Two-parameter harmonic analysis on the Heisenberg group

Fulvio Ricci
Scuola Normale Superiore, Pisa

Dispersive and subelliptic PDEs

Pisa, February 12, 2020

Calderón-Zygmund theory is one of the main tools in harmonic analysis on $\mathbb{R}^n$ and Lie groups, when the problem is to establish boundedness of given convolution operators on L^p spaces. Parts of it are

- singular integral operators,

Calderón-Zygmund theory is one of the main tools in harmonic analysis on $\mathbb{R}^n$ and Lie groups, when the problem is to establish boundedness of given convolution operators on L^p spaces. Parts of it are

- singular integral operators,
- maximal functions,

Calderón-Zygmund theory is one of the main tools in harmonic analysis on $\mathbb{R}^n$ and Lie groups, when the problem is to establish boundedness of given convolution operators on L^p spaces. Parts of it are

- singular integral operators,
- maximal functions,
- Littlewood-Paley square functions,

Calderón-Zygmund theory is one of the main tools in harmonic analysis on $\mathbb{R}^n$ and Lie groups, when the problem is to establish boundedness of given convolution operators on L^p spaces. Parts of it are

- singular integral operators,
- maximal functions,
- Littlewood-Paley square functions,
- *multiplier* operators, either in the sense, on $\mathbb{R}^n$, of operators defined by a Fourier multiplier m $(T_m f = \mathcal{F}^{-1}(m\widehat{f}))$, or operators defined by a spectral multiplier m on $\mathbb{R}$ applied to a given elliptic, or hypoelliptic, differential operator,

Calderón-Zygmund theory is one of the main tools in harmonic analysis on $\mathbb{R}^n$ and Lie groups, when the problem is to establish boundedness of given convolution operators on L^p spaces. Parts of it are

- singular integral operators,
- maximal functions,
- Littlewood-Paley square functions,
- *multiplier* operators, either in the sense, on $\mathbb{R}^n$, of operators defined by a Fourier multiplier m $\left(T_m f = \mathcal{F}^{-1}(m\widehat{f})\right)$, or operators defined by a spectral multiplier m on $\mathbb{R}$ applied to a given elliptic, or hypoelliptic, differential operator,
- Hardy and *BMO* spaces, as replacements for L^1 and L^∞, respectively, at the extreme values of p.

The triad maximal functions / singular integrals / square functions is tightly interconnected, in the sense that maximal operators control singular integral operators; these in turn determine properties of square functions, which finally allow more subtle extensions of the previous two.

The triad maximal functions / singular integrals / square functions is tightly interconnected, in the sense that maximal operators control singular integral operators; these in turn determine properties of square functions, which finally allow more subtle extensions of the previous two.

In classical contexts, each of the three allows an equivalent definition of Hardy spaces.

The triad maximal functions / singular integrals / square functions is tightly interconnected, in the sense that maximal operators control singular integral operators; these in turn determine properties of square functions, which finally allow more subtle extensions of the previous two.

In classical contexts, each of the three allows an equivalent definition of Hardy spaces.

While maximal functions and singular integrals only involve space variables, square functions involve spectral decompositions (i.e., in the frequency variables) of the function they are applied to. This establishes a deep connection with Fourier, or spectral, multipliers.

Metrics and dilations on $\mathbb{R}^n$

Part of Calderón-Zygmund theory can be deloped starting from a (pseudo)-distance d on the ambient space and on a measure m on it (for us, Lebesgue measure) that is doubling w.r. to d.

Typically, on $\mathbb{R}^n$, one considers distances $d = |x - y|$, where $|\ |$ is a homogeneous norm relative to a general one-parameter family of dilations

$$r \cdot (x_1, \dots, x_n) = \left(r^{\lambda_1} x_1, \dots, r^{\lambda_n} x_n\right) ,$$

e.g.,

$$|x| = \max\left\{|x_j|^{1/\lambda_j} : j = 1, \dots, n\right\} .$$

For given dilations, all choices of the norm are equivalent for our purposes.

Metrics and dilations on nilpotent groups

On nilpotent groups, a similar approach, based on dilations, is limited by the restrictions imposed by the pseudo-triangular inequality of homogeneous norms

$$|ab^{-1}| \leq C\big(|a| + |b|\big) \ ,$$

which holds if and only if the dilations are group authomorphisms.

Metrics and dilations on nilpotent groups

On nilpotent groups, a similar approach, based on dilations, is limited by the restrictions imposed by the pseudo-triangular inequality of homogeneous norms

$$|ab^{-1}| \leq C\big(|a| + |b|\big) ,$$

which holds if and only if the dilations are group authomorphisms.

For instance, on the Heisenberg group $\mathbb{H}_n \equiv \mathbb{C}^n \times \mathbb{R}$, with $a = (x + iy, t)$ and dilations

$$r \cdot (x, y, t) = \big(r^{\lambda_1}x, r^{\lambda_2}y, r^{\lambda_3}t\big) ,$$

one must impose that $\lambda_3 = \lambda_1 + \lambda_2$ (this can be relaxed to $\lambda_3 \leq \lambda_1 + \lambda_2$ if limited to small values of the norms).

(sub)-Riemannian metrics on Heisenberg groups

More interesting and flexible is the approach based on the choice of a generating subspace $\mathfrak{v}$ of the Lie algebra endowed with a euclidean norm, and on the induced Carnot-Carathéodory distance.

In particular, on the Heisenberg group $\mathbb{H}_n$ we will consider

- the sub-Riemannian distance d_{sr}, choosing the horizontal distribution as $\mathfrak{v}$;

(sub)-Riemannian metrics on Heisenberg groups

More interesting and flexible is the approach based on the choice of a generating subspace $\mathfrak{v}$ of the Lie algebra endowed with a euclidean norm, and on the induced Carnot-Carathéodory distance.

In particular, on the Heisenberg group $\mathbb{H}_n$ we will consider

- the sub-Riemannian distance d_{sr}, choosing the horizontal distribution as $\mathfrak{v}$;
- the Riemannian distance d_r, choosing $\mathfrak{v} = \mathfrak{h}_n$.

(sub)-Riemannian metrics on Heisenberg groups

More interesting and flexible is the approach based on the choice of a generating subspace $\mathfrak{v}$ of the Lie algebra endowed with a euclidean norm, and on the induced Carnot-Carathéodory distance.

In particular, on the Heisenberg group $\mathbb{H}_n$ we will consider

- the sub-Riemannian distance d_{sr}, choosing the horizontal distribution as $\mathfrak{v}$;
- the Riemannian distance d_r, choosing $\mathfrak{v} = \mathfrak{h}_n$.

(sub)-Riemannian metrics on Heisenberg groups

More interesting and flexible is the approach based on the choice of a generating subspace $\mathfrak{v}$ of the Lie algebra endowed with a euclidean norm, and on the induced Carnot-Carathéodory distance.

In particular, on the Heisenberg group $\mathbb{H}_n$ we will consider

- the sub-Riemannian distance d_{sr}, choosing the horizontal distribution as $\mathfrak{v}$;
- the Riemannian distance d_r, choosing $\mathfrak{v} = \mathfrak{h}_n$.

In terms of homogeneous norms, we have

$$d_{sr}(a,0) \sim |a| \ , \qquad d_r(a,0) = \begin{cases} \|a\| & \text{near } 0 \\ |a| & \text{near } \infty \ , \end{cases}$$

where $|a| = \max\left\{|z|, |t|^{\frac{1}{2}}\right\}$, $\|a\| = \max\left\{|z|, |t|\right\}$

In a paper of 1979 on the $\bar{\partial}$-Neumann problem on the Siegel domain in $\mathbb{C}^{n+1}$ (whose boundary is naturally identified with $\mathbb{H}_n$), D. Phong introduced operators on $\mathbb{H}_n$ that can be realized as compositions of the Green operators for the sub-Laplacian (of sub-Riemannian type) and ordinary pseudo-differential operators in the standard Hörmander classes (of Riemannian type).

Shortly later, Phong and E. Stein posed the problem of adapting Calderón-Zygmund theory to operators of this mixed (or multi-norm) type. In particular, they discussed, both on $\mathbb{R}^n$ and $\mathbb{H}_n$, weak-type 1-1 properties of operators

$$Tf = f * K_r * K_{sr} \ ,$$

where K_r and K_{sr} are CZ kernels of Riemannian and sub-Riemannian type respectively.

Operators of this type also intervene in the analysis of the (riemannian) Hodge Laplacian on H_n (Müller-Peloso-R., Mem. AMS 2015).

Recently, a systematic treatment of compositions of singular integral operators combining together multiple homogeneities has been developed in a joint paper by Nagel-R.-Stein-Wainger (Mem. AMS 2018), which covers any finite numbers of dilations and allowing (at least in a local form and with restrictions on admissible dilations) general homogeneous nilpotent groups (*multi-norm structures*).

Further work is in progress, jointly with A. Hejna and A. Nagel, on the notion of Hardy spaces adapted to multi-norm structures.

Operators of this type also intervene in the analysis of the (riemannian) Hodge Laplacian on H_n (Müller-Peloso-R., Mem. AMS 2015).

Recently, a systematic treatment of compositions of singular integral operators combining together multiple homogeneities has been developed in a joint paper by Nagel-R.-Stein-Wainger (Mem. AMS 2018), which covers any finite numbers of dilations and allowing (at least in a local form and with restrictions on admissible dilations) general homogeneous nilpotent groups (*multi-norm structures*).

Further work is in progress, jointly with A. Hejna and A. Nagel, on the notion of Hardy spaces adapted to multi-norm structures.

My presentation today is just an introductory overview on the basic aspects of this theory , with emphasis on the interactions between metric and spectral aspects.

One-parameter maximal functions

The one-parameter maximal functions considered in this context are smoothened variant of the Hardy-Littlewood maximal function, adapted to *automorphic* dilations with exponents $\lambda_j > 0$.
For fixed $\varphi \in C_c^\infty$ with $\int \varphi \neq 0$,

$$M_\varphi f(x) = \sup_{r>0} \left| f * \varphi_r(x) \right| ,$$

where $\varphi_r(x) = r^{-Q}\varphi(r^{-1} \cdot x)$, $Q = \sum_j \lambda_j$.
It is also convenient to use *grand-maximal functions* of the form

$$\mathcal{M} f(x) = \sup_{r>0\,,\,\varphi\in\mathcal{B}} \left| f * \varphi_r(x) \right| ,$$

where $\mathcal{B} = \left\{\varphi \in C_c^\infty(B_{R_0}) : \|\varphi\|_{C^1} \leq A\right\}$.

One-parameter maximal functions

The one-parameter maximal functions considered in this context are smoothened variant of the Hardy-Littlewood maximal function, adapted to *automorphic* dilations with exponents $\lambda_j > 0$.

For fixed $\varphi \in C_c^\infty$ with $\int \varphi \neq 0$,

$$M_\varphi f(x) = \sup_{r>0} \left| f * \varphi_r(x) \right| ,$$

where $\varphi_r(x) = r^{-Q}\varphi(r^{-1} \cdot x)$, $Q = \sum_j \lambda_j$.

It is also convenient to use *grand-maximal functions* of the form

$$\mathcal{M} f(x) = \sup_{r>0\,,\,\varphi\in\mathcal{B}} \left| f * \varphi_r(x) \right| ,$$

where $\mathcal{B} = \left\{ \varphi \in C_c^\infty(B_{R_0}) : \|\varphi\|_{C^1} \leq A \right\}$.

These maximal operators are bounded on L^p for $1 < p \leq \infty$ and weak-type 1-1.

Multi-parameter maximal functions on $\mathbb{R}^n$

The classical example is the *strong maximal function*,

$$M_s f(x) = \sup_{R \in \mathcal{R}_x} \fint_R |f| \ ,$$

where $\mathcal{R}_x$ is the set of all rectangles $R = I_1 \times \cdots \times I_n$ centered at x, or its smoothened "grand" analogue,

$$\mathcal{M}_s f(x) = \sup_{r_1,\dots,r_n>0\,,\,\varphi\in\mathcal{B}} \left| f * \varphi_{r_1,\dots,r_n}(x) \right| \ .$$

Maximal functions of this type are still L^p-bounded for $p > 1$, but no longer weak type 1-1. They belong to the *product theory*, which includes

- singular integrals of product type, e.g., the multiple Hilbert transform with kernel $p.v.(1/x_1 x_2 \cdots x_n)$,

Multi-parameter maximal functions on $\mathbb{R}^n$

The classical example is the *strong maximal function*,

$$M_s f(x) = \sup_{R \in \mathcal{R}_x} \fint_R |f| \ ,$$

where $\mathcal{R}_x$ is the set of all rectangles $R = I_1 \times \cdots \times I_n$ centered at x, or its smoothened "grand" analogue,

$$\mathcal{M}_s f(x) = \sup_{r_1,\dots,r_n > 0\,,\, \varphi \in \mathcal{B}} \left| f * \varphi_{r_1,\dots,r_n}(x) \right| \ .$$

Maximal functions of this type are still L^p-bounded for $p > 1$, but no longer weak type 1-1. They belong to the *product theory*, which includes

- singular integrals of product type, e.g., the multiple Hilbert transform with kernel $p.v.(1/x_1 x_2 \cdots x_n)$,
- the product Hardy spaces of R. Fefferman and A. Chang.

Compositions of one-parameter settings

Assume for simplicity that we are in $\mathbb{R}^2$ and are considering the *isotropic* dilations, with exponents $(1,1)$, together with the *parabolic* dilations, with exponents $(1,2)$.

The natural (grand)-maximal functions adapted to the multi-norm context have the form

$$\mathcal{M}_{\mathrm{iso/par}} f(x) = \sup_{r,s>0\,,\,\varphi,\psi\in\mathcal{B}} \left| f * \varphi_{r,r} * \psi_{s,s^2}(x) \right| \,,$$

where $\varphi_{a,b} = a^{-1}b^{-1}\varphi(x_1/a, x_2/b)$.

Restriction of parameters

Consider the convolution $\varphi_{r,r} * \psi_{s,s^2}$. If $I_r = [-r, r]$, this function is supported in

$$(I_r \times I_r) + (I_s \times I_{s^2}) = I_{r+s} \times I_{r+s^2} \sim I_{r\vee s} \times I_{r\vee s^2}\ .$$

Notice that

- the two scales $r \vee s$ and $r \vee s^2$ are equal, unless $s^2 < r < s$ or $s < r < s^2$,

Restriction of parameters

Consider the convolution $\varphi_{r,r} * \psi_{s,s^2}$. If $I_r = [-r, r]$, this function is supported in

$$(I_r \times I_r) + (I_s \times I_{s^2}) = I_{r+s} \times I_{r+s^2} \sim I_{r \vee s} \times I_{r \vee s^2} \ .$$

Notice that

- the two scales $r \vee s$ and $r \vee s^2$ are equal, unless $s^2 < r < s$ or $s < r < s^2$,
- if this is the case and $a = r \vee s$, $b = r \vee s^2$, then $a^2 < b < a$ or $a < b < a^2$ (that we write as $b \in (a, a^2)$),

Restriction of parameters

Consider the convolution $\varphi_{r,r} * \psi_{s,s^2}$. If $I_r = [-r, r]$, this function is supported in

$$(I_r \times I_r) + (I_s \times I_{s^2}) = I_{r+s} \times I_{r+s^2} \sim I_{r\vee s} \times I_{r \vee s^2} \ .$$

Notice that

- the two scales $r \vee s$ and $r \vee s^2$ are equal, unless $s^2 < r < s$ or $s < r < s^2$,
- if this is the case and $a = r \vee s$, $b = r \vee s^2$, then $a^2 < b < a$ or $a < b < a^2$ (that we write as $b \in (a, a^2)$),
- $\varphi_{r,r} * \psi_{s,s^2} = \eta_{r\vee s, r\vee s^2}$ with $\eta \in \mathcal{B}$ (or rather in some fixed $\mathcal{B}' \supset \mathcal{B}$).

Restriction of parameters

Consider the convolution $\varphi_{r,r} * \psi_{s,s^2}$. If $I_r = [-r, r]$, this function is supported in

$$(I_r \times I_r) + (I_s \times I_{s^2}) = I_{r+s} \times I_{r+s^2} \sim I_{r \vee s} \times I_{r \vee s^2} \ .$$

Notice that

- the two scales $r \vee s$ and $r \vee s^2$ are equal, unless $s^2 < r < s$ or $s < r < s^2$,
- if this is the case and $a = r \vee s$, $b = r \vee s^2$, then $a^2 < b < a$ or $a < b < a^2$ (that we write as $b \in (a, a^2)$),
- $\varphi_{r,r} * \psi_{s,s^2} = \eta_{r \vee s, r \vee s^2}$ with $\eta \in \mathcal{B}$ (or rather in some fixed $\mathcal{B}' \supset \mathcal{B}$).

Restriction of parameters

Consider the convolution $\varphi_{r,r} * \psi_{s,s^2}$. If $I_r = [-r, r]$, this function is supported in

$$(I_r \times I_r) + (I_s \times I_{s^2}) = I_{r+s} \times I_{r+s^2} \sim I_{r\vee s} \times I_{r\vee s^2} \ .$$

Notice that

- the two scales $r \vee s$ and $r \vee s^2$ are equal, unless $s^2 < r < s$ or $s < r < s^2$,
- if this is the case and $a = r \vee s$, $b = r \vee s^2$, then $a^2 < b < a$ or $a < b < a^2$ (that we write as $b \in (a, a^2)$),
- $\varphi_{r,r} * \psi_{s,s^2} = \eta_{r\vee s, r\vee s^2}$ with $\eta \in \mathcal{B}$ (or rather in some fixed $\mathcal{B}' \supset \mathcal{B}$).

Hence

$$\sup_{r,s>0\, ,\, \varphi,\psi\in\mathcal{B}} \left| f * \varphi_{r,r} * \psi_{s,s^2}(x) \right| \sim \sup_{a>0\, ,\, b\in(a,a^2)\, ,\, \eta\in\mathcal{B}} \left| f * \eta_{a,b}(x) \right| \ .$$

Two-parameter maximal functions on $\mathbb{H}_n$

The one-parameter grand-maximal function associated to the sub-Riemannian distance is modeled on the parabolic one in $\mathbb{R}^2$,

$$\mathcal{M}_{\mathrm{sr}} f(z,t) = \sup_{r>0\,,\,\varphi\in\mathcal{B}} \left| f * \varphi_{r,r^2}(z,t) \right| \,.$$

The Riemannian one requires a more careful definition because the distance is not homogeneous. However, due to the estimates given before,

$$d_{\mathrm{r}}\big((z,t),0\big) : \qquad \sim |z| + |t| \ \text{ near } 0\ , \qquad \sim |z| + |t|^{\frac{1}{2}} \ \text{ near } \infty\ ,$$

any reasonable definition amounts to be equivalent to

$$\mathcal{M}_{\mathrm{r}} f(z,t) = \sup_{0<r<1\,,\,\varphi\in\mathcal{B}} \left| f * \varphi_{r,r}(z,t) \right| + \sup_{r\geq 1\,,\,\varphi\in\mathcal{B}} \left| f * \varphi_{r,r^2}(z,t) \right| \,.$$

Take into account that, for small radii, the Heisenberg product of two balls (each of either type) has the same size, both in z and t, as their vector sum.

We are so led to define

$$\mathcal{M}_{\mathrm{r/sr}} f(z,t) = \sup_{r<1\,,\, r^2<s<r\,,\, \varphi\in\mathcal{B}} \left| f * \varphi_{r,s}(z,t) \right| + \sup_{r\geq 1\,,\, \varphi\in\mathcal{B}} \left| f * \varphi_{r,r^2}(z,t) \right| .$$

Square functions on $\mathbb{R}^n$

Consider a partition of unity on $\mathbb{R}^n \setminus \{0\}$ (regarded as the frequency space) of the form

$$1 = \sum_{j \in \mathbb{Z}} \eta(2^{-j} \cdot \xi) \ ,$$

with η smooth and supported where $|\xi| \in [1, 4]$. If $\psi = \mathcal{F}^{-1}\eta \in \mathcal{S}$, in the space variable we have

$$\delta_0 = \sum_{j \in \mathbb{Z}} 2^{jQ} \psi(2^j \cdot x) = \sum_{j \in \mathbb{Z}} \psi_{2^{-j}}(x)$$

in the sense of distributions, and, for every $f \in \mathcal{S}$,

$$f = \sum_{j \in \mathbb{Z}} f * \psi_{2^{-j}} \ .$$

We must think of $f * \psi_{2^{-j}}$ as the result of extracting from f its components at frequencies ξ with $|\xi| \sim 2^j$.

The basic theorem of Littlewood-Paley theory says that, if $1 < p < \infty$, the functions f and

$$Sf(x) = \Big(\sum_{j \in \mathbb{Z}} \big| f * \psi_{2^{-j}}(x) \big|^2 \Big)^{\frac{1}{2}}$$

have equivalent L^p-norms:

$$\|Sf\|_p \sim \|f\|_p \ .$$

The same equivalence holds for the *product square function*

$$S_{\mathrm{prod}} f(x) = \Big(\sum_{J \in \mathbb{Z}^n} \big| f * \psi_{2^{-J}}(x) \big|^2 \Big)^{\frac{1}{2}} \ ,$$

where $f * \psi_J$ extracts from f the components at frequencies ξ with $|\xi_i| \sim 2^{j_i}$ for all i.

Superposing two one-parameter decompositions of δ_0

Restricting matters to $\mathbb{R}^2$, from two dyadic decompositions

$$\delta_0 = \sum_{j\in\mathbb{Z}} \psi_{2^{-j},2^{-j}}(x)\ , \qquad \delta_0 = \sum_{k\in\mathbb{Z}} \widetilde{\psi}_{2^{-k},2^{-2k}}(x)\ ,$$

we derive the two parameter decomposition

$$\delta_0 = \delta_0 * \delta_0 = \sum_{j,k\in\mathbb{Z}} \psi_{2^{-j},2^{-j}} * \widetilde{\psi}_{2^{-k},2^{-2k}}(x)\ .$$

Superposing two one-parameter decompositions of δ_0

Restricting matters to $\mathbb{R}^2$, from two dyadic decompositions

$$\delta_0 = \sum_{j\in\mathbb{Z}} \psi_{2^{-j},2^{-j}}(x) \ , \qquad \delta_0 = \sum_{k\in\mathbb{Z}} \widetilde{\psi}_{2^{-k},2^{-2k}}(x) \ ,$$

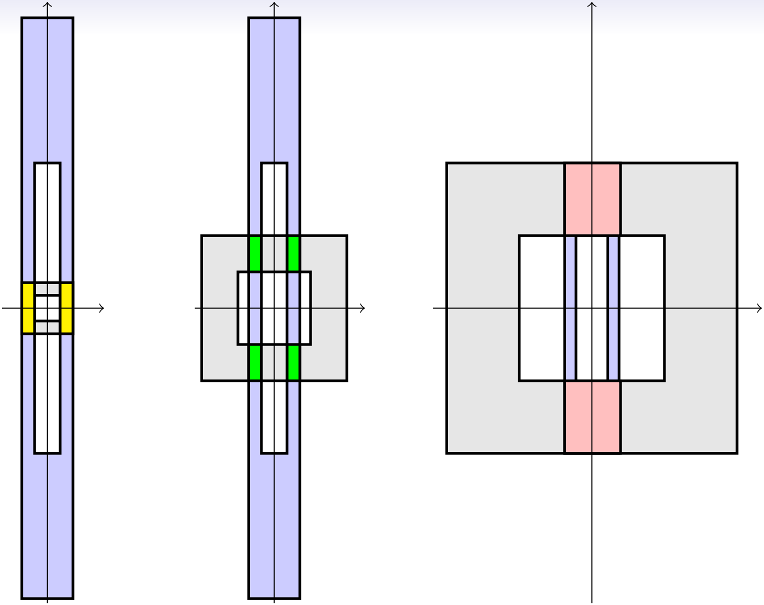
we derive the two parameter decomposition

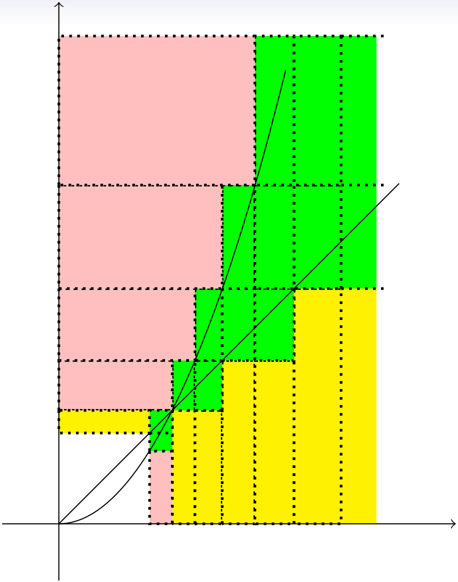
$$\delta_0 = \delta_0 * \delta_0 = \sum_{j,k\in\mathbb{Z}} \psi_{2^{-j},2^{-j}} * \widetilde{\psi}_{2^{-k},2^{-2k}}(x) \ .$$

However, this full double sum is quite redundant. Observe in fact that

$$\mathcal{F}(\psi_{2^{-j},2^{-j}} * \widetilde{\psi}_{2^{-k},2^{-2k}}) = \eta(2^{-j}\xi_1, 2^{-j}\xi_2)\widetilde{\eta}(2^{-k}\xi_1, 2^{-2k}\xi_2) \ ,$$

which is 0 unless 2^j is between $c2^k$ and $c'2^{2k}$.





In the resulting dyadic decomposition of the frequency space, we have three types of regions:

- *product regions*: $\{\xi : |\xi_1| \sim 2^j\,,\, |\xi_2| \sim 2^{2k}\}$, $j \in \mathbb{Z}$, $k \in (j/2, j)$;

In the resulting dyadic decomposition of the frequency space, we have three types of regions:

- *product regions*: $\left\{\xi : |\xi_1| \sim 2^j\, , \, |\xi_2| \sim 2^{2k}\right\}$, $j \in \mathbb{Z}$, $k \in (j/2, j)$;
- *isotropic regions*: $\left\{\xi : |\xi_1| \sim 2^j\, , \, |\xi_2| \lesssim 2^j\right\}$, (resp. $\lesssim, \sim$ if $j < 0$);

In the resulting dyadic decomposition of the frequency space, we have three types of regions:

- *product regions*: $\left\{\xi : |\xi_1| \sim 2^j \,,\, |\xi_2| \sim 2^{2k}\right\}$, $j \in \mathbb{Z}$, $k \in (j/2, j)$;
- *isotropic regions*: $\left\{\xi : |\xi_1| \sim 2^j \,,\, |\xi_2| \lesssim 2^j\right\}$, (resp. $\lesssim, \sim$ if $j < 0$);
- *parabolic regions*: $\left\{\xi : |\xi_1| \lesssim 2^k \,,\, |\xi_2| \sim 2^{2k}\right\}$, (resp. $\sim, \lesssim$ if $j < 0$).

In the resulting dyadic decomposition of the frequency space, we have three types of regions:

- *product regions*: $\left\{\xi : |\xi_1| \sim 2^j \,,\, |\xi_2| \sim 2^{2k}\right\}$, $j \in \mathbb{Z}$, $k \in (j/2, j)$;
- *isotropic regions*: $\left\{\xi : |\xi_1| \sim 2^j \,,\, |\xi_2| \lesssim 2^j\right\}$, (resp. $\lesssim, \sim$ if $j < 0$);
- *parabolic regions*: $\left\{\xi : |\xi_1| \lesssim 2^k \,,\, |\xi_2| \sim 2^{2k}\right\}$, (resp. $\sim, \lesssim$ if $j < 0$).

In the resulting dyadic decomposition of the frequency space, we have three types of regions:

- *product regions*: $\{\xi : |\xi_1| \sim 2^j\,,\ |\xi_2| \sim 2^{2k}\}$, $j \in \mathbb{Z}$, $k \in (j/2, j)$;
- *isotropic regions*: $\{\xi : |\xi_1| \sim 2^j\,,\ |\xi_2| \lesssim 2^j\}$, (resp. $\lesssim, \sim$ if $j < 0$);
- *parabolic regions*: $\{\xi : |\xi_1| \lesssim 2^k\,,\ |\xi_2| \sim 2^{2k}\}$, (resp. $\sim, \lesssim$ if $j < 0$).

The Dirac delta decomposes as a corresponding sum:

$$\delta_0 = \sum_{j/2<k/j<j} \psi_{2^{-j},2^{-2k}} + \sum_{j\geq 0} \psi'_{2^{-j},2^{-j}} + + \sum_{k\geq 0} \psi''_{2^{-k},2^{-2k}}$$
$$+ \sum_{j<k/j<j/2} \psi_{2^{-j},2^{-2k}} + \sum_{k<0} \psi'_{2^{-k},2^{-2k}} + \sum_{j\geq 0} \psi''_{2^{-j},2^{-j}}\ ,$$

where the three functions ψ, ψ', ψ'' have the following cancellations:

- $\int \psi\, dx_1 = 0 \ \ \forall\, x_2$ and $\int \psi\, dx_2 = 0 \ \ \forall\, x_1$,

In the resulting dyadic decomposition of the frequency space, we have three types of regions:

- *product regions*: $\left\{\xi : |\xi_1| \sim 2^j\,,\ |\xi_2| \sim 2^{2k}\right\}$, $j \in \mathbb{Z}$, $k \in (j/2, j)$;
- *isotropic regions*: $\left\{\xi : |\xi_1| \sim 2^j\,,\ |\xi_2| \lesssim 2^j\right\}$, (resp. $\lesssim, \sim$ if $j < 0$);
- *parabolic regions*: $\left\{\xi : |\xi_1| \lesssim 2^k\,,\ |\xi_2| \sim 2^{2k}\right\}$, (resp. $\sim, \lesssim$ if $j < 0$).

The Dirac delta decomposes as a corresponding sum:

$$\delta_0 = \sum_{j/2<k/j<j} \psi_{2^{-j},2^{-2k}} + \sum_{j\geq 0} \psi'_{2^{-j},2^{-j}} + + \sum_{k\geq 0} \psi''_{2^{-k},2^{-2k}} \\ + \sum_{j<k/j<j/2} \psi_{2^{-j},2^{-2k}} + \sum_{k<0} \psi'_{2^{-k},2^{-2k}} + \sum_{j\geq 0} \psi''_{2^{-j},2^{-j}}\ ,$$

where the three functions ψ, ψ', ψ'' have the following cancellations:

- $\int \psi\, dx_1 = 0 \ \ \forall\, x_2$ and $\int \psi\, dx_2 = 0 \ \ \forall\, x_1$,
- $\int \psi'\, dx_1 = 0 \ \ \forall\, x_2$,

In the resulting dyadic decomposition of the frequency space, we have three types of regions:

- *product regions*: $\left\{\xi : |\xi_1| \sim 2^j \,,\, |\xi_2| \sim 2^{2k}\right\}$, $j \in \mathbb{Z}$, $k \in (j/2, j)$;
- *isotropic regions*: $\left\{\xi : |\xi_1| \sim 2^j \,,\, |\xi_2| \lesssim 2^j\right\}$, (resp. $\lesssim, \sim$ if $j < 0$);
- *parabolic regions*: $\left\{\xi : |\xi_1| \lesssim 2^k \,,\, |\xi_2| \sim 2^{2k}\right\}$, (resp. $\sim, \lesssim$ if $j < 0$).

The Dirac delta decomposes as a corresponding sum:

$$\delta_0 = \sum_{j/2<k/j<j} \psi_{2^{-j},2^{-2k}} + \sum_{j\geq 0} \psi'_{2^{-j},2^{-j}} + + \sum_{k\geq 0} \psi''_{2^{-k},2^{-2k}}$$
$$+ \sum_{j<k/j<j/2} \psi_{2^{-j},2^{-2k}} + \sum_{k<0} \psi'_{2^{-k},2^{-2k}} + \sum_{j\geq 0} \psi''_{2^{-j},2^{-j}} \ ,$$

where the three functions ψ, ψ', ψ'' have the following cancellations:

- $\int \psi \, dx_1 = 0 \ \ \forall \, x_2$ and $\int \psi \, dx_2 = 0 \ \ \forall \, x_1$,
- $\int \psi' \, dx_1 = 0 \ \ \forall \, x_2$,
- $\int \psi'' \, dx_2 = 0 \ \ \forall \, x_1$.

Singular kernels

The singular kernels belonging to our two-norm setting are those that can be represented as

$$K = \sum_{j/2<k/j<j} \psi^{j,k}_{2^{-j},2^{-2k}} + \sum_{j\geq 0} {\psi'}^{j}_{2^{-j},2^{-j}} + + \sum_{k\geq 0} {\psi''}^{k}_{2^{-k},2^{-2k}}$$
$$+ \sum_{j<k/j<j/2} \psi^{j,k}_{2^{-j},2^{-2k}} + \sum_{k<0} {\psi'}^{k}_{2^{-k},2^{-2k}} + \sum_{j\geq 0} {\psi''}^{j}_{2^{-j},2^{-j}} \ ,$$

where the functions ψ, ψ', ψ'' satisfy uniform bounds and the same cancellations as in the previous slide. It turns out that, away from 0, they are functions satisfying the inequalities

$$\left|\partial^{\alpha}_{x_1}\partial^{\beta}_{x_2} K(x)\right| \leq C_{\alpha,\beta} \begin{cases} (|x_1|+|x_2|)^{-1-\alpha}(|x_1|^2+|x_2|)^{-1-\beta} & \text{if } |x|<1 \ , \\ (|x_1|+|x_2|^{\frac{1}{2}})^{-1-\alpha}(|x_1|+|x_2|)^{-1-\beta} & \text{if } |x|>1 \ . \end{cases}$$

The "frequency" space for $\mathbb{H}_n$

On $\mathbb{H}_n$ it is necessary to compensate for the lack of commutativity of the product, which does not allow a classical kind of Fourier analysis.
The point of view that some form of Fourier transform can be recovered using spectral analysis of the two main differential operators which are relevant in the two geometries, i.e., the sub-Laplacian

$$L = -\sum_j (X_j^2 + Y_j^2) = d_H^* d_H \ ,$$

and the Riemannian Laplacian

$$\Delta = L - T^2 = d^* d \ , \qquad (T = \partial_t) \ .$$

Each operator is associated to a resolution of the identity on $L^2(H_n)$ based on $\mathbb{R}_+$, denoted by $dE_{\mathrm{sr}}(\xi)$ and $dE_{\mathrm{r}}(\xi')$ respectively:

$$\int_0^\infty dE_{\mathrm{sr}}(\xi) = I \ , \qquad \int_0^\infty \xi \, dE_{\mathrm{sr}}(\xi) = L \ ,$$
$$\int_0^\infty dE_{\mathrm{r}}(\xi') = I \ , \qquad \int_0^\infty \xi' \, dE_{\mathrm{r}}(\xi') = \Delta \ .$$

Take η is smooth, nonnegative, supported on $[1,4]$ and such that $\sum_{j\in\mathbb{Z}} \eta(2^{-j}\xi) = 1$ for $\xi > 0$,

$$I = \sum_{j\in\mathbb{Z}} \int_0^\infty \eta(2^{-j}\xi)\, dE_{\mathrm{sr}}(\xi) \stackrel{\mathrm{def}}{=} \sum_{j\in\mathbb{Z}} \eta(2^{-j}L)$$
$$I = \sum_{j\in\mathbb{Z}} \int_0^\infty \eta(2^{-j}\xi)\, dE_{\mathrm{r}}(\xi) \stackrel{\mathrm{def}}{=} \sum_{j\in\mathbb{Z}} \eta(2^{-j}\Delta) \ .$$

By left-translation invariance,

$$\eta(2^{-j}L)f = f * \varphi_j \ , \qquad \eta(2^{-j}\Delta)f = f * \psi_j \ ,$$

where $\varphi_j, \psi_j \in \mathcal{S}(H_n)$ (due to hypoellipticity). In other words, we have two (spectrally) dyadic decompositions of the Dirac delta,

$$\delta_0 = \sum_{j\in\mathbb{Z}} \varphi_j \ , \qquad \delta_0 = \sum_{j\in\mathbb{Z}} \psi_j \ .$$

In the space variables (z, t),

$$\varphi_j(z,t) = 2^{(n+1)j}\varphi_0(2^{j/2}z, 2^j t) = (\varphi_0)_{2^{-j/2},2^{-j}} \ .$$

Some consideration is needed concerning the ψ_j, since Δ is not homogeneous. One way to deal with this problem is by "lifting" the setting to higher dimensions.

Consider the group $N = \mathbb{H}_n \times \mathbb{R}$ with dilations $r \cdot (z, t, u) = (rz, r^2 t, ru)$ and the homomorphism

$$\tau : (z, t, u) \longmapsto (z, t - u) ,$$

that identifies $\mathbb{H}_n \cong N / \ker \tau$.

The sublaplacian $\widetilde{L} = L - \partial_u^2$ on N is homogeneous and $d\tau(\widetilde{L}) = \Delta$.

The operators $\eta(2^{-j}\widetilde{L})$ involve convolution on N with functions $\widetilde{\psi}_j$ at scale $(2^j, 2^{2j}, 2^j)$ and

$$\psi_j(z, t) = \int_{\mathbb{R}} \widetilde{\psi}_j(z, t + u, u) \, du .$$

- for $j > 0$, the ψ_j are uniformly at scale $(2^{-j}, 2^{-j})$, i.e., the set $\left\{2^{-(2n+1)j}\psi_j(2^{-j}z, 2^{-j}t)\right\}$ is bounded in $\mathcal{S}$ (isotropic scaling);

- for $j > 0$, the ψ_j are uniformly at scale $(2^{-j}, 2^{-j})$, i.e., the set $\left\{2^{-(2n+1)j}\psi_j(2^{-j}z, 2^{-j}t)\right\}$ is bounded in $\mathcal{S}$ (isotropic scaling);
- for $j \leq 0$, the ψ_j are uniformly at scale $(2^{-j/2}, 2^{-j})$ (parabolic-automorphic scaling).

- for $j > 0$, the ψ_j are uniformly at scale $(2^{-j}, 2^{-j})$, i.e., the set $\left\{2^{-(2n+1)j}\psi_j(2^{-j}z, 2^{-j}t)\right\}$ is bounded in $\mathcal{S}$ (isotropic scaling);
- for $j \leq 0$, the ψ_j are uniformly at scale $(2^{-j/2}, 2^{-j})$ (parabolic-automorphic scaling).

- for $j > 0$, the ψ_j are uniformly at scale $(2^{-j}, 2^{-j})$, i.e., the set $\left\{2^{-(2n+1)j}\psi_j(2^{-j}z, 2^{-j}t)\right\}$ is bounded in $\mathcal{S}$ (isotropic scaling);
- for $j \leq 0$, the ψ_j are uniformly at scale $(2^{-j/2}, 2^{-j})$ (parabolic-automorphic scaling).

From this we derive the two-parameter decomposition

$$\delta_0 = \sum_{j,k\in\mathbb{Z}} \varphi_j * \psi_k \ ,$$

but, like in $\mathbb{R}^2$, many terms can be eliminated or simplified.

- for $j > 0$, the ψ_j are uniformly at scale $(2^{-j}, 2^{-j})$, i.e., the set $\{2^{-(2n+1)j}\psi_j(2^{-j}z, 2^{-j}t)\}$ is bounded in $\mathcal{S}$ (isotropic scaling);
- for $j \leq 0$, the ψ_j are uniformly at scale $(2^{-j/2}, 2^{-j})$ (parabolic-automorphic scaling).

From this we derive the two-parameter decomposition

$$\delta_0 = \sum_{j,k\in\mathbb{Z}} \varphi_j * \psi_k \ ,$$

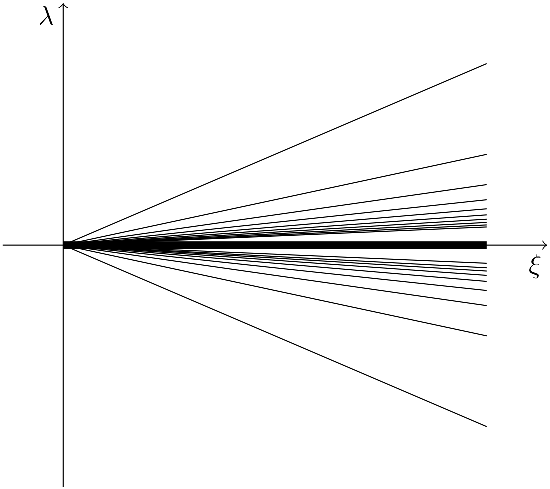
but, like in $\mathbb{R}^2$, many terms can be eliminated or simplified.

To see this, it is convenient to refer to the pair of symmetric operators L and iT and to the *joint spectrum* Σ of their self-adjoint extensions.
This is possible because L and iT commute.
Their joint resolution of the identity on $L^2(H_n)$, $dE(\xi,\lambda)$, is based on $\mathbb{R}^2$ and allows to define

$$m(L, iT) = \int_{\mathbb{R}^2} m(\xi,\lambda)\, dE(\xi,\lambda) \ .$$

The support Σ of $E = E_L \times E_{iT}$ (the joint L^2-spectrum) is known as the *Heisenberg fan*.

What is relevant for us is that $\Sigma \subset \left\{(\xi, \lambda) : |\lambda| < \xi/n\right\}$.



From the formulas

$$\eta(2^{-j}L)f = f * \varphi_j \ , \qquad \eta(2^{-k}\Delta)f = f * \psi_k \ ,$$

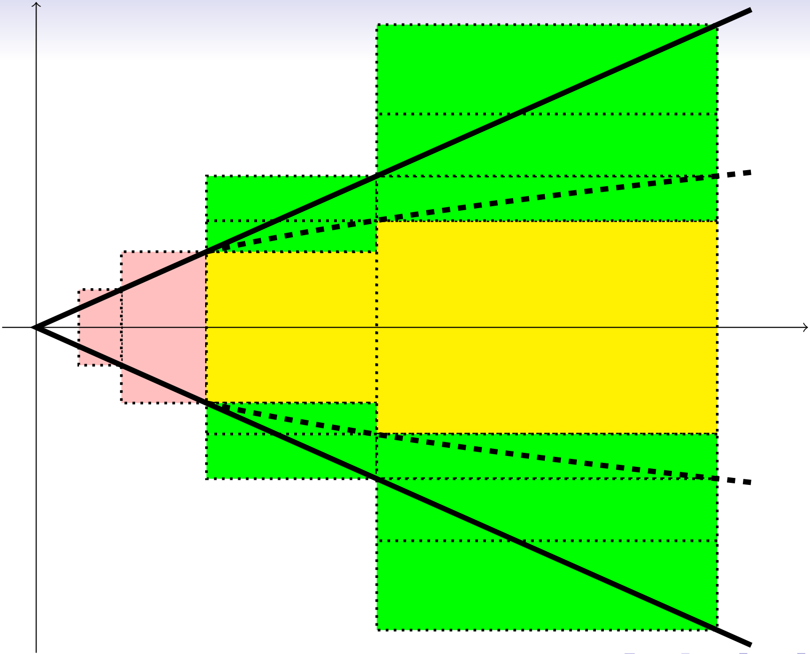
we derive that

$$f * \varphi_j * \psi_k = \eta\big(2^{-k}(L - T^2)\big)\eta(2^{-j}L) = m_{j,k}(L, iT) \ ,$$

with

$$m_{j,k}(\xi, \lambda) = \eta\big(2^{-k}(\xi + \lambda^2)\big)\eta(2^{-j}\xi) \ .$$

The support of $m_{j,k} = \big\{(\xi, \lambda) : \xi \sim 2^j \, , \, \xi + \lambda^2 \sim 2^k\big\}$ does not always intersect Σ.



Ultimately,

$$\delta_0 = \sum_{j,k\in\mathbb{Z}} \varphi_j * \psi_k$$
$$= \sum_{j<0} (\widetilde{\varphi}_j)_{2^{-j/2},2^{-j}} + \sum_{j>0} (\widetilde{\varphi}_j)_{2^{-j},2^{-j}} + \sum_{j/2<k<j} (\widetilde{\varphi}_{j,k})_{2^{-j},2^{-k}} \ .$$

Ultimately,

$$\delta_0 = \sum_{j,k\in\mathbb{Z}} \varphi_j * \psi_k$$
$$= \sum_{j<0} (\widetilde{\varphi}_j)_{2^{-j/2},2^{-j}} + \sum_{j>0} (\widetilde{\varphi}_j)_{2^{-j},2^{-j}} + \sum_{j/2<k<j} (\widetilde{\varphi}_{j,k})_{2^{-j},2^{-k}} \ .$$

In comparison with the $\mathbb{R}^2$ case, there is an issue about cancellations. On $\mathbb{R}^n$ the cancellations of the dyadic terms guarantee that their sum converges in the sense of distributions. Cancellations follow from vanishing of the Fourier transform near the coordinate axes.

Ultimately,

$$\delta_0 = \sum_{j,k\in\mathbb{Z}} \varphi_j * \psi_k$$
$$= \sum_{j<0} (\widetilde{\varphi}_j)_{2^{-j/2},2^{-j}} + \sum_{j>0} (\widetilde{\varphi}_j)_{2^{-j},2^{-j}} + \sum_{j/2<k<j} (\widetilde{\varphi}_{j,k})_{2^{-j},2^{-k}} \ .$$

In comparison with the $\mathbb{R}^2$ case, there is an issue about cancellations. On $\mathbb{R}^n$ the cancellations of the dyadic terms guarantee that their sum converges in the sense of distributions. Cancellations follow from vanishing of the Fourier transform near the coordinate axes.

On $\mathbb{H}_n$, the fact that a multiplier $m(\xi, \lambda)$ is supported where $\xi \sim 2^j$, $\lambda \sim 2^k$, does not imply that integration in dz is zero.
What saves the day is that there is however a weak form of cancellation which implies that integrals on non-vertical hyperplanes in $\mathbb{H}_n$ decay exponentially in j.

The Hardy space H^1

In the one-parameter setting (either in $\mathbb{R}^n$ or $\mathbb{H}_n$ and for any fixed automorphic dilations or metric) the condition $f \in H^1$ is defined by any of the following equivalent conditions (Fefferman-Stein, 1972):

(i) $M_\varphi f \in L^1$ for some (and then every) $\varphi \in \mathcal{S}$ with $\int \varphi \neq 0$;

The Hardy space H^1

In the one-parameter setting (either in $\mathbb{R}^n$ or $\mathbb{H}_n$ and for any fixed automorphic dilations or metric) the condition $f \in H^1$ is defined by any of the following equivalent conditions (Fefferman-Stein, 1972):

(i) $M_\varphi f \in L^1$ for some (and then every) $\varphi \in \mathcal{S}$ with $\int \varphi \neq 0$;

(ii) the grand-maximal function $\mathcal{M}f$ is in L^1;

The Hardy space H^1

In the one-parameter setting (either in $\mathbb{R}^n$ or $\mathbb{H}_n$ and for any fixed automorphic dilations or metric) the condition $f \in H^1$ is defined by any of the following equivalent conditions (Fefferman-Stein, 1972):

(i) $M_\varphi f \in L^1$ for some (and then every) $\varphi \in \mathcal{S}$ with $\int \varphi \neq 0$;

(ii) the grand-maximal function $\mathcal{M} f$ is in L^1;

(iii) some (and then every) square function Sf is in L^1;

The Hardy space H^1

In the one-parameter setting (either in $\mathbb{R}^n$ or $\mathbb{H}_n$ and for any fixed automorphic dilations or metric) the condition $f \in H^1$ is defined by any of the following equivalent conditions (Fefferman-Stein, 1972):

(i) $M_\varphi f \in L^1$ for some (and then every) $\varphi \in \mathcal{S}$ with $\int \varphi \neq 0$;

(ii) the grand-maximal function $\mathcal{M}f$ is in L^1;

(iii) some (and then every) square function Sf is in L^1;

(iv) (atomic decomposition, Coifman-Fefferman, Coifman-Weiss) $f(x) = \sum_{j\in\mathbb{N}} c_j a_j(x)$, where $\sum_{j\in\mathbb{N}} |c_j| < \infty$ and each function a_j is supported in a ball B_j and satisfies

$$\int_{B_j} a_j = 0 \ , \qquad \|a_j\|_2 \leq m(B_j)^{-\frac{1}{2}} \ .$$

One has the following properties:

- CZ operators map H^1 into iself continuously;

One has the following properties:

- CZ operators map H^1 into iself continuously;
- the dual space of H^1 is BMO, the space of locally integrable functions g, modulo constants, such that, for some $C > 0$ and any ball B,

$$\fint_B |g(x) - g_B|\, dx \leq C \ , \qquad \left(g_B = \fint_B g \right) .$$

One has the following properties:

- CZ operators map H^1 into iself continuously;
- the dual space of H^1 is BMO, the space of locally integrable functions g, modulo constants, such that, for some $C > 0$ and any ball B,

$$\fint_B \big|g(x) - g_B\big|\, dx \le C\ , \qquad \left(g_B = \fint_B g\right)\ .$$

- In some cases ($\mathbb{R}^n$ with isotropic dilations or $\mathbb{H}_n$ with sub-riemannian distance) there is a further characterization of H^1 as the space of those $f \in L^1$ whose Riesz transforms

$$R_j f = \partial_{x_j}(-\Delta)^{-\frac{1}{2}} f \qquad \left(\text{or } X_j(-L)^{-\frac{1}{2}} f \text{ and } Y_j(-L)^{-\frac{1}{2}} f \text{ on } \mathbb{H}_n\right)$$

are also in L^1.

The product Hardy space on $\mathbb{R}^n$

Integrability of maximal functions and of square functions (conditions i-iii above appropriately modified) remain equivalent in the product setting on $\mathbb{R}^n$ and define the "product H^1 space".

An analogue of condition iv (characterization via atomic decompostions) requires a considerable modification of the notion of "atom" $a(x)$ (Chang-R. Fefferman 1980).

The product Hardy space on $\mathbb{R}^n$

Integrability of maximal functions and of square functions (conditions i-iii above appropriately modified) remain equivalent in the product setting on $\mathbb{R}^n$ and define the "product H^1 space".

An analogue of condition iv (characterization via atomic decompostions) requires a considerable modification of the notion of "atom" $a(x)$ (Chang-R. Fefferman 1980).

A product atom is a sum of *pre-atoms* $a(x_1, x_2) = \sum_i b_i(x_1, x_2)$, where

- a is supported on a *general set* Ω of finite measure and $\|a\|_2 \leq m(\Omega)^{-\frac{1}{2}}$,

The product Hardy space on $\mathbb{R}^n$

Integrability of maximal functions and of square functions (conditions i-iii above appropriately modified) remain equivalent in the product setting on $\mathbb{R}^n$ and define the "product H^1 space".

An analogue of condition iv (characterization via atomic decompostions) requires a considerable modification of the notion of "atom" $a(x)$ (Chang-R. Fefferman 1980).

A product atom is a sum of *pre-atoms* $a(x_1, x_2) = \sum_i b_i(x_1, x_2)$, where

- a is supported on a *general set* Ω of finite measure and $\|a\|_2 \leq m(\Omega)^{-\frac{1}{2}}$,
- each b_i is supported on a rectangle $R_i = I_i \times J_i$ with $\frac{1}{2}R_i \subset \Omega$,

The product Hardy space on $\mathbb{R}^n$

Integrability of maximal functions and of square functions (conditions i-iii above appropriately modified) remain equivalent in the product setting on $\mathbb{R}^n$ and define the "product H^1 space".

An analogue of condition iv (characterization via atomic decompostions) requires a considerable modification of the notion of "atom" $a(x)$ (Chang-R. Fefferman 1980).

A product atom is a sum of *pre-atoms* $a(x_1, x_2) = \sum_i b_i(x_1, x_2)$, where

- a is supported on a *general set* Ω of finite measure and $\|a\|_2 \leq m(\Omega)^{-\frac{1}{2}}$,
- each b_i is supported on a rectangle $R_i = I_i \times J_i$ with $\frac{1}{2} R_i \subset \Omega$,
- each b_i satisfies the conditions

$$\int b_i(x_1, x_2)\, dx_1 = \int b_i(x_1, x_2)\, dx_2 = 0 \ ,$$

The product Hardy space on $\mathbb{R}^n$

Integrability of maximal functions and of square functions (conditions i-iii above appropriately modified) remain equivalent in the product setting on $\mathbb{R}^n$ and define the "product H^1 space".

An analogue of condition iv (characterization via atomic decompostions) requires a considerable modification of the notion of "atom" $a(x)$ (Chang-R. Fefferman 1980).

A product atom is a sum of *pre-atoms* $a(x_1, x_2) = \sum_i b_i(x_1, x_2)$, where

- a is supported on a *general set* Ω of finite measure and $\|a\|_2 \le m(\Omega)^{-\frac{1}{2}}$,
- each b_i is supported on a rectangle $R_i = I_i \times J_i$ with $\frac{1}{2}R_i \subset \Omega$,
- each b_i satisfies the conditions

$$\int b_i(x_1, x_2)\, dx_1 = \int b_i(x_1, x_2)\, dx_2 = 0 \ ,$$

- $\sum_i \|b_i\|_2^2 \le m(\Omega)^{-1}$.

In the product setting of Chang-Fefferman, it is convenient to start from the square-function definition of H^1_{prod}, using a square function

$$S_{\mathrm{prod}} f(x_1, x_2) = \Big(\sum_{(j,k)\in\mathbb{Z}^2} \big| f * \varphi_{j,k}(x_1, x_2) \big|^2 \Big)^{\frac{1}{2}} ,$$

where the functions $\varphi_{j,k}(x_1, x_2) = \psi_j(x_1)\psi_k(x_2)$ add up to the Dirac delta at 0.

Multi-norm H^1 spaces

A discrete amount of work has been done in the last decade on Hardy spaces for two-parameter structures that are not product. The most frequented family of structures are the so-called *flag structures* introduced by Nagel-R.-Stein in 2001. For flag structures, the most relevant papers are

- Y. Han, G. Lu, E. Sawyer, 2014, containing a definition of flag H^p, $0 < p \leq 1$, both on $\mathbb{R}^n$ and H_n, through several equivalent square function, and the proof of boundedness on H^1 of singular integrals of flag type;

Multi-norm H^1 spaces

A discrete amount of work has been done in the last decade on Hardy spaces for two-parameter structures that are not product. The most frequented family of structures are the so-called *flag structures* introduced by Nagel-R.-Stein in 2001. For flag structures, the most relevant papers are

- Y. Han, G. Lu, E. Sawyer, 2014, containing a definition of flag H^p, $0 < p \leq 1$, both on $\mathbb{R}^n$ and H_n, through several equivalent square function, and the proof of boundedness on H^1 of singular integrals of flag type;
- X. Wu, 2014, containing an equivalent description of flag H^p on $\mathbb{R}^n$ via appropriate atomic decompositions;

Multi-norm H^1 spaces

A discrete amount of work has been done in the last decade on Hardy spaces for two-parameter structures that are not product. The most frequented family of structures are the so-called *flag structures* introduced by Nagel-R.-Stein in 2001. For flag structures, the most relevant papers are

- Y. Han, G. Lu, E. Sawyer, 2014, containing a definition of flag H^p, $0 < p \leq 1$, both on $\mathbb{R}^n$ and H_n, through several equivalent square function, and the proof of boundedness on H^1 of singular integrals of flag type;
- X. Wu, 2014, containing an equivalent description of flag H^p on $\mathbb{R}^n$ via appropriate atomic decompositions;
- a recent, still unpublished, work of Chen, Cowling, Lee, Li, Ottazzi on H_n, based on square functions associated to heat semigroups.

As to multi-norm structures, H^p-spaces have been introduced by Han-Li-Lin-Lu-Ruan-Sawyer in 2013 in the isotropic/parabolic case.

As to multi-norm structures, H^p-spaces have been introduced by Han-Li-Lin-Lu-Ruan-Sawyer in 2013 in the isotropic/parabolic case.

In work still in progress, in the last year A. Hejna, A. Nagel and myself have introduced multi-norm square functions on $\mathbb{R}^n$ with arbitrary norms, defined the corresponding Hardy space H^1, proved boundedness of multi-norm singular integral operators and obtained equivalence with a definition via atomic decompositions.

In the isotropic/parabolic case, our square functions and our atoms are not the same as in by Han-Li-Lin-Lu-Ruan-Sawyer, though the two Hardy spaces are the same.

We believe that the most valuable part of our contribution is the appropriate identification of the scales that should intervene in the decomposition of the Dirac delta in each case.

Our approach also includes flag structures as a limiting case.

Our theorem, stated here only in the isotropic/parabolic case on $\mathbb{R}^2$, establishes that the following three conditions are equivalent for $f \in L^1$:

(i) $Sf \in L^1$, where

$$
\begin{aligned}
Sf(x)^2 = & \sum_{j/2<k/j<j} \left| f * \psi_{2^{-j},2^{-2k}} \right|^2 + \sum_{j\geq 0} \left| f * \psi'_{2^{-j},2^{-j}} \right|^2 + \sum_{k\geq 0} \left| f * \psi''_{2^{-k},2^{-2k}} \right|^2 \\
& + \sum_{j<k/j<j/2} \left| f * \psi_{2^{-j},2^{-2k}} \right|^2 + \sum_{k<0} \left| f * \psi'_{2^{-k},2^{-2k}} \right|^2 + \sum_{j\geq 0} \left| f * \psi''_{2^{-j},2^{-j}} \right|^2 \ ;
\end{aligned}
$$

(ii) f can be writtem as $\sum_n \lambda_n a_n$, where $\sum |\lambda_n| < \infty$ and the a_n are atoms of the following three types:

- product (Chang-Fefferman) atom: $a_n = \sum_m b_{n,m}$, with the $b_{n,m}$ supported on rectangles $I \times J$ with $|J| \in \left[|I|, |I|^2\right]$;

(ii) f can be writtem as $\sum_n \lambda_n a_n$, where $\sum |\lambda_n| < \infty$ and the a_n are atoms of the following three types:

- product (Chang-Fefferman) atom: $a_n = \sum_m b_{n,m}$, with the $b_{n,m}$ supported on rectangles $I \times J$ with $|J| \in \left[|I|, |I|^2\right]$;
- isotropic (Coifman-Weiss) atom: a_n supported on a square of side ℓ and $\int a\, dx_1 = 0$ if $\ell > 1$, $\int a\, dx_2 = 0$ if $\ell < 1$;

(ii) f can be writtem as $\sum_n \lambda_n a_n$, where $\sum |\lambda_n| < \infty$ and the a_n are atoms of the following three types:

- product (Chang-Fefferman) atom: $a_n = \sum_m b_{n,m}$, with the $b_{n,m}$ supported on rectangles $I \times J$ with $|J| \in \left[|I|, |I|^2\right]$;
- isotropic (Coifman-Weiss) atom: a_n supported on a square of side ℓ and $\int a\, dx_1 = 0$ if $\ell > 1$, $\int a\, dx_2 = 0$ if $\ell < 1$;
- parabolic (Coifman-Weiss) atom: a_n supported on a rectangle $I \times J$ with $|J|^2 = |I| = \ell$ and $\int a\, dx_2 = 0$ if $\ell > 1$, $\int a\, dx_1 = 0$ if $\ell > 1$;

(ii) f can be writtem as $\sum_n \lambda_n a_n$, where $\sum |\lambda_n| < \infty$ and the a_n are atoms of the following three types:

- product (Chang-Fefferman) atom: $a_n = \sum_m b_{n,m}$, with the $b_{n,m}$ supported on rectangles $I \times J$ with $|J| \in \left[|I|, |I|^2\right]$;
- isotropic (Coifman-Weiss) atom: a_n supported on a square of side ℓ and $\int a\, dx_1 = 0$ if $\ell > 1$, $\int a\, dx_2 = 0$ if $\ell < 1$;
- parabolic (Coifman-Weiss) atom: a_n supported on a rectangle $I \times J$ with $|J|^2 = |I| = \ell$ and $\int a\, dx_2 = 0$ if $\ell > 1$, $\int a\, dx_1 = 0$ if $\ell > 1$;

(ii) f can be writtem as $\sum_n \lambda_n a_n$, where $\sum |\lambda_n| < \infty$ and the a_n are atoms of the following three types:

- product (Chang-Fefferman) atom: $a_n = \sum_m b_{n,m}$, with the $b_{n,m}$ supported on rectangles $I \times J$ with $|J| \in \left[|I|, |I|^2\right]$;
- isotropic (Coifman-Weiss) atom: a_n supported on a square of side ℓ and $\int a\, dx_1 = 0$ if $\ell > 1$, $\int a\, dx_2 = 0$ if $\ell < 1$;
- parabolic (Coifman-Weiss) atom: a_n supported on a rectangle $I \times J$ with $|J|^2 = |I| = \ell$ and $\int a\, dx_2 = 0$ if $\ell > 1$, $\int a\, dx_1 = 0$ if $\ell > 1$;

(iii) the grand-maximal function

$$\mathcal{M} f(x) = \sup_{s \in (r, r^2)\,,\, \varphi \in \mathcal{B}} \left| f * \varphi_{r,s}(x) \right|$$

is in L^1.