SMOOTHING EFFECT FOR TIME-DEGENERATE SCHÖDINGER OPERATORS

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Introduction

The local smoothing effect is a property typical of dispersive equations consisting in the local gain of derivatives of the solution with respect to the initial datum. This property was shown to be very powerful both to derive existence results for rough data and to prove regularity results for the solution. Moreover, when available, local smoothing estimates can be exploited in the analysis of variable coefficients Schrödinger (and not only Schrödinger) operators.

Here we shall investigate the local smoothing effect for time-degenerate Schrödinger operators of the form

$$\mathcal{L}_\alpha = i\partial_t + t^\alpha \Delta_x + b(t,x) \cdot \nabla_x, \quad \alpha > 0,$$

and use it to obtain the local well-posedness of the related nonlinear Cauchy problem.

Outline

1. Local smoothing estimates and related results in the variable coefficients case
2. A time-degenerate Schrödinger operator: the case $b \equiv 0$.
3. Local smoothing estimates and local well-posedness
4. The case $b \not\equiv 0$: local smoothing estimates
5. Local well-posedness of the nonlinear Cauchy problem
6. Final remarks

Local smoothing estimates

Smoothing estimates were proved to be satisfied for general dispersive equations.

- In 1983 Kato showed the local smoothing effect for the Korteweg-de Vries equation;
- The homogeneous local smoothing effect for the Schrödinger equation was simultaneously established by Sjolin in 1987 and by Vega in 1988;
- In 1989 Constantin-Saut proved the homogeneous local smoothing effect for general dispersive equations;
- In 1991 Kenig-Ponce-Vega improved and generalized the previous results and derived inhomogeneous smoothing estimates.

Variable coefficients

Craig-Kappeler-Strauss and Doi obtained, under suitable regularity assumptions on the coefficients, the homogeneous version of the local smoothing effect for variable coefficients Schrödinger operators of the form

$$\mathcal{L} = i\partial_t + a(x, D) + b(x) \cdot \nabla u,$$

where $a(x, D)$ is a second order elliptic operator.
The inhomogeneous smoothing effect for $\mathcal{L}$ was then derived by Kenig-Ponce-Vega, while smoothing estimates (both homogeneus and inhomogeneous) for ultra-hyperbolic variable coefficients Schrödinger operators was proved by Kenig-Ponce-Rolvung-Vega.
The gain of one derivative in the inhomogeneous case allowed to prove l.w.p. results for the nonlinear Cauchy problem with derivatives nonlinearities associated with the aforementioned nondegenerate variable coefficients Schrödinger operators.

Time-degenerate Schrödinger operators

In what follows we shall show the local smoothing effect for degenerate Schrödinger operators of the form

$$\mathcal{L}_\alpha = i\partial_t + t^\alpha \Delta_x + b(t,x)\cdot\nabla_x, \quad \alpha > 0,$$

where the coefficients $b(t,x) = (b_1(t,x), \ldots, b_n(t,x))$ are *complex valued* functions satisfying suitable decay assumptions.
Operators of the form $\mathcal{L}_\alpha$ have been previously analyzed by Cicognani and Reissig who proved local well-posedness results for the associated **homogeneous** Cauchy problem both in Sobolev and Gevrey spaces.
However we want to stress that our approach is different, since, in particular, we derive **smoothing estimates** for the operator $\mathcal{L}_\alpha$ and apply them to the resolution of the associated **nonlinear** Cauchy problem.

We studied the operator $\mathcal{L}_\alpha = i\partial_t + t^\alpha \Delta_x + b(t,x)\cdot\nabla_x, \quad \alpha > 0$, in the following two different cases.

Cases

- $b \equiv 0$. By Fourier analysis methods we derived weighted smoothing effect with a gain of $1/2$ derivative with respect to the initial datum;
- $b \not\equiv 0$. We used the pseudo-differential calculus to get a weighted smoothing effect with a gain of 1 derivative for the inhomogeneous part with respect to the initial datum;

Remark. The results obtained in the general case $b \not\equiv 0$ still hold in the particular case $b \equiv 0$. However we decided to analyze the two cases separately in order to show how different techniques can/must be used depending on the presence of a gradient term.

The case $b \equiv 0$: $\quad \mathcal{L}_\alpha = i\partial_t + t^\alpha \Delta_x, \quad \alpha > 0$

We considerd the IVP

$$\begin{cases} \mathcal{L}_\alpha u = f \\ u(s,t) = u_s(x), \quad s \geq 0. \end{cases}$$

By using Fourier analysis we get that the function u solving the problem above is

$$u(t,x) = W_\alpha(t,s)u_s(x) + \int_0^t W_\alpha(t',s)f(t',x)dt',$$

where the so called **solution operator** $W_\alpha(t,s)$, $t,s \geq 0$, is given by

$$W_\alpha(t,s)u_s(x) := e^{i\frac{t^{\alpha+1}-s^{\alpha+1}}{\alpha+1}\Delta} u_s := \int_{\mathbb{R}^n} e^{-i(\frac{t^{\alpha+1}-s^{\alpha+1}}{\alpha+1}|\xi|^2 - x\cdot\xi)} \widehat{u_s}(\xi)d\xi.$$

Notation: $W_\alpha(t,0) =: W_\alpha(t)$.
Notice that for $\alpha = 0$ we have the standard Schrödinger group.

The case $b \equiv 0$: $\quad \mathcal{L}_\alpha = i\partial_t + t^\alpha \Delta_x, \quad \alpha > 0$

Observe that the solution operator satisfies the following properties.

(i) $W_\alpha(t,t) = I$;

(ii) $W_\alpha(t,s) = W_\alpha(t,r)W_\alpha(r,s)$ for every $s,t,r \in [0,T]$;

(iii) $W_\alpha(t,s)\Delta_x u = \Delta_x W_\alpha(t,s)u$.

Moreover it is easy to see that

$$\|W_\alpha(t,s)u_s\|_{H^s_x} = \|u_s\|_{H^s_x}.$$

Smoothing effect when $b \equiv 0$

By exploiting the properties of $W_\alpha(t,s)$ we derived the following smoothing effect for $\mathcal{L}_\alpha$.

Theorem. (F.-Staffilani) Let $W_\alpha(t) := W_\alpha(t,0)$, with $\alpha > 0$, then If $n = 1$ for all $f \in L^2(\mathbb{R})$,

$$\sup_x \| t^{\alpha/2} D_x^{1/2} W_\alpha(t) f \|^2_{L^2_t([0,T])} \lesssim \|f\|^2_{L^2(\mathbb{R})}; \tag{1}$$

If $n \geq 2$, on denoting by $\{Q_\beta\}_{\beta\in\mathbb{Z}^n}$ the family of non overlapping cubes of unit size such that $\mathbb{R}^n = \bigcup_{\beta\in\mathbb{Z}^n} Q_\beta$, then for all $f \in L^2_x(\mathbb{R}^n)$,

$$\sup_{\beta\in\mathbb{Z}^n} \left(\int_{Q_\beta} \int_0^T |t^{\alpha/2} D_x^{1/2} W_\alpha(t) f(x)|^2 dt\, dx \right)^{1/2} \lesssim \|f\|_{L^2(\mathbb{R}^n)}, \tag{2}$$

where $D_x^\gamma f(x) = (|\xi|^\gamma \widehat{f}(\xi))^\vee(x)$.

Theorem. (F.-Staffilani) Let $n = 1$ and $g \in L^1_x L^2_t([0, T] \times \mathbb{R})$, then

$$\|D_x^{1/2} \int_{\mathbb{R}_+} t^{\alpha/2} W_\alpha(0,t) g(t) dt\|_{L^2_x(\mathbb{R})} \lesssim \|g\|_{L^1_x L^2_t(\mathbb{R}\times[0,T])}, \tag{3}$$

and, for all $g \in L^1_t L^2_x([0, T] \times \mathbb{R})$,

$$\|t^{\alpha/2} D_x^{1/2} \int_0^t W_\alpha(t,\tau) g(\tau) d\tau\|_{L^\infty_x(\mathbb{R}) L^2_t([0,T])} \lesssim \|g\|_{L^1_t L^2_x([0,T]\times\mathbb{R})}. \tag{4}$$

If $n \geq 2$, denoting by $\{Q_\beta\}_{\beta\in\mathbb{Z}^n}$ a family of non overlapping cubes of unit size such that $\mathbb{R}^n = \bigcup_{\beta\in\mathbb{Z}^n} Q_\beta$, then, for all $g \in L^1_t L^2_x([0, T] \times \mathbb{R}^n)$,

$$\sup_{\beta\in\mathbb{Z}^n} \left(\int_{Q_\beta} \left\| t^{\alpha/2} D_x^{1/2} \int_0^t W_\alpha(t,\tau) g(\tau) d\tau \right\|^2_{L^2_t([0,T])} dx \right)^{1/2} \lesssim \|g\|_{L^1_t L^2_x([0,T]\times\mathbb{R}^n)} \tag{5}$$

Local well-posedness result

Theorem. (F.-Staffilani) Let $k \geq 1$, then the IVP

$$\begin{cases} \mathcal{L}_\alpha u = \pm u|u|^{2k} \\ u(0,x) = u_0(x), \end{cases}$$

is locally well-posed in H^s for $s > n/2$ and its solution satisfies smoothing estimates.

Remark. The local well-posedness of the Cauchy problem above but with derivative nonlinearities follows from the equivalent result given in the general case $b \not\equiv 0$, which, as we already said, is true even in the particular case $b \equiv 0$.

Local well-posedness.

The proof of the previous theorem is obtained by using the standard fixed point argument.

In particular (when $n = 1$) we considered the metric space

$$X = \{u : [0, T] \times \mathbb{R} \to \mathbb{C}; \|t^{\alpha/2} D_x^{1/2+s} u\|_{L_x^\infty L_t^2([0,T])} < \infty, \\ \|u\|_{L_t^\infty([0,T]) H_x^s} < \infty\},$$

equipped with the distance

$$d(u, v) = \|t^{\alpha/2} D_x^{1/2+s}(u - v)\|_{L_x^\infty L_t^2([0,T])} + \|u - v\|_{L_t^\infty([0,T]) \dot{H}_x^s} \\ + \|u - v\|_{L_t^\infty([0,T]) L_x^2}$$

in which $\dot{H}_x^s$ stands for the homogeneous Sobolev space, and considered

$$\Phi : X \to X, \quad \Phi(u) = W_\alpha(t) u_0 + \int_0^t W_\alpha(t, \tau) u |u|^{2k}(\tau) d\tau.$$

By using the smoothing estimates given before we proved that Φ is a contraction which, after application of standard fixed point arguments, gives the result.

The case $b \not\equiv 0$. Local weighted smoothing effect

We considered the IVP

$$\begin{cases} \partial_t u = it^\alpha \Delta_x u + ib(t,x) \cdot \nabla_x u + f(t,x) \\ u(0,x) = u_0(x). \end{cases} \tag{6}$$

Theorem. (F.-Staffilani)
Let $u_0 \in H^s(\mathbb{R}^n), s \in \mathbb{R}$. Assume that, for all $j = 1, ..., n$, b_j is such that $b_j \in C([0,T], C_b^\infty(\mathbb{R}^n))$ and there exists $\sigma > 1$ such that

$$|\mathrm{Im}\, \partial_x^\gamma b_j(t,x)|, |\mathrm{Re}\, \partial_x^\gamma b_j(t,x)| \lesssim t^\alpha \langle x \rangle^{-\sigma-|\gamma|}, \quad x \in \mathbb{R}^n, \tag{7}$$

and denote by $\lambda(|x|) := \langle x \rangle^{-\sigma}$.
Then

(i) If $f \in L^1([0,T]; H^s(\mathbb{R}^n))$ then the IVP (6) has a unique solution $u \in C([0,T]; H^s(\mathbb{R}^n))$ and there exist positive constants C_1, C_2 such that

$$\sup_{0 \le t \le T} \|u(t)\|_s \le C_1 e^{C_2(\frac{T^{\alpha+1}}{\alpha+1}+T)} \left(\|u_0\|_s + \int_0^T \|f(t)\|_s dt \right);$$

(ii) If $f \in L^2([0,T]; H^s(\mathbb{R}^n))$ then the IVP (6) has a unique solution $u \in C([0,T]; H^s(\mathbb{R}^n))$ and there exist two positive constants C_1, C_2 such that

$$\sup_{0\leq t\leq T} \|u(t)\|_s^2 + \int_0^T \int_{\mathbb{R}^n} t^\alpha \left|\Lambda^{s+1/2} u\right|^2 \lambda(|x|) dx\, dt$$

$$\leq C_1 e^{C_2(\frac{T^{\alpha+1}}{\alpha+1}+T)} \left(\|u_0\|_s^2 + \int_0^T \|f(t)\|_s^2 dt \right);$$

(iii) If $\Lambda^{s-1/2} f \in L^2([0,T]\times\mathbb{R}^n; t^{-\alpha}\lambda(|x|)^{-1} dtdx)$ then the IVP (6) has a unique solution $u \in C([0,T]; H^s(\mathbb{R}^n))$ and there exist positive constants C_1, C_2 such that

$$\sup_{0\leq t\leq T} \|u(t)\|_s^2 + \int_0^T \int_{\mathbb{R}^n} t^\alpha \left|\Lambda^{s+1/2} u\right|^2 \lambda(|x|) dx\, dt$$

$$\leq C_1 e^{C_2 \frac{T^{\alpha+1}}{\alpha+1}} \left(\|u_0\|_s^2 + \int_0^T \int_{\mathbb{R}^n} t^{-\alpha} \lambda(|x|)^{-1} \left|\Lambda^{s-1/2} f\right|^2 dx\, dt \right).$$

Remarks on the conditions and strategy of the proof

We remark that it is natural to require the previous conditions on the term b in order to have the l.w.p. of the IVP. Indeed, even in the nondegenerate case (with $b = b(x)$), decay conditions on $\mathrm{Re}\, b$ are necessary for the local well-posedness of the linear Cauchy problem to hold.
The main problems when proving the smoothing effect for $\mathcal{L}_\alpha$ are given by the presence of the time degeneracy t^α in the second order term and by the presence of the first order term $b \cdot \nabla_x$. Our strategy aims to get rid of the gradient term by exploiting the form of the second order term and the decay properties of b.

Remarks on the conditions

In particular the time degeneracy has to be managed in order to apply a method similar to that of Mizohata and Doi to absorb the space-time variable coefficients first order term through the application of the Sharp Gårding inequality. The presence of the term $b(t,x)\cdot\nabla$ affects the applicability of the Sharp Gårding inequality as well, and, in addition, determines, in general, a loss of derivatives. Due to these considerations it is clear that conditions on $b(t,x)$ are necessary to control the behavior of the operator, and, specifically, conditions relating the coefficient t^α and the coefficients $b_j(t,x)$.

Strategy of the proof. Doi's lemma

A key point in the proof of the smoothing theorem above is represented by Doi's lemma below.

Let $a^w(t,x,\xi)$ be the Weyl symbol of a pseudo-differential operator $A = A(t,x,D_x)$. We shall say that $a^w =: a$ satisfies $(B1)$, $(B2)$ and $(A6)$ if

(B1) $a(t,x,\xi) = ia_2(x,\xi) + a_1(t,x,\xi) + a_0(t,x,\xi)$, *where $a_2 \in S^2_{1,0}$ is real-valued and $a_j \in S^j_{1,0}$,* for $j = 0, 1$;

(B2) $|a_2(x,\xi)| \geq \delta|\xi|^2$ *with* $x \in \mathbb{R}^n$, $|\xi|^2 \geq C$, *and* $\delta, C > 0$;

(A6) *There exists a real-valued function $q \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that, with $C_{\alpha\beta}, C_1, C_2 > 0$,*

$$|\partial_\xi^\alpha \partial_x^\beta q(x,\xi)| \leq C_{\alpha\beta}\langle x\rangle\langle\xi\rangle^{-|\alpha|}, \quad x,\xi \in \mathbb{R}^n,$$

$$H_{a_2}q(x,\xi) = \{a_2, q\}(x,\xi) \geq C_1|\xi| - C_2, \quad x,\xi \in \mathbb{R}^n,$$

where $S^j_{1,0} = S^j_{\rho=1,\delta=0} =: S^j$ is standard class of pseudo-differential symbols of order j, and by $\{\cdot,\cdot\}$ the Poisson bracket.

Strategy of the proof

Lemma (Doi's lemma)

Assume (B1), (B2) and (A6). Let $\lambda(s)$ be a positive non increasing function in $C([0,\infty))$. Then

1. *If $\lambda \in L^1([0,\infty))$ there exists a real-valued symbol $p \in S^0$ and $C > 0$ such that*
$$H_{a_2} p \geq \lambda(|x|)|\xi| - C, \quad x, \xi \in \mathbb{R}^n; \tag{8}$$
2. *If $\int_0^t \lambda(\tau)d\tau \leq C \log(t+1) + C'$, $t \geq 0$, $C, C' > 0$, then there exists a real-valued symbol $p \in S_1^0(\log\langle\xi\rangle)$ such that*
$$H_{a_2} p \geq \lambda(|x|)|\xi| - C_1 \log\langle\xi\rangle - C_2, \quad x, \xi \in \mathbb{R}^n. \tag{9}$$

This lemma permits to absorb the gradient term in $\mathcal{L}_\alpha$ by means of the use of a suitable weighted norm together with the sharp Gårding inequality.

Strategy of the proof

The proof can be summarized in the following steps.

Step 1: We apply Doi's lemma on the symbol $a^w =: a = a_2 + ia_1 + a_0$ with $a_2(x,\xi) = |\xi|^2$ and $a_1 = a_0 = 0$. In this case conditions (B1) and (B2) of the lemma are trivially satisfied, while (A6) holds with $q(x,\xi) = x \cdot \xi \langle \xi \rangle^{-1}$. Therefore, choosing $\lambda'(|x|) = C' \langle x \rangle^{-\sigma}$, with C' to be chosen later, we get that there exists $p \in S^0$ and $C > 0$ such that (8) holds.

Step 2: We define a norm equivalent to the H^s-Sobolev norm as follows. We consider the pseudo-differential operator K with symbol $K(x,\xi) = e^{p(x,\xi)}\Lambda^s$, where $\Lambda^s := \langle \xi \rangle^s$ and $p(x,\xi)$ is given by Doi's lemma, and define the norm N on $H^s(\mathbb{R}^n)$, equivalent to the standard one, given by

$$N(u)^2 = \|Ku\|_0^2 + \|u\|_{s-1}^2, \tag{10}$$

where $\|\cdot\|_s$ stands for the standard norm in the Sobolev space $H^s(\mathbb{R}^n)$.

Strategy of the proof

Step 3: We estimate $\partial_t N(u)^2$ by using commutator estimates and exploiting the properties of K (i.e. of $p(x,\xi)$) to control the gradient term by means of the Sharp Gårding inequality. In particular

$$\partial_t N(u)^2 = \partial_t \|Ku\|_0^2 + \partial_t \|u\|_{s-1}^2$$

$$\leq Ct^\alpha N(u)^2 + C' N(u)^2 + 2\mathrm{Re}\langle (it^\alpha [p,\Delta_x](x,D) - b(t,x)\cdot D_x)Ku, Ku\rangle$$
$$+C''' N(f)N(u) + C_3 \min\{N(f)N(u); \langle t^{-\alpha}\lambda(|x|)^{-1}\Lambda^{s-1/2}f, \Lambda^{s-1/2}f\rangle\}.$$

Since

$$\mathrm{Re}\left(it^\alpha(-i)\{p,-|\xi|^2\}(x,\xi) - b(t,x)\cdot\xi\right) \leq t^\alpha(-C\lambda(|x|)\langle\xi\rangle + C_3) + C_4,$$

by the Sharp Gårding inequality we get

$$\partial_t N(u)^2 \leq Ct^\alpha N(u)^2 + C' N(u)^2 - C'' t^\alpha \|\lambda(|x|)^{1/2}\Lambda^{1/2}Ku\|_0^2 + C''' N(f)N(u) +$$
$$+C_3 \min\{N(f)N(u); \langle t^{-\alpha}\lambda(|x|)^{-1}\Lambda^{s-1/2}f, \Lambda^{s-1/2}f\rangle\}.$$

Step 4: From the previous estimate, by using different upper bounds for the term

$$\min\{N(f)N(u); \langle t^{-\alpha}\lambda(|x|)^{-1}\Lambda^{s-1/2}f, \Lambda^{s-1/2}f\rangle\}$$

we get two different inequalities.

Integrating in time the resulting inequalities the weighted local smoothing estimates follow.

Finally, by using the smoothing estimates together with standard functional analysis arguments, the local well-posedness follows.

Local well-posedness of the NLIVP

We then considered the NLIVP

$$\begin{cases} \mathcal{L}_\alpha u = \pm u|u|^{2k} \\ u(0,x) = u_0(x), \end{cases} \tag{11}$$

and

$$\begin{cases} \mathcal{L}_\alpha u = \pm t^\beta \nabla u \cdot u^{2k}, \quad \beta \geq \alpha > 0, \\ u(0,x) = u_0(x). \end{cases} \tag{12}$$

Theorem. (F.-Staffilani) Let $\mathcal{L}_\alpha$ be such that condition (7) is satisfied. Then the IVP (11) is locally well posed in H^s for $s > n/2$.

Theorem. (F.-Staffilani) Let $\mathcal{L}_\alpha$ be such that condition (7) is satisfied with $\sigma = 2N$ (thus $\lambda(|x|) = \langle x \rangle^{-2N}$) for some $N \geq 1$, and $s > n + 4N + 3$ such that $s - 1/2 \in 2\mathbb{N}$. Let $H^s_\lambda := \{u_0 \in H^s(\mathbb{R}^n); \lambda(|x|)u_0 \in H^s(\mathbb{R}^n)\}$, then the IVP (12) with $\beta \geq \alpha > 0$, is locally well posed in H^s_λ.

Local well-posedness of the NLIVP

The proof follows, once again, by using the contraction argument. This is possible thanks to the application of the smoothing estimates given above together with some commutator estimates.
We want to remark that we did not focus here on the optimality of the exponent s for the H^s local well-posedness and stress the the solutions of the nonlinear problem satisfy the weighted smoothing estimates given before. In fact we prove the map $\Phi : X^s_T \mapsto X^s_T$ is a contraction, where

$$X^s_T := \{u : [0, T] \times \mathbb{R}^n \to \mathbb{C};\, \|u\|_{L^\infty_t H^s_x} < \infty,$$
$$\left(\int_0^T \int_{\mathbb{R}^n} t^\alpha \lambda(|x|) |\Lambda^{s+1/2} u|^2 dx\, dt\right)^{1/2} < \infty,$$
$$\|\lambda(|x|)^{-1} u\|_{L^\infty_t H^{s-2N-3/2}_x} < \infty\},$$

and

$$\|u\|^2_{X^s_T} = \|u\|^2_{L^\infty_t H^s_x} + \int_0^T \int_{\mathbb{R}^n} t^\alpha \lambda(|x|) |\Lambda^{s+1/2} u|^2 dx\, dt + \|\lambda^{-1} u\|^2_{L^\infty_t H^{s-2N-3/2}_x}.$$

Final remarks

- Possibly by using the technique used by Kenig-Ponce-Rolvung-Vega, one can obtain the same smoothing and local well-posedness results for the equation

  $$\mathcal{L}_\alpha u = i\partial_t + t^\alpha \Delta_x + b(t,x) \cdot \nabla_x u, + c(t,x) \cdot \nabla_x \overline{u},$$

  with $c(t,x)$ possibly satisfying conditions similar to that assumed for $b(t,x)$. In this case one should reduce the problem to a system, that, after diagonalization, satisfies the desired estimates.
- By using our procedure one should be able to prove a suitable weighted local smoothing effect and local well-posedness results for degenerate Schrödinger operators of the form

  $$\mathcal{L}_g = i\partial_t + g(t)\Delta_x + b(t,x) \cdot \nabla_x,$$

  provided that g and b satisfy suitable decay and regularity assumptions, as, for instance, g having constant sign (for $t > 0$), g vanishing at $t = 0$ and b such that $|\partial_x^\gamma b_j(t,x)| \lesssim \lambda(|x|)|g(t)|$ for all $j = 1, ..., n$.

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