(t,x,\xi,\gamma)| &\leq C_\alpha(\gamma+|\xi|)^{m-|\alpha|}\log(\frac{1}{\varepsilon}+1),\\
|\partial_x^\beta\partial_\xi^\alpha \sigma_{\partial_t a_\varepsilon}(t,x,\xi,\gamma)| &\leq C_{\beta,\alpha}(\gamma+|\xi|)^{m-|\alpha|+|\beta|-1}\,\frac{1}{\varepsilon},\\
|\partial_\xi^\alpha \sigma_{\partial_t^2 a_\varepsilon}(t,x,\xi,\gamma)| &\leq C_\alpha(\gamma+|\xi|)^{m-|\alpha|}\,\frac{1}{\varepsilon},\\
|\partial_x^\beta\partial_\xi^\alpha \sigma_{\partial_t^2 a_\varepsilon}(t,x,\xi,\gamma)| &\leq C_{\beta,\alpha}(\gamma+|\xi|)^{m-|\alpha|+|\beta|-1}\,\frac{1}{\varepsilon^2},
\end{aligned}
$$

where $|\beta|\geq 1$ and all the constants C_α and $C_{\beta,\alpha}$ don't depend on γ.

Tools: low regularity symbols and calculus/3

In particular

$$|\partial_\xi^\alpha \sigma_{\partial_t a_\varepsilon}(t,x,\xi,\gamma)| \le C_\alpha(\gamma+|\xi|)^{m-|\alpha|}\log(\frac{1}{\varepsilon}+1)$$

is the analogue *(remember Tarama's proof)* of

$$\sup_t |a_\varepsilon'| \le C\log(\frac{1}{\varepsilon}+1)$$

and

$$|\partial_\xi^\alpha \sigma_{\partial_t^2 a_\varepsilon}(t,x,\xi,\gamma)| \le C_\alpha(\gamma+|\xi|)^{m-|\alpha|}\,\frac{1}{\varepsilon}$$

is the analogue of

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Proof: approximate energy/1

Let $u \in C^2([0,T], H^\infty)$.

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$$\partial_t^2 u = \sum_{j,k} \partial_j (a_{jk}(t,x)\partial_k u) + Lu = \sum_{j,k} \partial_j (T_{a_{jk}} \partial_k u) + \tilde{L}u,$$

where

$$\tilde{L}u = Lu + \sum_{j,k} \partial_j ((a_{jk} - T_{a_{jk}})\partial_k u).$$

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$$\tilde{L}u = Lu + \sum_{j,k} \partial_j((a_{jk} - T_{a_{jk}})\partial_k u).$$

We apply the operator Δ_ν and we obtain

$$\partial_t^2 u_\nu = \sum_{j,k} \partial_j(T_{a_{jk}}\partial_k u_\nu) + \sum_{j,k} \partial_j([\Delta_\nu, T_{a_{jk}}]\partial_k u) + (\tilde{L}u)_\nu,$$

where $u_\nu = \Delta_\nu u$, $(\tilde{L}u)_\nu = \Delta_\nu(\tilde{L}u)$ and $[\Delta_\nu, T_{a_{jk}}]$ is the commutator between the localization operator Δ_ν and the paramultiplication operator $T_{a_{jk}}$.

Proof: approximate energy/2

We consider the 0-th order symbol

$$\alpha_\varepsilon(t,x,\xi,\gamma) = (\gamma^2 + |\xi|^2)^{-\frac{1}{2}}(\gamma^2 + \sum_{j,k} a_{jk,\varepsilon}(t,x)\xi_j\xi_k)^{\frac{1}{2}}.$$

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We fix

$$\varepsilon = 2^{-\nu},$$

and we write α_ν and $a_{jk,\,\nu}$ instead of $\alpha_{2^{-\nu}}$ and $a_{jk,\,2^{-\nu}}$ respectively.

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We fix

$$\varepsilon = 2^{-\nu},$$

and we write α_ν and $a_{jk,\,\nu}$ instead of $\alpha_{2^{-\nu}}$ and $a_{jk,\,2^{-\nu}}$ respectively. We set

$$v_\nu(t,x) = T_{\alpha_\nu^{-1/2}}\partial_t u_\nu - T_{\partial_t(\alpha_\nu^{-1/2})} u_\nu,$$

$$w_\nu(t,x) = T_{\alpha_\nu^{1/2}(\gamma^2+|\xi|^2)^{1/2}}\ u_\nu,$$

$$z_\nu(t,x) = u_\nu,$$

Proof: approximate energy/3

We define

$$e_\nu(t) = \|v_\nu(t,\cdot)\|^2_{L^2} + \|w_\nu(t,\cdot)\|^2_{L^2} + \|z_\nu(t,\cdot)\|^2_{L^2}$$

(note that this is the analogue of Tarama's energy, where the role of ξ is now played by 2^ν Tarama's energy)

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$$E_s(t) = \sum_{\nu=0}^{+\infty} 2^{2\nu s} e_\nu(t).$$

It is possible to prove that there exist constants C_s and C_s', depending only on s, such that

$$E_s(0)^{\frac{1}{2}} \leq C_s(\|u(0,\cdot)\|_{H^{s+1}} + \|\partial_t u(0,\cdot)\|_{H^s}),$$
$$E_s(t)^{\frac{1}{2}} \geq C_s'(\|u(t,\cdot)\|_{H^{s+1}} + \|\partial_t u(t,\cdot)\|_{H^s}).$$

Proof: time derivative of the approximate energy/1

We obtain

$$
\begin{aligned}
\frac{d}{dt}\|v_\nu(t)\|^2_{L^2} \quad &= 2\,\mathrm{Re}\Big(v_\nu, \sum_{j,k} T_{\alpha_\nu^{-1/2}} \partial_j (T_{a_{jk}} \partial_k u_\nu)\Big)_{L^2} \\
&\qquad + 2\,\mathrm{Re}\Big(v_\nu, \sum_{j,k} T_{\alpha_\nu^{-1/2}} \partial_j ([\Delta_\nu,\ T_{a_{jk}}] \partial_k u)\Big)_{L^2} \\
&\qquad\qquad + 2\,\mathrm{Re}\Big(v_\nu,\, T_{\alpha_\nu^{-1/2}} (\tilde{L}u)_\nu\Big)_{L^2} + Q_1,
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with $|Q_1| \leq Ce_\nu(t)$,

Proof: time derivative of the approximate energy/1

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with $|Q_1| \le Ce_\nu(t)$,

$$\frac{d}{dt}\|w_\nu(t)\|^2_{L^2} = 2\,\mathrm{Re}\Big(v_\nu,\ T_{\alpha_\nu^{-1/2}}T_{\alpha_\nu^2(\gamma^2+|\xi|^2)}u_\nu\Big)_{L^2} + Q_2,$$

with $|Q_2| \le Ce_\nu(t)$ and

$$\frac{d}{dt}\|z_\nu(t)\|^2_{L^2} \le |2\,\mathrm{Re}\big(u_\nu, \partial_t u_\nu\big)_{L^2}| \le Ce_\nu(t).$$

Proof: time derivative of the approximate energy/2

Putting all together some terms cancel (due to the form of the energy) and we have

$$\begin{aligned}\frac{d}{dt}\, e_\nu(t) \quad &\leq \; C_1 e_\nu(t) + C_2 (e_\nu(t))^{\frac{1}{2}} \, \|(\tilde{L}u)_\nu\|_{L^2} \\ &\quad + \Big|2\,\mathrm{Re}\Big(v_\nu, \sum_{j,k} T_{\alpha_\nu^{-1/2}} \partial_j([\Delta_\nu,\, T_{a_{jk}}]\partial_k u)\Big)_{L^2}\Big|.\end{aligned}$$

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It remains to estimate the term containing $\tilde{L}u$ and that one with the commutator.

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\end{aligned}
$$

It remains to estimate the term containing $\tilde{L}u$ and that one with the commutator. In this computation it is used a result due to *Coifman and Meyer '78*.
We conclude that

$$
\frac{d}{dt} E_s(t) \leq C(E_s(t) + (E_s(t))^{\frac{1}{2}} \|Lu(t)\|_{H^s}).
$$

The energy estimate easily follows from this last inequality and Grönwall Lemma.

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Thank you for your attention!

integral condition

It is interesting to remark that the original condition of *Colombini, De Giorgi, and Spagnolo* is an integral condition weaker than the pontwise one, i.e

$$\int_0^{T-\tau} |a_{j,k}(t+\tau) - a_{j,k}(t)| \, dt \leq C|\tau| \log(\frac{1}{|\tau|} + 1).$$

back

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Similarly the conditions given by *Tarama* are

$$\int_\tau^{T-\tau} |a_{j,k}(t+\tau) + a_{j,k}(t-\tau) - 2a_{j,k}(t)|\, dt \leq C|\tau|,$$

and

$$\int_\tau^{T-\tau} |a_{j,k}(t+\tau) + a_{j,k}(t-\tau) - 2a_{j,k}(t)|\, dt \leq C|\tau| \log(\frac{1}{|\tau|} + 1).$$

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