Average solutions to linear second order semielliptic PDEs

Ermanno Lanconelli
Bologna

We speak about the Dirichlet problem

$$\mathcal{L}u = 0 \text{ in } \Omega \;, \quad u|_{\partial\Omega} = \varphi$$

- $\mathcal{L}$ is a divergence form semielliptic PDE :

$$\mathcal{L} = \operatorname{div}(A(x)\, D) \;, \quad x \in \mathbb{R}^N$$

$A(x)$ real, symmetric, ≥ 0 $N \times N$ matrix

- Ω bounded open $\subseteq \mathbb{R}^N$
- $f \in C(\overline{\Omega}, \mathbb{R})$
- $\varphi \in C(\partial\Omega, \mathbb{R})$

Hypotheses

- $\mathcal{L} - \varepsilon$ is hypoelliptic, for every $\varepsilon > 0$
- trace $(A(x)) > 0$, $\forall x \in \mathbb{R}^N$
- $\mathcal{L}$ has a nonnegative global fundamental solution

$$(x, y) \longmapsto \Gamma(x, y) = \Gamma(y, x)$$

of class C^∞ in $\{x \neq y\}$ and s.t.

$$\lim_{x \to y} \Gamma(x, y) = \infty, \quad \lim_{x \to \infty} \Gamma(x, y) = 0$$

$$\forall y \in \mathbb{R}^N$$

Fundamental solution

- $\Gamma(x,\cdot) \in L^1_{loc}(\mathbb{R}^N) \quad \forall x \in \mathbb{R}^N$
- For every $\varphi \in C_0^\infty(\mathbb{R}^N, \mathbb{R})$:

$$\int_{\mathbb{R}^N} \Gamma(x,y)\, \mathcal{L}\varphi(y)\, dy = -\varphi(x) \quad \forall x \in \mathbb{R}^N,$$

$$\mathcal{L} \int_{\mathbb{R}^N} \Gamma(x,y)\, \varphi(y)\, dy = -\varphi(x) \quad \forall x \in \mathbb{R}^N.$$

Main difficulties

(I) The presence of $\mathcal{L}$- characteristic points

on $\partial\Omega$, even for smooth domains

(II) The lack of an ad hoc

Syngular Integrals Theory

for the Poisson- type equation

$$\mathcal{L}u = f$$

Perron-Wiener and average solutions

We have bypassed these difficulties by combining

(I) a Perron-Wiener approach to the "homogeneous" Dirichlet problem

$$\mathcal{L}u = 0 \text{ in } \Omega, \quad u|_{\partial\Omega} = \varphi$$

(II) a notion of average solution to

$$\mathcal{L}u = f \quad \text{for every } f \text{ continuous}$$

Pizzetti

Regarding (II): we have extended to $\mathcal{L}$ the Pizzetti's notion of <u>average solution</u> to the classical Poisson equation : $\Delta u = f$.

- (Pizzetti) Let $u \in C(\Omega, \mathbb{R})$. Then u is an <u>average solution</u> to

$$\Delta u = f \quad \text{in } \Omega \quad (f : \Omega \longrightarrow \mathbb{R})$$

Pizzetti, continuation

if $$\lim_{r \to 0} \frac{M_r(u)(x) - u(x)}{r^2} = \frac{1}{2(N+2)} f(x) \,, \quad \forall x \in \Omega .$$

- $M_r(u)(x) =$ Gauss average : $\displaystyle \fint_{B(x,r)} u(y)\, dy$

In this case we let $\underline{\Delta_a u = f}$

<u>NOTE</u>. If $f \in C^2$, then

$$\Delta_a u = \Delta u$$

Pizzetti Theorem

Let $f \in C_c(\mathbb{R}^N, \mathbb{R})$. Define

$$u(x) = \int_{\mathbb{R}^N} \Gamma(x-y)\, f(y)\, dy \quad , \quad x \in \mathbb{R}^N$$

Γ = fundamental solution of Δ.

Then : $\quad \Delta_a u = -f$.

<u>NOTE</u> 1.- In general: $u \notin C^2$

<u>NOTE</u> 2 (Calderón - Zygmund) $u \in W^{2,p}_{loc} \;\; \forall\, p \in \,]1, \infty[$
and $\Delta u(x) = -f(x)$ a.e.

The level set of Γ

Our extension of the notion of average solution to semielliptic operator is based on a representation formula on the super level set

Γ = fondamental solution of $\mathcal{L}$.

- If $x \in \mathbb{R}^N$ and $r > 0$, define

$$\Omega_r(x) := \left\{ y \in \mathbb{R}^N \,/\, \Gamma(x,y) > \frac{1}{r} \right\}$$

Level set of Γ, continuation

- $\Omega_r(x) \neq \emptyset$, open, bounded

- $\bigcap_{r>0} \Omega_r(x) = \{x\} \quad \forall x \in \mathbb{R}^N$

- $\dfrac{|\Omega_r(x)|}{r} \longrightarrow 0$ as $r \to 0$

Ex. If $\mathcal{L} = \Delta$ and $N \geq 3$, then

$$\Omega_r(x) = B(x, \rho), \quad \rho = (c_N r)^{\frac{1}{N-2}}$$

Euclidean ball

Representation formula

- Let $\alpha > -1$. Let $u \in C^{(2)}(\Omega, \mathbb{R})$. Let $\overline{\Omega_r(x)} \subseteq \Omega$

- (PJ) $u(x) = M_r(u)(x) - N_r(\mathcal{L}u)(x)$, $x \in \Omega$

- $M_r(u)(x) := \dfrac{\alpha+1}{r^{\alpha+1}} \displaystyle\int_{\Omega_r(x)} u(y)\, K(x,y)\, dy$

$$K(x,y) := \frac{\langle A(x) \nabla_y \Gamma(x,y), \nabla_y \Gamma(x,y) \rangle}{(\Gamma(x,y))^{\alpha+1}}$$

- $N_r(w)(x) =: \dfrac{\alpha+1}{r^{\alpha+1}} \displaystyle\int_0^r \rho^{\alpha+1} \left(\int_{\Omega_\rho(x)} \left(\Gamma(x,y) - \frac{1}{\rho} \right) w(y)\, dy \right) d\rho$

Remark 1 :- $\mathcal{L} = \Delta$, $\alpha = \dfrac{2}{N-2}$ $(N > 2) \Longrightarrow$

$K =$ constant, $M_r =$ Gauss average on the Euclidean ball $B_\rho(x)$, $\rho = c_N\, r^{\frac{1}{N-2}}$

Remark 2 . $\mathcal{L} = \Delta_{\mathbb{G}}$ sublaplacian on $\mathbb{G}$.

If $Q :=$ hom. dim. $\mathbb{G} \geq 3$; $\alpha = \dfrac{2}{Q-2}$

$\Downarrow$

$K =$ homogeneous function of degree $= 0$

<u>Remark 3</u>.- . When $\mathcal{L} = \sum_{j=1}^{p} X_j^2$ formula (PJ)

was proved in [Citti - Garofalo - Lanc (1993), Am. J. of Math]

• A proof of (PJ) in the present setting can be found:

[Bonfiglioli - Lanc (2013), Eur. Math. J.]

$\uparrow$

See References therein

for a bibliography on

Mean Value Theorems

Average solutions

- Define:

$$\underline{q_r(x)} := N_r(1)(x) = \frac{\alpha+1}{r^{\alpha+1}} \int_0^r \rho^{\alpha+1} \int_{\Omega_\rho(x)} (\Gamma(x,y) - \frac{1}{\rho})$$

An easy computation gives:

$$q_r(x) = \int_0^r \frac{|\Omega_\rho(x)|}{\rho^2} \left(1 - \left(\frac{\rho}{r}\right)^{\alpha+1}\right) d\rho .$$

Ex. $\mathcal{L} = \Delta_G$, $Q = \text{hom. dim}(G) \geq 3$, $\alpha = \frac{2}{Q-2}$

$$\Downarrow$$

$$\underline{q_r(x) = c_N \, \rho^2} \quad , \ \rho = r^{\frac{1}{Q-2}}$$

- $u \in C^2(\Omega, \mathbb{R}) \Longrightarrow$

$$\lim_{r \to 0} \frac{M_r(u)(x) - u(x)}{q_r(x)} = \mathcal{L}u(x) \,, \quad \forall x \in \Omega$$

- <u>Definition</u> (<u>Average solution</u>).

Let $u, f \in C(\Omega, \mathbb{R})$. Then u is an <u>average solution to $\mathcal{L}u = f$</u> if

$$\mathcal{L}_a(u)(x) := \lim_{r \to 0} \frac{M_r(u)(x) - u(x)}{q_r(x)}$$

exists at any point $x \in \Omega$ and

$$\underline{\mathcal{L}_a(u)(x) = f(x)} \quad \forall x \in \Omega.$$

- <u>Remark.</u> We will see that

$$\mathcal{L}_\alpha u = f \text{ in } \Omega \overset{!}{\Longleftrightarrow} \mathcal{L} u = f \text{ in } \mathcal{D}'(\Omega)$$

- <u>Theorem</u> (Pizzetti-type Theorem)

Let $f \in C_c(\mathbb{R}^N, \mathbb{R})$. Define

$$\underline{u(x) = \Gamma_f(x) := \int_{\mathbb{R}^N} \Gamma(x,y)\, f(y)\, dy}, \quad x \in \mathbb{R}^N.$$

Then

$$\underline{\mathcal{L}_\alpha u = -f \quad \text{in } \mathbb{R}^N}$$

Average = Weak

Let $f \in C_0(\mathbb{R}^N, \mathbb{R})$, $u \in C(\mathbb{R}^N, \mathbb{R})$. Then

$$\mathcal{L}_a u = f \iff \mathcal{L} u = f \text{ in } \mathcal{D}'.$$

Proof.- $\mathcal{L}_a u = f \iff \mathcal{L}_a (u + \Gamma_f) = 0$

[Gutierrez - Lanc (2004)] $\rightarrow \Updownarrow$

$u + \Gamma_f \in C^\infty$, $\mathcal{L}(u + \Gamma_f) = 0 \iff$

$\mathcal{L}(u + \Gamma_f) = 0$ in $\mathcal{D}' \iff \mathcal{L}u = -\mathcal{L}(\Gamma_f)$

$= f$ in $\mathcal{D}'$

To the "homogeneous" Dirichlet problem

Let $F \in C(\overline{\Omega}, \mathbb{R})$ and $\phi \in C(\partial\Omega, \mathbb{R})$, Ω bounded.
Choose $f \in C_c(\mathbb{R}^n, \mathbb{R})$: $f|_{\overline{\Omega}} = F$. Then:

$$\mathcal{L}_a v = F \text{ in } \Omega, \quad v|_{\partial\Omega} = \phi \quad \Longleftrightarrow$$

(letting $u = v + \Gamma_f$, $\varphi = (\phi + \Gamma_f)|_{\partial\Omega}$)

$$\mathcal{L} u = 0 \text{ in } \Omega, \quad u|_{\partial\Omega} = \varphi$$

$$\varphi \in C(\partial\Omega, \mathbb{R})$$

PW - method

- u $\mathcal{L}$-harmonic in Ω $\overset{\text{def}}{\Longleftrightarrow}$ $\mathcal{L}u = 0$ in Ω

- $u : \Omega \longrightarrow [-\infty, \infty[$, u.s.c. is $\mathcal{L}$-subharmonic

if $\{u > -\infty\}$ is dense in Ω and

for every open $V \subseteq \overline{V} \subseteq \Omega$,

for every $h \in C(\overline{V}, \mathbb{R})$ and $\mathcal{L}$- harmonic in V

$$\Downarrow$$

$u \leq h$ in V if $u \leq h$ on ∂V

PW-method, continuation

- Ω bounded, $\varphi \in C(\partial\Omega, \mathbb{R})$.

$$H_\varphi^\Omega = \underline{\text{generalized solution}} \text{ to } \begin{cases} \mathcal{L}u = 0 \text{ in } \Omega \\ u|_{\partial\Omega} = \varphi \end{cases}$$

$$:= \sup \{ u \ \mathcal{L}\text{-}\underline{\text{subharmonic in } \Omega} : \limsup_{\partial\Omega} u \leq \varphi \}$$

- <u>Theorem</u>: H_φ^Ω is $\mathcal{L}$-harmonic in Ω

Boundary behavior of H_φ^Ω.

- In general: $H_\varphi^\Omega(x) \not\longrightarrow \varphi(y)$ as $x \to y \in \partial\Omega$
- A point $y \in \partial\Omega$ is called $\mathcal{L}$-regular for Ω

  if $\lim\limits_{x \to y} H_\varphi^\Omega(x) = \varphi(y) \quad \forall \varphi \in C(\partial\Omega, \mathbb{R})$.

- Theorem (Bouligand-type Theorem for $\mathcal{L}$).

$$y \in \partial\Omega \text{ is } \mathcal{L}\text{-regular for } \Omega$$

$$\Updownarrow$$

$$\text{There exists an } \mathcal{L}\text{-barrier for } \Omega \text{ at } y.$$

• <u>Definition</u> An <u>$\mathcal{L}$-barrier</u> for Ω at y

is a function

h $\mathcal{L}$-<u>superharmonic</u> in $\Omega \cap V$,

V neigh. of y, s.t.

(i) $h > 0$ in $\Omega \cap V$

(ii) $\lim_{x \to y} h(x) = 0$

• h $\mathcal{L}$-<u>superharmonic</u> iff $-h$ is <u>$\mathcal{L}$-subharmonic</u>

Back to $\mathcal{L}$- subharmonic functions

Let $u : \Omega \longrightarrow [-\infty, \infty[$ be u.s.c. and s.t.

$\Omega(u) := \{ x \,/\, u(x) > -\infty \}$ is dense in Ω.

For $x \in \Omega$ define $R(x) := \sup\{ r : \Omega_r(x) \subseteq \Omega \}$.

Then:

- u is $\mathcal{L}$- subharmonic

$$\Updownarrow$$

- $u(x) \le M_r(u)(x) \quad \forall x \in \Omega$ and $0 < r < R(x)$

$$\Updownarrow$$

- $\overline{\mathcal{L}_a} u(x) := \limsup_{r \to 0} \dfrac{M_r(u)(x) - u(x)}{q_r(x)} \ge 0 \quad \forall x \in \Omega$

$$\Updownarrow$$

- $r \longmapsto M_r(u)(x)$ is increasing in $]0, R(x)[$, $\forall x \in \Omega$

  and $M_r(u)(x) \longrightarrow u(x)$ as $r \to 0$

$$\Updownarrow$$

- $u \in L^1_{loc}(\Omega)$, $\mathcal{L}u \geq 0$ in $\mathcal{D}'(\Omega)$,

  $M_r(u)(x) \longrightarrow u(x)$ as $r \to 0$, $\forall x \in \Omega$

[Bonfiglioli - Lanc (2011), J. Eur. Math. Soc.]

Smoothing of $\mathcal{L}$-subharmonic functions

. u $\mathcal{L}$-subharmonic in Ω

$$\Downarrow$$

$x \longmapsto M_r(u)(x)$ $\mathcal{L}$-subharmonic

$$\Downarrow$$

There exists a sequence (u_n) of $\mathcal{L}$-subharmonic functions s.t.

(i) $u_n \downarrow u$

(ii) $u_n \in C^\infty$

In summary

Our solution to

$$\begin{cases} \mathcal{L}u = f & \text{in } \Omega \\ u|_{\partial\Omega} = \varphi \end{cases}$$

is given by

$$u = \underbrace{H^{\Omega}_{\varphi + \Gamma_f}}_{\text{PW solution}} - \Gamma_f$$

PW $\overset{?}{=}$ Variational

- Assume the Poincaré inequality

$$\int_\Omega u^2\,dx \le C \int_\Omega |\nabla_A u|^2\,dx \quad \forall u \in \mathcal{D}(\Omega)$$

$$\langle ADu, Du\rangle$$

Define

- $\|u\|_{\mathcal{L}} := \|\nabla_A u\|_{L^2(\Omega)}$

- $\overset{\circ}{H}_{\mathcal{L}}(\Omega) =$ closure of $\mathcal{D}(\Omega)$ w.r.t. $\|\cdot\|_{\mathcal{L}}$

- $H^{-1}_{\mathcal{L}}$ = dual space of $\mathring{H}_{\mathcal{L}}$

- Let $\varphi \in C(\partial\Omega, \mathbb{R})$. Assume there exists

$$\phi \in C(\bar{\Omega}, \mathbb{R}) \text{ s.t. } \phi\big|_{\partial\Omega} = \varphi \qquad \mathcal{L}\phi \in H^{-1}_{\mathcal{L}}(\Omega)$$

$$\Big| \int_\Omega \phi \mathcal{L} v \Big| \le c \, \|v\|_{\mathcal{L}} \quad \forall v \in \mathcal{D}(\Omega)$$

Then

$$\begin{cases} \mathcal{L}u = -\mathcal{L}\phi \\ u \in \mathring{H}_{\mathcal{L}} \end{cases}$$ has a unique variational solution

Theorem. $w = u + \phi$ is the PW solution to

$$\begin{cases} \mathcal{L} w = 0 \text{ in } \Omega \\ w|_{\partial\Omega} = \varphi \end{cases}$$

- $\mathcal{L} = \Delta$: [Arendt and Daners (2008), Bull. London Math. Soc.]
- $\mathcal{L} = \Delta_{\mathbb{G}}$: [Abbondanza (2016), Math. Nachr.]

Lebesgue approach to the Dirichlet problem

- For $x \in \Omega$ define $r(x) := \frac{1}{2} \sup\{ r \,/\, \Omega_r(x) \subset \Omega \}$
- For $v \in C(\Omega, \mathbb{R})$ define

$$T(v) : \Omega \to \mathbb{R},$$

$$T(v)(x) = M_{r(x)}(v)(x).$$

Let $\varphi \in C(\partial\Omega, \mathbb{R})$ and let

$$\phi \in C(\overline{\Omega}, \mathbb{R}) : \phi\big|_{\partial\Omega} = \varphi$$

Define the sequence (u_n):

$$u_0 = \phi, \quad u_1 = T(\phi),$$

$$u_{n+1} = T(u_n).$$

Then

$$u_n \longrightarrow H^{\Omega}_{\varphi} \quad \text{as } n \to \infty$$

PW solution to $\begin{cases} \mathcal{L}u = 0 & \text{in } \Omega \\ u|_{\partial\Omega} = \varphi \end{cases}$

[Abbondanza - Bonfiglioli (2012), J. London Math. Soc.]

A final comment

In recent years asymptotic average solutions for nonlinear PDEs have been study by many authors. Let me only mention

Manfredi - Parviainen - Rossi (2010) Proc. AMS

Ferrari (2015) Comm. Pure Appl. Anal.