Dispersive and subelliptic PDEs
Centro de Giorgi

Γ-convergence for integral functionals depending on vector fields

Andrea Pinamonti

Joint work with Alberto Maione and Francesco Serra Cassano
Department of Mathematics, University of Trento

February 12, 2020

Plan of the talk

Framework

We assume that $X_1, \dots, X_m$ are locally Lipschitz continuous vector fields on an open set $\Omega \subset \mathbb{R}^n$, i.e., $X_j = (c_{j1}, \dots, c_{jn})$, with $c_{ji} \in Lip_{\mathrm{loc}}(\Omega)$ for $j = 1, \dots, m$, $i = 1, \dots, n$. We identify

$$X_j = \sum_{i=1}^{n} c_{ji}(x)\partial_i \,.$$

Moreover, we define the *X-gradient*

$$X := (X_1, \dots, X_m)$$

and the *coefficient matrix of the X-gradient* as the $m \times n$ matrix

$$C(x) = [c_{ji}(x)]_{\substack{j=1,\dots,m \\ i=1,\dots,n}} \,.$$

Framework

Definition - Linear Independence Condition

We say that $X = (X_1, \dots, X_m)$ satisfies the *linear independence condition* (LIC) on an open set $\Omega \subset \mathbb{R}^n$, if there exists a set $\mathcal{N}_X \subset \Omega$, closed in the topology of Ω, such that $\mathcal{L}^n(\mathcal{N}_X) = 0$ and, for each $x \in \Omega_X := \Omega \setminus \mathcal{N}_X$, $X_1(x), \dots, X_m(x)$ are linearly independent as vectors of $\mathbb{R}^n$.

Framework

Definition - Linear Independence Condition

We say that $X = (X_1, \ldots, X_m)$ satisfies the *linear independence condition* (LIC) on an open set $\Omega \subset \mathbb{R}^n$, if there exists a set $\mathcal{N}_X \subset \Omega$, closed in the topology of Ω, such that $\mathcal{L}^n(\mathcal{N}_X) = 0$ and, for each $x \in \Omega_X := \Omega \setminus \mathcal{N}_X$, $X_1(x), \ldots, X_m(x)$ are linearly independent as vectors of $\mathbb{R}^n$.

Rmk. Notice that if $X = (X_1, \ldots, X_m)$ satisfies (LIC), then $m \leq n$.

Examples of relevant vector fields satisfying LIC

(i) (Euclidean gradient) Let $X = (X_1, \ldots, X_n) = (\partial_{x_1}, \ldots, \partial_{x_n})$.
Rmk. $\mathcal{N}_X = \emptyset$ and $m = n$.

Examples of relevant vector fields satisfying LIC

(i) (Euclidean gradient) Let $X = (X_1, \dots, X_n) = (\partial_{x_1}, \dots, \partial_{x_n})$.
Rmk. $\mathcal{N}_X = \emptyset$ and $m = n$.

(ii) (Grushin) Let $X = (X_1, X_2)$ be the vector fields on $\mathbb{R}^2$ defined as:

$$X_1(x) := \partial_{x_1}, \quad X_2(x) := x_1\, \partial_{x_2} \text{ if } x = (x_1, x_2) \in \mathbb{R}^2 .$$

Rmk. $\mathcal{N}_X = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 = 0\}$ and $m = n$.

Examples of relevant vector fields satisfying LIC

(i) (Euclidean gradient) Let $X = (X_1, \dots, X_n) = (\partial_{x_1}, \dots, \partial_{x_n})$.
Rmk. $\mathcal{N}_X = \emptyset$ and $m = n$.

(ii) (Grushin) Let $X = (X_1, X_2)$ be the vector fields on $\mathbb{R}^2$ defined as:

$$X_1(x) := \partial_{x_1}, \quad X_2(x) := x_1 \, \partial_{x_2} \text{ if } x = (x_1, x_2) \in \mathbb{R}^2 .$$

Rmk. $\mathcal{N}_X = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 = 0\}$ and $m = n$.

(iii) (Heisenberg) Let $X = (X_1, X_2)$ be the vector fields on $\mathbb{R}^3$ defined as:

$$X_1(x) := \partial_{x_1} - \frac{x_2}{2}\partial_{x_3}, \; X_2(x) := \partial_{x_2} + \frac{x_1}{2}\partial_{x_3} \text{ if } x = (x_1, x_2, x_3) \in \mathbb{R}^3 .$$

Rmk. $\mathcal{N}_X = \emptyset$ and $m < n$.

Examples of relevant vector fields satisfying LIC

(i) (Euclidean gradient) Let $X = (X_1, \ldots, X_n) = (\partial_{x_1}, \ldots, \partial_{x_n})$.
Rmk. $\mathcal{N}_X = \emptyset$ and $m = n$.

(ii) (Grushin) Let $X = (X_1, X_2)$ be the vector fields on $\mathbb{R}^2$ defined as:

$$X_1(x) := \partial_{x_1}, \quad X_2(x) := x_1\, \partial_{x_2} \text{ if } x = (x_1, x_2) \in \mathbb{R}^2 .$$

Rmk. $\mathcal{N}_X = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 = 0\}$ and $m = n$.

(iii) (Heisenberg) Let $X = (X_1, X_2)$ be the vector fields on $\mathbb{R}^3$ defined as:

$$X_1(x) := \partial_{x_1} - \frac{x_2}{2}\partial_{x_3},\ X_2(x) := \partial_{x_2} + \frac{x_1}{2}\partial_{x_3} \text{ if } x = (x_1, x_2, x_3) \in \mathbb{R}^3 .$$

Rmk. $\mathcal{N}_X = \emptyset$ and $m < n$.

(iv) (Vector Fields not satisfying the *Hörmander condition*) Let $X = (X_1, X_2)$ be the vector fields on $\mathbb{R}^3$ defined as:

$$X_1(x) := \partial_{x_1}, \quad X_2(x) := \partial_{x_2} \text{ if } x = (x_1, x_2, x_3) \in \mathbb{R}^3 .$$

Rmk. $\mathcal{N}_X = \emptyset$, $m < n$.

We will deal with integral functionals $F: L^p(\Omega) \to [0,\infty]$, $1 < p < \infty$, of the form

$$F(u) := \begin{cases} \int_\Omega f(x, Xu(x))dx & \text{if } u \in \mathrm{C}^1(\Omega) \\ +\infty & \text{if } u \in L^p(\Omega) \setminus \mathrm{C}^1(\Omega) \end{cases},$$

with *integrand function* $f: \Omega \times \mathbb{R}^m \to [0,\infty]$ in the class $I_{m,p}(\Omega, c_0, c_1, a_0, a_1)$, composed by Borel functions verifying the following assumptions:

(I_1) for a.e. $x \in \Omega$, the function $f(x,\cdot): \mathbb{R}^m \to [0,\infty)$ is convex;

(I_2) there exists constants $c_1 > c_0 \geq 0$ and two nonnegative functions $a_0, a_1 \in L^1(\Omega)$ such that

$$c_0 |\eta|^p - a_0(x) \leq f(x,\eta) \leq c_1 |\eta|^p + a_1(x),$$

for a.e. $x \in \Omega$ and for each $\eta \in \mathbb{R}^m$.

Functionals depending on vector fields

We will deal with integral functionals $F : L^p(\Omega) \to [0, \infty]$, $1 < p < \infty$, of the form

$$F(u) := \begin{cases} \int_\Omega f(x, Xu(x))dx & \text{if } u \in \mathrm{C}^1(\Omega) \\ +\infty & \text{if } u \in L^p(\Omega) \setminus \mathrm{C}^1(\Omega) \end{cases},$$

with *integrand function* $f : \Omega \times \mathbb{R}^m \to [0, \infty]$ in the class $I_{m,p}(\Omega, c_0, c_1, a_0, a_1)$, composed by Borel functions verifying the following assumptions:

(I_1) for a.e. $x \in \Omega$, the function $f(x, \cdot) : \mathbb{R}^m \to [0, \infty)$ is convex;

(I_2) there exists constants $c_1 > c_0 \geq 0$ and two nonnegative functions $a_0, a_1 \in L^1(\Omega)$ such that

$$c_0 |\eta|^p - a_0(x) \leq f(x, \eta) \leq c_1 |\eta|^p + a_1(x),$$

for a.e. $x \in \Omega$ and for each $\eta \in \mathbb{R}^m$.

Rmk. We will denote $I_{m,p}(\Omega, c_0, c_1) = I_{m,p}(\Omega, c_0, c_1, 0, 0)$.

Functionals depending on vector fields

Let $f \in I_{m,p}(\Omega, c_0, c_1, a_0, a_1)$ and $u \in \mathrm{C}^1(\Omega)$. One can write

$$F(u) = \int_\Omega f(x, Xu(x))dx = \int_\Omega f_e(x, Du(x))dx\,,$$

where $D = (\partial_{x_1}, \ldots, \partial_{x_n})$ and

$$f_e(x, \xi) := f(x, C(x)\xi) \text{ if } \xi \in \mathbb{R}^n$$

will be called Euclidean integrand associated to F.

Functionals depending on vector fields

Let $f \in I_{m,p}(\Omega, c_0, c_1, a_0, a_1)$ and $u \in \mathrm{C}^1(\Omega)$. One can write

$$F(u) = \int_\Omega f(x, Xu(x))dx = \int_\Omega f_e(x, Du(x))dx\,,$$

where $D = (\partial_{x_1}, \ldots, \partial_{x_n})$ and

$$f_e(x, \xi) := f(x, C(x)\xi) \text{ if } \xi \in \mathbb{R}^n$$

will be called Euclidean integrand associated to F.

Rmk. One can prove that the opposite representation may not hold.

Rmk. We proved that the opposite representation holds if and only

$$f_e(x,\xi) = f_e(x, \Pi_x(\xi)) \quad \text{for a.e. } x \in \Omega,\ \forall \xi \in \mathbb{R}^n,$$

where $\{V_x : x \in \Omega_X\}$ is the distribution of m-planes in $\mathbb{R}^n$ $V_x = \mathrm{span}_{\mathbb{R}}\{X_1(x), \dots, X_m(x)\}$ and $\Pi_x : \mathbb{R}^n \to V_x$ denotes the projection of $\mathbb{R}^n$ on V_x. Let X be the Heisenberg vector fields in $\mathbb{R}^3$, let $\Omega \subset \mathbb{R}^3$ be a bounded open set containing the origin and $p = 2$. Let $F : L^2(\Omega) \times \Omega \to [0, \infty]$ be the local functional defined as

$$F(u) := \begin{cases} \int_\Omega |Du|^2\, dx & \text{if } u \in W^{1,2}(\Omega) \\ \infty & \text{otherwise} \end{cases}.$$

If there is some integrand $f : \Omega \times \mathbb{R}^2 \to [0, \infty]$ for which the representation holds then,

$$|\xi|^2 = f_e(x,\xi) = f_e(x, \Pi_x(\xi)) = |\Pi_x(\xi)|^2$$

for a.e. $x \in \Omega, \forall \xi \in \mathbb{R}^3$.

Since the function $\Omega \ni x \mapsto \Pi_x(\xi)$ is continuous, the previous identity must hold for each $x \in \Omega$ and $\xi \in \mathbb{R}^3$. Let $x = 0$, then a simple calculation yields that $\Pi_0(\xi) = (\xi_1, \xi_2, 0)$ for each $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$. Thus, if we choose $\xi = (0, 0, 1)$, the previous identity is not satisfied and then we have a contradiction.

Examples of functionals depending on vector fields

Let $f(x,\eta) = |\eta|^2$ and let $u \in \mathrm{C}^1(\Omega)$. Then

(i) (**Grushin**)

$$F(u) = \int_\Omega f(x, Xu)\,dx = \int_\Omega \left(\partial_{x_1} u^2 + x_1^2 \partial_{x_2} u^2\right) dx\,.$$

(ii) (**Heisenberg**)

$$F(u) = \int_\Omega f(x, Xu)\,dx = \int_\Omega \left((\partial_1 u - \frac{x_2}{2}\partial_{x_3} u)^2 + (\partial_{x_2} u + \frac{x_1}{2}\partial_{x_3} u)^2\right) dx\,.$$

Examples of functionals depending on vector fields

Let $f(x,\eta) = |\eta|^2$ and let $u \in \mathrm{C}^1(\Omega)$. Then

(i) (**Grushin**)

$$F(u) = \int_\Omega f(x, Xu)\, dx = \int_\Omega \left(\partial_{x_1} u^2 + x_1^2 \partial_{x_2} u^2\right) dx\,.$$

(ii) (**Heisenberg**)

$$F(u) = \int_\Omega f(x, Xu)\, dx = \int_\Omega \left((\partial_1 u - \frac{x_2}{2}\partial_{x_3} u)^2 + (\partial_{x_2} u + \frac{x_1}{2}\partial_{x_3} u)^2\right) dx\,.$$

Rmk. Observe that the previous functionals are not coercive w.r.t. the Euclidean gradient, that is, the coercivity condition

$$f_e(x,\xi) \geq c_0\, |\xi|^2 \text{ a.e. } x \in \Omega,\ \forall\, \xi \in \mathbb{R}^n\,,$$

for a suitable constant $c_0 > 0$, may fail.

Sobolev spaces depending on vector fields

For $1 \leq p \leq \infty$ we set

$$W_X^{1,p}(\Omega) := \{u \in L^p(\Omega) : X_j u \in L^p(\Omega) \text{ for } j = 1, \dots, m\}$$

Rmk. It holds:

$$W^{1,p}(\Omega) \subset W_X^{1,p}(\Omega) \quad \forall\, p \in [1, \infty] \text{ and, for any } u \in W^{1,p}(\Omega)\,,$$
$$Xu(x) = C(x)\, Du(x) \quad \text{for a.e. } x \in \Omega\,,$$

where $W^{1,p}(\Omega)$ denotes the classical Sobolev space, or, equivalently, the space $W_X^{1,p}(\Omega)$ associated to $X = D := (\partial_{x_1}, \dots, \partial_{x_n})$. The inclusion can be strict.

Moreover, we will denote by $W_{X,0}^{1,p}(\Omega)$ the closure of $\mathrm{C}_c^1(\Omega) \cap W_X^{1,p}(\Omega)$ in $W_X^{1,p}(\Omega)$.

G.B. Folland, E.M. Stein, *Hardy spaces on homogeneous groups*, Princeton University Press, Princeton, 1982

Quick introduction to Γ convergence I

The theory of Γ-convergence was introduced in the '70 by E.De Giorgi. Among the precursors of the theory, one should mention:

- the Mosco convergence (for convex functions and their duals);
- the G-convergence of Spagnolo for elliptic operators in divergence form.

But, it is only with De Giorgi and with the examples worked out by his school that the theory reached a mature stage.

Let (X, d) be a metric space, $F_n : X \to (-\infty, +\infty)$ lower semicontinuous. As in many other cases, to define convergence we pass through the intermediate notions of upper and lower limits:

$$\Gamma - \limsup_{n\to\infty} F_n(x) := \inf\{\limsup_{n\to\infty} F_n(x_n) \mid x_n \to x\}$$
$$\Gamma - \liminf_{n\to\infty} F_n(x) := \inf\{\liminf_{n\to\infty} F_n(x_n) \mid x_n \to x\}$$

It is obvious that $\Gamma - \liminf_{n\to\infty} F_n \leq \Gamma - \limsup_{n\to\infty} F_n$, and it is not too difficult to check that they are both lower semicontinuous. We say that F_n $\Gamma-$converge if

$$\Gamma - \liminf_{n\to\infty} F_n \geq \Gamma - \limsup_{n\to\infty} F_n$$

and we denote the common value of the upper and lower Γ limits by $\Gamma - \lim_{n\to\infty} F_n$.

As soon as we have a guess F for the Γ-limit, we have to prove that

$$\Gamma - \limsup_{n\to\infty} F_n \leq F(x) \qquad \text{and} \qquad F(x) \leq \Gamma - \liminf_{n\to\infty} F_n.$$

The first inequality means that we should be able to find $(x_n) \subset X$ convergent to x with $\limsup_{n\to\infty} F_n(x_n) \leq F(x)$. Any sequence (x_n) with this property is called recovery sequence. The second inequality means that we should be able to prove, for any $(x_n) \subset X$ convergent to x, the lower bound for the $\liminf$, namely $\liminf_{n\to\infty} F_n(x_n) \geq F(x)$. In general pointwise convergence has nothing to do with Γ-convergence, for instance $F_n(x) = \sin(nx)$ Γ-converges to -1. In this case

$$x_n = -\frac{\pi}{2n} + \frac{2[nx/2]\pi}{n} \quad \text{is a recovery sequence.}$$

Theorem

If $\Gamma\text{-}\lim_{n\to\infty} F_n = F$ *and* $(x_n) \subset X$ *is s.t.*

$$F_n(x_n) \leq \inf_X F_n + \varepsilon_n$$

with $\varepsilon_n \to 0$*, then any limit point* x *of* (x_n) *minimizes* F*. In addition, under the equi-coercitivity assumption*

$$\inf_X F_n = \inf_K F_n \quad \textit{for some compact set } K \subset X \textit{ independent of } n,$$

one has that F_n *attain their minimum value, and*

$$\lim_{n\to\infty} \min_X F_n = \min_X F.$$

A Γ-compactness problem

Let $X = (X_1, \ldots, X_m)$ be a given family of locally Lipschitz vector fields on a bounded open set $\Omega \subset \mathbb{R}^n$, let $(f_h)_h \subset I_{m,p}(\Omega, c_0, c_1, a_0, a_1)$ and let $F_h : L^p(\Omega) \to [0, \infty]$ be defined as

$$F_h(u) := \begin{cases} \int_\Omega f_h(x, Xu(x))dx & \text{if } u \in W_X^{1,p}(\Omega) \\ +\infty & \text{if } u \in L^p(\Omega) \setminus W_X^{1,p}(\Omega) \end{cases} .$$

Question

Are there a function $f \in I_{m,p}(\Omega, c_0, c_1, a_0, a_1)$ and a functional $F : L^p(\Omega) \to [0, \infty]$ such that, up to a subsequence,

- $F = \Gamma(L^p(\Omega)) - \lim_{h\to\infty} F_h$,
- $F(u) = \int_\Omega f(x, Xu(x))\, dx$ for each $u \in W_X^{1,p}(\Omega)$?

Moreover, how can we characterize

$$\mathrm{dom}\, F := \{u \in L^p(\Omega) : F(u) < \infty\}\,?$$

The starting point

Assume that $f_h = f \in I_{m,p}(\Omega, c_0, c_1, a_0, a_1)$ for each $h \in \mathbb{N}$. Then, it is well-known that

$$\left(\Gamma(L^p(\Omega)) - \lim_{h\to\infty} F_h\right)(u) = \overline{F}(u),$$

where

$$\overline{F}(u) := \inf\left\{\liminf_{h\to\infty} F(u_h) : (u_h)_h \subset L^p(\Omega), u_h \to u \text{ in } L^p(\Omega)\right\}.$$

is the relaxed functional of F, w.r.t. the L^p-topology.

The starting point

Assume that $f_h = f \in I_{m,p}(\Omega, c_0, c_1, a_0, a_1)$ for each $h \in \mathbb{N}$. Then, it is well-known that

$$\left(\Gamma(L^p(\Omega)) - \lim_{h\to\infty} F_h\right)(u) = \overline{F}(u),$$

where

$$\overline{F}(u) := \inf\left\{\liminf_{h\to\infty} F(u_h) : (u_h)_h \subset L^p(\Omega), u_h \to u \text{ in } L^p(\Omega)\right\}.$$

is the relaxed functional of F, w.r.t. the L^p-topology.

Theorem (Franchi-Serapioni-Serra Cassano, 1996)

Let $X = (X_1, \ldots, X_m)$ be a given family of vector fields on a open set $\Omega \subset \mathbb{R}^n$ and let $1 < p < \infty$. Then:

- $\mathrm{dom}\,\overline{F} = W^{1,p}_X(\Omega)$;
- $\overline{F}(u) = \int_\Omega f(x, Xu(x))\,dx$ for every $u \in W^{1,p}_X(\Omega)$.

The Γ-compactness result

Theorem (Maione-P.-Serra Cassano

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and let $X = (X_1, \dots, X_m)$ satisfy (LIC) on Ω. Let $(f_h)_h \subset I_{m,p}(\Omega, c_0, c_1, a_0, a_1)$ and let $(F_h)_h$ be the associated sequence of integral functionals on $L^p(\Omega)$, $1 < \ p < \ \infty$. Then, up to a subsequence, there exist a $F : L^p(\Omega) \to [0, \infty]$ and $f \in I_{m,p}(\Omega, c_0, c_1, a_0, a_1)$ such that

- $F = \Gamma(L^p(\Omega)) - \lim_{h\to\infty} F_h$
- For each $u \in L^p(\Omega)$

$$\left(\Gamma(L^p(\Omega)) - \lim_{h\to\infty} F_h\right)(u) = \begin{cases} \int_\Omega f(x, Xu(x))dx & \text{if } u \in W^{1,p}_X(\Omega) \\ \infty & \text{otherwise} \end{cases}$$

A. Maione, A. Pinamonti, F. Serra Cassano, Γ*-convergence for functionals depending on vector fields I. Integral representation and compactness*, Journal de Mathématiques Pures et Appliquées, (2020)

Proof outline

Proof's strategy consists in two steps.

1st step. Let $\mathcal{A}$ be the class of all open subsets of Ω and let $(F_h)_h$ be a sequence of integral functionals on $L^p(\Omega) \times \mathcal{A}$, $1 < p < \infty$, of the form

$$F_h(u, A) := \begin{cases} \int_A f_{h,e}(x, Du(x))dx & \text{if } A \in \mathcal{A},\ u \in W^{1,1}_{\mathrm{loc}}(A) \\ +\infty & \text{otherwise} \end{cases}$$

where

$$f_{h,e}(x,\xi) := f_h(x, C(x)\xi) \quad x \in \Omega,\ \xi \in \mathbb{R}^n .$$

Proof outline

Proof's strategy consists in two steps.

1st step. Let $\mathcal{A}$ be the class of all open subsets of Ω and let $(F_h)_h$ be a sequence of integral functionals on $L^p(\Omega) \times \mathcal{A}$, $1 < p < \infty$, of the form

$$F_h(u, A) := \begin{cases} \int_A f_{h,e}(x, Du(x))dx & \text{if } A \in \mathcal{A},\ u \in W^{1,1}_{\mathrm{loc}}(A) \\ +\infty & \text{otherwise} \end{cases}$$

where

$$f_{h,e}(x, \xi) := f_h(x, C(x)\xi) \quad x \in \Omega,\ \xi \in \mathbb{R}^n .$$

Then, applying classical results from the Euclidean setting, up to a subsequence, there exists $F : L^p(\Omega) \times \mathcal{A} \to [0, \infty]$ such that

$$F(\cdot, A) = \left(\Gamma(L^p(\Omega)) - \lim_{h\to\infty} F_h \right) (\cdot, A) \quad \text{for each } A \in \mathcal{A} . \tag{1}$$

Proof outline

Moreover, F can be represented by an integral form on $W^{1,p}(A)$ by means of an **Euclidean integrand function**, that is,

$$F(u,A) := \int_A f_e(x, Du(x))\,dx \tag{2}$$

for every $A \in \mathcal{A}$, for every $u \in L^p(\Omega)$ such that $u|_A \in W^{1,p}(A)$, and for a suitable Borel function $f_e : \Omega \times \mathbb{R}^n \to [0,\infty]$.

Proof outline

Moreover, F can be represented by an integral form on $W^{1,p}(A)$ by means of an **Euclidean integrand function**, that is,

$$F(u,A) := \int_A f_e(x, Du(x))\,dx \tag{2}$$

for every $A \in \mathcal{A}$, for every $u \in L^p(\Omega)$ such that $u|_A \in W^{1,p}(A)$, and for a suitable Borel function $f_e : \Omega \times \mathbb{R}^n \to [0,\infty]$.

2nd step. We prove the following closure property w.r.t. the Γ-convergence: let $(f_h)_h \subset I_{m,p}(\Omega, c_0, c_1, a_0, a_1)$ such that (1) and (2) hold, then F can be represented in the following integral form

$$F(u) = \begin{cases} \int_\Omega f(x, Xu(x))dx & \text{if } u \in W^{1,p}_X(\Omega) \\ \infty & \text{otherwise} \end{cases}$$

for a suitable function $f \in I_{m,p}(\Omega, c_0, c_1, a_0, a_1)$.

Two interesting compact subclasses of integrands

- the subclass $J_1(\Omega, c_0, c_1)$ composed by integrand functions $f \in I_{m,p}(\Omega, c_0, c_1)$ such that $f = f(\eta)$, that is, f independent of x.

Two interesting compact subclasses of integrands

- the subclass $J_1(\Omega, c_0, c_1)$ composed by integrand functions $f \in I_{m,p}(\Omega, c_0, c_1)$ such that $f = f(\eta)$, that is, f independent of x.
- the subclass of $J_2(\Omega, c_0, c_1) := I_{m,2}(\Omega, c_0, c_1)$ composed of integrand functions $f \in I_{m,2}(\Omega, c_0, c_1)$ which are quadratic forms with respect to η, that is,

$$f(x, \eta) = \langle a(x)\eta, \eta\rangle = \sum_{i,j=1}^{m} a_{ij}(x)\eta_i\eta_j \quad \text{a.e. } x \in \Omega, \forall\, \eta \in \mathbb{R}^m,$$

with $a(x) = [a_{ij}(x)]$ $m \times m$ symmetric matrix.

Possible extensions

- Study functionals depending also on u;

Possible extensions

- Study functionals depending also on u;
- Study what happens if $p \in \{1, \infty\}$

A H-compactness problem

Let X be defined on a bounded open neighbourhood Ω_0 of Ω, let $H^1_{X,0}(\Omega)$ be $W^{1,2}_{X,0}(\Omega)$ and let $H^{-1}_X(\Omega)$ denote the dual space of $H^1_{X,0}(\Omega)$. Moreover, let

$$X_j^T\varphi := -\sum_{i=1}^{n}\partial_{x_i}(c_{j,i}\,\varphi) = -(\mathrm{div}(X_j) + X_j)\,\varphi \quad \forall\,\varphi\in \mathrm{C}^\infty_c(\Omega)$$

denote the (formal) adjoint of X_j in $L^2(\Omega)$, and $a(x) := [a_{ij}(x)]$ be a matrix in $J_2(\Omega, c_0, c_1)$, such that $c_0|\eta|^2 \le \langle a(x)\eta,\eta\rangle \le c_1|\eta|^2$ a.e. $x\in\Omega$ for all $\eta\in\mathbb{R}^m$.

A H-compactness problem

Let X be defined on a bounded open neighbourhood Ω_0 of Ω, let $H^1_{X,0}(\Omega)$ be $W^{1,2}_{X,0}(\Omega)$ and let $H^{-1}_X(\Omega)$ denote the dual space of $H^1_{X,0}(\Omega)$. Moreover, let

$$X_j^T\varphi := -\sum_{i=1}^n \partial_{x_i}(c_{j,i}\,\varphi) = -(\mathrm{div}(X_j) + X_j)\,\varphi \quad \forall\,\varphi \in \mathrm{C}_c^\infty(\Omega)$$

denote the (formal) adjoint of X_j in $L^2(\Omega)$, and $a(x) := [a_{ij}(x)]$ be a matrix in $J_2(\Omega, c_0, c_1)$, such that $c_0|\eta|^2 \le \langle a(x)\eta, \eta\rangle \le c_1|\eta|^2$ a.e. $x \in \Omega$ for all $\eta \in \mathbb{R}^m$.

Question

Can we infer a H-compactness result for the class $\mathcal{E}(\Omega, c_0, c_1)$, of linear partial differential operators in X-divergence form

$$\mathcal{L} = \mathrm{div}_X(a(x)X) := \sum_{j,i=1} X_j^T(a_{ij}(x)X_i),$$

whose domain $D(\mathcal{L})$ is the set of functions $u \in W^{1,2}_X(\Omega)$ such that the distribution defined by the right hand side belongs to $L^2(\Omega)$?

Additional hypotheses on X

Let Ω be a bounded open set and let X be defined and Lipschitz continuous in a neighbourhood Ω_0 of $\overline{\Omega}$, satisfying (LIC) and such that:

(H1) Let $d : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty]$ be the so-called Carnot-Carathéodory distance function induced by X. We assume $d(x, y) < \infty$ for any $x, y \in \Omega_0$, so that d is a standard distance in Ω_0. Moreover, the distance d is continuous with respect to the usual topology of $\mathbb{R}^n$.

(H2) For any compact $K \subset \Omega_0$ and for any $r < r_K$ and any $x \in K$ there exists a constant $C_K > 0$ such that $|B_d(x, 2r)| \leq C_K |B_d(x, r)|$ where $B_d(x, r)$ is the (open) metric ball with respect to d.

A. Sánchez-Calle, *Fundamental solutions and geometry of the sum of squares of vector fields*, Invent. Math. **78** (1) (1985), 103–147

A. Nagel, E. M. Stein, S. Wainger, *Balls and metrics defined by vector fields I: basic properties*, Acta Math. **155** (1984), 143–160

N. Garofalo, DM. Nhieu, *Isoperimetric and Sobolev inequalities for Carnot-Carathéodory spaces and the existence of minimal surfaces, Comm. on Pure and Applied Mathematics 49* **10** *(1996), 1081–1144*

Additional hypotheses on X

(H3) There exist geometric constants $c, C > 0$ such that for any $B = B_d(\overline{x}, r)$ with $cB := B_d(\overline{x}, cr) \subseteq \Omega_0$, for any $f \in \mathrm{Lip}(\overline{cB})$ and $x \in \overline{B}$

$$\left| f(x) - \frac{1}{|B|} \int_B f(y)dy \right| \leq C \int_{cB} |Xf(y)| \frac{d(x,y)}{|B_d(x, d(x,y))|} dy.$$

B. Franchi, E. Lanconelli, *Hölder regularity theorem for a class of linear nonuniformly elliptic operators with measurable coefficients* , Ann. Scuola Norm. Sup. Pisa **10** (4) (1983), 523–541

B. Franchi, C.E. Gutiérrez, R.L. Wheeden, *Weighted Sobolev-Poincaré inequalities for Grushin type operators*, Comm. Partial Differential Equations **19** (3–4) (1994), 523–604

B. Franchi, G. Lu, R. L. Wheeden, *Representation formulas and weighted Poincaré inequalities for Hörmander vector fields*, Ann. Inst. Fourier, Grenoble 45**2** (1995), 577–604

B. Franchi, R. Serapioni, F. Serra Cassano, *Approximation and Imbedding Theorems for Weighted Sobolev Spaces Associated with Lipschitz Continuous Vector Fields*, Bolletino U.M.I, **11-B** (1997), 83–117

Compact embeddings

Theorem (Franchi-Serapioni-Serra Cassano, 1997)

Let Ω and Ω_0 be respectively a bounded open and an open set with $\bar{\Omega} \subset \Omega_0$, let $1 \leq p < \infty$ and $X = (X_1, \dots, X_m)$ be a family of Lipschitz continuous vector fields defined in Ω_0. If X satisfies conditions (H1), (H2) and (H3), then for each metric ball $B = B_d(x, r) \subset \Omega$ and $u \in W_X^{1,p}(\Omega)$ there exist constants $c(u, B) \in \mathbb{R}$ and $C \in \mathbb{R}$

$$\int_B |u(x) - c(u, B)|^p \, dx \leq \; C \, r^p \int_B |Xu|^p \, dx \quad \forall \, u \in W_X^{1,p}(\Omega) \, ,$$

where the constant C is independent of u.

Theorem (Franchi-Serapioni-Serra Cassano, 1997)

Let $\Omega \Subset \Omega_0$ be a bounded open set, $1 \leq p < \infty$ and $X = (X_1, \dots, X_m)$ be a family of Lipschitz continuous vector fields defined in Ω_0. If X satisfies conditions (H1), (H2) and (H3), then $W_{X,0}^{1,p}(\Omega)$ is compactly embedded in $L^p(\Omega)$.

Poincaré inequality

Theorem

Let $\Omega \Subset \Omega_0$ be open, bounded and connected, $1 \le p < \infty$ and let $X = (X_1, \dots, X_m)$ be a family of Lipschitz continuous vector fields defined in Ω_0 such that X satisfies conditions (H1), (H2) and (H3). Then, there exists a positive constant $c_{p,\Omega} > 0$ such that

$$\int_\Omega |u|^p \, dx \le c_{p,\Omega} \int_\Omega |Xu|^p \, dx \text{ for each } u \in W^{1,p}_{X,0}(\Omega)\,.$$

Corollary

Let p, Ω and X as above. Then the function

$$\|u\|_{W^{1,p}_{X,0}} := \left(\int_\Omega |Xu|^p \, dx \right)^{\frac{1}{p}}$$

is a norm in $W^{1,p}_{X,0}(\Omega)$ equivalent to $\|\cdot\|_{W^{1,p}_X(\Omega)}$.

The H-compactness result

Theorem (Maione-P.-Serra Cassano, 2019)

Let Ω and Ω_0 be respectively a bounded open and an open set with $\bar{\Omega} \subset \Omega_0$ and let X be defined in Ω_0 satisfying conditions (H1), (H2), (H3), (LIC) on Ω. Let $a_h(x) = [a_{h,ij}(x)] \in J_2(\Omega, c_0, c_1)$ and let $(\mathcal{L}_h)_h$ be the associate operators in $\mathcal{E}(\Omega, c_0, c_1)$. Then, up to a subsequence, there exists an operator $\mathcal{L} := \mathrm{div}_X(a(x)X) \in \mathcal{E}(\Omega, c_0, c_1)$, such that, for all $g \in L^2(\Omega)$ and $\mu \geq 0$, if $(u_h)_h$ and u denote, respectively, the (unique) solutions of

$$\begin{cases} \mu v + \mathcal{L}_h(v) = g \text{ in } \Omega \\ v \in H^1_{X,0}(\Omega) \end{cases} \quad \text{and} \quad \begin{cases} \mu v + \mathcal{L}(v) = g \text{ in } \Omega \\ v \in H^1_{X,0}(\Omega) \end{cases}$$

then, as $h \to \infty$

- $u_h \to u$ in $L^2(\Omega)$;
- $a_h X u_h \to a X u$ weakly in $L^2(\Omega)^m$.

A. Maione, P., F. Serra Cassano, *Γ-convergence for functionals depending on vector fields II. Convergence of minimizers and H-convergence*, forthcoming

- In this general framework it is not possible to apply classic H-compactness techniques, because no definition of a curl is given (and even possible!).

- In this general framework it is not possible to apply classic H-compactness techniques, because no definition of a curl is given (and even possible!).

Rmk. An appropriate definition of curl, as well as a generalization of the Div-Curl lemma, have been given in the context of Carnot groups in the following paper:

A. Baldi, B. Franchi, N. Tchou, M.C. Tesi, *Compensated compactness for differential forms in Carnot groups*, Adv. in Math. (2010), 1555–1607

- In this general framework it is not possible to apply classic H-compactness techniques, because no definition of a curl is given (and even possible!).

  Rmk. An appropriate definition of curl, as well as a generalization of the Div-Curl lemma, have been given in the context of Carnot groups in the following paper:

  A. Baldi, B. Franchi, N. Tchou, M.C. Tesi, *Compensated compactness for differential forms in Carnot groups*, Adv. in Math. (2010), 1555–1607

- The techniques adopted here are an adaptation, to the symmetric case, of the ones of:

  N. Ansini, G. Dal Maso, C. I. Zeppieri, *Γ-convergence and H-convergence of linear elliptic operators*, Journal de Mathématiques Pures et Appliquées. Neuvième Série, 99 (3), (2013), 321–329

Sketch of the proof

Let us prove that, up to a subsequence, there exists an operator $\mathcal{L} = \mathrm{div}_X(a(x)X) \in \mathcal{E}(\Omega)$ for which the convergence of the solutions holds. Let $(a_h) \subset J_1(\Omega, c_0, c_1)$ be the sequence of matrices associated to $(\mathcal{L}_h)$ and let $F_h : L^2(\Omega) \times \mathcal{A} \to [0, \infty]$ be the quadratic functionals defined by

$$F_h(u, A) := \begin{cases} \frac{1}{2} \int_A \langle a_h(x) X u(x), X u(x) \rangle \, dx & \text{if } A \in \mathcal{A},\ u \in W_X^{1,2}(A) \\ \infty & \text{otherwise} \end{cases} .$$

By our compactness theorem, there exist a subsequence (F_{h_k}) of (F_h) and $a = (a_{ij}) \in J_1(\Omega, c_0, c_1)$ such that $(F_{h_k}(\cdot, \Omega))$ $\Gamma-$converges in $L^2(\Omega)$ to

$$F(u, \Omega) := \begin{cases} \frac{1}{2} \int_\Omega \langle a(x) X u(x), X u(x) \rangle \, dx & \text{if } u \in W_X^{1,2}(\Omega) \\ \infty & \text{otherwise} \end{cases} .$$

Let $\mathcal{L}$ be the operator associated to $F^0 : L^2(\Omega) \to [0, \infty]$

$$F^0(u) = \begin{cases} \frac{1}{2} \int_\Omega \langle a(x) Xu(x), Xu(x) \rangle \, dx & \text{if } u \in H^1_{X,0}(\Omega) \\ \infty & \text{otherwise} \end{cases} .$$

Let us consider the sequence of functionals $F^0_h : L^2(\Omega) \to [0, \infty]$ defined by

$$F^0_h(u) = \begin{cases} \frac{1}{2} \int_\Omega \langle a_h(x) Xu(x), Xu(x) \rangle \, dx & \text{if } u \in H^1_{X,0}(\Omega) \\ \infty & \text{otherwise} \end{cases}$$

whose associated operators are the functionals $\mathcal{L}_h$. It is possible to prove that

$$(F^0_h)_h \ \Gamma\text{-converges to } F^0 \text{ in } L^2(\Omega) .$$

Let $\mu \geq 0$ and $g \in L^2(\Omega)$, we denote by $G : L^2(\Omega) \to \mathbb{R}$ the functional

$$G(u) := \int_\Omega (\frac{\mu}{2}u^2 - gu)\, dx.$$

Since G is (strongly) continuous in $L^2(\Omega)$,it follows that

$$(F_h^0 + G)_h \ \Gamma\text{-converges to } F^0 + G \text{ in } L^2(\Omega)\,.$$

It is easy to prove that for any $h \in \mathbb{N}$ the functions u_h and u are the unique elements of the sets

$$\operatorname{argmin}\left\{F_h^0(u) + G(u) \mid u \in H^1_{X,0}(\Omega)\right\}$$

and

$$\operatorname{argmin}\left\{F^0(u) + G(u) \mid u \in H^1_{X,0}(\Omega)\right\}$$

respectively. The Poincaré inequality and the compact immersion gives that $(F_h^0 + G)$ is equicoercive in $H^1_{X,0}(\Omega)$ which gives the thesis.

To prove the convergence of the momenta we proved the following:

Theorem (Convergence of momenta)

Let $(f_h)_h \subset I_{m,p}(c_0, c_1, a_0, a_1)$ *and let* $F_h : L^p(\Omega) \to [0, \infty], \mathcal{F}_h : L^p(\Omega)^m \to [0, \infty]$ *be the sequence of functionals defined by*

$$F_h(u) = F_h(u, \Omega) := \begin{cases} \int_\Omega f_h(x, Xu(x))\, dx & \text{if } u \in W^{1,p}_X(\Omega) \\ \infty & \text{otherwise} \end{cases},$$

$$\mathcal{F}_h(\Phi) := \int_\Omega f_h(x, \Phi(x))\, dx\,,$$

respectively.

Assume that:

(i) $f_h(x,\cdot):\mathbb{R}^m\to[0,\infty)$ belongs to $C^1(\mathbb{R}^m)$, for each h, for a.e. $x\in\Omega$ and there exist $c_2>0$, $0<\alpha<\min\{1,p-1\}$ and a non negative function $a_3\in L^p(\Omega)$ such that

$$|\partial_\eta f_h(x,\eta_1)-\partial_\eta f_h(x,\eta_2)|\le c_2|\eta_1-\eta_2|^\alpha\left(|\eta|+a_3(x)\right)^{p-1}$$

for a.e. $x\in\Omega$, for each h;

(ii) there exists $F = \Gamma(L^p(\Omega)) - \lim_{h\to\infty} F_h$, with

$$F(u) = F(u,\Omega) := \begin{cases} \int_\Omega f(x, Xu(x))\,dx & \text{if } u \in W^{1,p}_X(\Omega) \\ \infty & \text{otherwise} \end{cases},$$

and $f(x,\cdot) : \mathbb{R}^m \to [0,\infty)$ belongs to $C^1(\mathbb{R}^m)$ for a.e. $x \in \Omega$;

(iii) there exist a sequence $(u_h)_h$ and a function u in $W_X^{1,p}(\Omega)$ such that

$$u_h \to u \text{ in } L^p(\Omega) \text{ and } \mathcal{F}_h(Xu_h) \to \mathcal{F}(Xu), \text{ as } h \to \infty\,,$$

where $\mathcal{F}(\Phi) := \int_\Omega f(x, \Phi(x))\,dx$, if $\Phi \in L^p(\Omega)^m$.

Then

$$\partial_\Phi \mathcal{F}_h(Xu_h) \to \partial_\Phi \mathcal{F}(Xu) \text{ weakly in } L^{p'}(\Omega)^m, \text{ as } h \to \infty\,.$$

where

$$\partial_\Phi \mathcal{F} : \; L^p(\Omega)^m \to L^{p'}(\Omega)^m$$

is given by

$$\partial_\Phi \mathcal{F}(\Phi) = \partial_\eta f(x, \Phi)$$

To prove the convergence of the momenta in our case it suffices to apply the previous theorem with

$$f_h(x,\eta) := \langle a_h(x)\eta, \eta\rangle \text{ and } f(x,\eta) := \langle a(x)\eta, \eta\rangle \,,$$

if $x \in \Omega$, $\eta \in \mathbb{R}^m$. In this case

$$\partial_\Phi \mathcal{F}_h(Xu_h) = \, a_h\, Xu_h \text{ and } \partial_\Phi \mathcal{F}(Xu) = \, a\, Xu\,.$$

Thank you