Measure preserving homeomorphisms of the torus

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TABLE DES MATIÈRES

CHAPTER 1

INTRODUCTION

The subject of the lecture is the dynamics of volume preserving torus homeomorphisms. We will have successively various points of view on this subject, with a common technique, the use of periodic approximations, which is the main theme of this School. One striking feature is that we will mainly use *discontinuous* periodic approximations to understand the dynamics of continuous systems. The main reference is a book by Alpern and Prasad, *Typical dynamics of volume preserving homeomorphisms*. Another reference that we use is the classical book by Halmos, *Lectures on ergodic theory*.

We consider the torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. It is equipped with the metric and volume induced by the usual euclidean metric and volume in $\mathbb{R}^n$. In other words, if $\Pi : \mathbb{R}^n \to \mathbb{T}^n$ denotes the projection, we have

$$d_{\mathbb{T}^n}(x, y) = \inf\{d_{\mathbb{R}^n}(\tilde{x}, \tilde{y}) | \Pi(\tilde{x}) = x, \Pi(\tilde{y}) = y\}$$

and

$$\mathrm{Vol}_{\mathbb{T}^n}(A) = \mathrm{Vol}_{\mathbb{R}^n}(\Pi^{-1}(A) \cap [0, 1]^n)$$

for any points x, y and borelian subset A of the torus.

We denote by $\mathrm{Homeo}(\mathbb{T}^n, \mathrm{Vol})$ the set of homeomorphisms of the torus (bijective bicontinuous maps $\mathbb{T}^n \to \mathbb{T}^n$) that preserve the volume, that is, satisfy $\mathrm{Vol}(h^{-1}(A)) = \mathrm{Vol}(A)$ for any borelian set A. Easy examples include translations $x \mapsto x + \alpha$ where α is any element of the torus, and linear automorphisms $T \in \mathrm{SL}(n, \mathbb{Z})$.

The group $\mathrm{Homeo}(\mathbb{T}^n, \mathrm{Vol})$ is endowed with the sup metric,

$$||f - g|| = \sup_{x \in \mathbb{T}^n} d(f(x), g(x)),$$

which induces the topology of uniform convergence. It turns this group into a topological group.(1) The connected component of the identity is denoted by $\mathrm{Homeo}_0(\mathbb{T}^n, \mathrm{Vol})$; an element h is in $\mathrm{Homeo}_0(\mathbb{T}^n, \mathrm{Vol})$ if and only if there is an *isotopy* from the identity to h, i.e. a continuous family $(h_t)_{t \in [0,1]}$ with $h_0 = \mathrm{Id}$ and $h_1 = h$. On this subgroup there is a fundamental dynamical invariant, called the *mean rotation vector* (see chapter 2 for the definition). The set of $\mathrm{Homeo}_{0,0}(\mathbb{T}^n, \mathrm{Vol})$ of homeomorphisms whose mean rotation vector is zero is a normal subgroup of $\mathrm{Homeo}(\mathbb{T}^n, \mathrm{Vol})$.(2) In dimension two, Patrice Le Calvez proved that any element of this subgroup has at least three fixed point. This generalized the Arnol'd conjecture, that involved diffeomorphisms and was proved by Conley and Zehnder, and Franks' theorem, that stated the existence of one fixed point for homeomorphisms. We will

(1) Note that this metric is right-invariant, *i.e.* satisfies $||f - g|| = ||f\varphi - g\varphi||$.

(2) Note that it is unknown whether this is a simple group.

see that periodic approximations provide a nice proof of Franks' theorem (as shown by Alpern and Prasad).

***Theorem* (Franks).** — *Any volume preserving homeomorphisms of $\mathbb{T}^2$, isotopic to the identity, with vanishing mean rotation vector, has at least one fixed point.*

When endowed with the sup metric, the group $\mathrm{Homeo}(\mathbb{T}^n, \mathrm{Vol})$ is not complete. But the metric $d(f,g) = ||f - g|| + ||f^{-1} - g^{-1}||$ induces the same topology, and is complete. (The space is not locally compact, though). The Baire category theorem applies, so we may ask: which properties are generic?

This question was first attacked by Oxtoby and Ulam: in 1941 they proved that ergodicity is generic. Then in 1970 Katok and Stepin obtained the stronger result that weak mixing is generic. In the interval between those two results, Halmos proved that ergodicity and even weak mixing are generic in the setting of ergodic theory, *i. e.* for volume preserving bijections. Finally, in 1978 Alpern found a deep link between the two settings, proving the following theorem.

***Theorem*.** — *Any dynamical property which is generic in* $\mathrm{Auto}(\mathbb{T}^n, \mu)$ *is also generic in* $\mathrm{Homeo}(\mathbb{T}^n, \mu)$*.*

For a precise definition of genericity in both contexts, see below.

The previous theorem gives a beautiful solution to the problem of generic ergodic properties. It says that, when you translate the study from topological dynamics to ergodic theory, no new generic property appears. What about non generic ergodic properties? Lind and Thouvenot proved that any (finite entropy) ergodic system is conjugate to a volume preserving torus homeomorphism (topologically conjugate to a linear map). But it is natural to ask for unique ergodicity, that is, to impose the statistics of *all* the orbits. The conjecture is that we should have a similar result: any automorphism is conjecturally conjugate to a uniquely ergodic volume preserving torus homeomorphism. This has been proved for a special (non generic!) but large class of automorphisms, namely those whose L^2-operator admits an irrational eigenvalue.

***Theorem* (Uniquely ergodic realizations, Béguin-Crovisier-Le Roux)**

Any automorphism which is an extension of an irrational circle rotation is conjugate to a uniquely ergodic volume preserving homeomorphism of $\mathbb{T}^2$.

Note that the conjecture is known to be true if we replace $\mathbb{T}^2$ by the Cantor set: this is the Jewett-Krieger theorem.(3)

Her is what we plan to do in this course. We will deduce the first theorem from the plane translation theorem of Brouwer (which will be taken for granted). Then we will give a complete proof of the second theorem. The last one is more technical, we will only explain a very special but basic case.

Generalizations. — The fixed points results are specific to the dimension 2, and very dependent on the topology of the manifold.

The genericity result actually holds on any (smooth) manifold, see the book by Alpern and Prasad. The idea is to prove it on the cube, and then to see any manifold as a quotient of the cube, obtained by identifying some points on the boundary.

The realization of automorphisms as uniquely ergodic is a special case of a theorem that holds on many more manifolds.

(3) The theorem also holds for $\mathbb{T}^n$.

Let us finally note that the generic ergodic properties of C^1-diffeomorphisms are unknown, for example it is conjectured that generic C^1-diffeomorphisms on manifolds of dimension greater than 1 are ergodic, but this is unknown even in dimension 2.

Examples in dimension two: hamiltonian dynamics. — We have already mentioned two families of examples, namely the translations and the linear automorphisms. The first one is a n-parameter family, while the second one is a discrete family. These examples do not give a right idea of how large is the group $\mathrm{Homeo}(\mathbb{T}^n, \mathrm{Vol})$, which should be thought of as an infinite dimensional analogue of the classical Lie groups. Here we describe a bigger (infinite dimensional) family on $\mathbb{T}^2$. Furthermore, the construction will provide examples in $\mathrm{Homeo}_{0,0}(\mathbb{T}^n, \mathrm{Vol})$ (for the time being we only have the identity...).

In order to define hamiltonian diffeomorphisms, we endow the torus $\mathbb{T}^2$ with its smooth manifold structure, and with the (symplectic) area form $dx \wedge dy$. Let $H : \mathbb{T}^2 \to \mathbb{R}$ be any smooth function. To this function is associated a vector field X_H, called the hamiltonian vector field associated to H. One practical concrete way to define this vector field is as follow: at each point x, consider the gradient (dual of df_x with respect to the euclidean metric $dx^2 + dy^2$), and define $X_H(x)$ to be the vector obtained from the gradient of f at x by applying a rotation of $\pi/2$. More abstractly (and this generalizes on $\mathbb{T}^{2n}$), $X_H(x)$ is the vector dual to df_x with respect to the anti-symetric 2-form $dx \wedge dy$.

We may integrate this vector field into a smooth flow $(\Phi_t)_{t\in\mathbb{R}}$ which is a one-parameter subgroup of $\mathrm{Homeo}(\mathbb{T}^2)$. Then a basic computation shows that the vector field X_H has vanishing divergence, which amounts to saying that the flow Φ_t preserves the symplectic form $dx \wedge dy$. Thus every time of this flow provide an example of a (smooth) element of $\mathrm{Homeo}(\mathbb{T}^2, \mathrm{Vol})$.

Clearly every element Φ_t belongs to $\mathrm{Homeo}_0(\mathbb{T}^2, \mathrm{Vol})$. Actually, Φ_t even belongs to the normal subgroup $\mathrm{Homeo}_{0,0}(\mathbb{T}^2, \mathrm{Vol})$. This fact is linked to the interpretation of the quantity $H(x) - H(y)$ as the flux of the vector field X_H through any curve from x to y.

An easy generalization consists in considering time-dependent hamiltonian function H_t, which induce time-dependent divergence free vector fields $X_{H,t}$. By integrating we again get a one-parameter family (not subgroup) in $\mathrm{Homeo}_{0,0}(\mathbb{T}^2, \mathrm{Vol})$. Now it can be proved that any C^∞ diffeomorphism in $\mathrm{Homeo}_{0,0}(\mathbb{T}^2, \mathrm{Vol})$ may be constructed this way. For this reason, the elements of $\mathrm{Homeo}_{0,0}(\mathbb{T}^2, \mathrm{Vol})$ are sometimes called *hamiltonian homeomorphisms.*

Exercise Construct a counter-example to Franks' theorem in dimension 3. *Hint:* f may be obtained as the composition of 2 diffeomorphisms. The first one preserves the x and y coordinates, and rotation each circle $\{(x, y)\} \times \mathbb{S}^1$. The second one do the same for x, z. They both belongs to $\mathrm{Homeo}_{0,0}(\mathbb{T}^n, \mathrm{Vol})$, and their fixed point sets are disjoint.

CHAPTER 2

APPROXIMATION BY PERMUTATIONS

2.1. Lax's theorem

We denote by $\mathcal{D}_m$ the set of *dyadic cubes of order* m, that is, products of intervals of the form $[k/2^m, (k+1)/2^m]$ (there are 2^{mn} such cubes). A permutation σ of the set $\mathcal{D}_m$ induces a map $\mathbb{T}^n \to \mathbb{T}^n$, still denoted by σ, and defined by demanding that on each dyadic cube c, σ is the translation that sends c on $\sigma(c)$. The map is not defined on the boundary of the cubes, which does not matter, since we see σ as an element of $\mathrm{Auto}(\mathbb{T}^n, \mathrm{Vol})$, the set of bijections of $\mathbb{T}^n$ that are bi-measurable and preserves the volume, where we identify two maps that coincides on a subset of volume 1. Note that we may see $\mathrm{Homeo}(\mathbb{T}^n, \mathrm{Vol})$ as a subspace of $\mathrm{Auto}(\mathbb{T}^n, \mathrm{Vol})$, and the sup metric extend to a metric on $\mathrm{Auto}(\mathbb{T}^n, \mathrm{Vol})$, where $||g - h||$ is defined as the *essential* upper bound of $d(g(x), h(x))$ (thus $||g - h||$ is the least number α satisfying $d(g(x), h(x)) \leq \alpha$ for almost every x).

According to Lax, the following theorem was motivated by the problem of the discretisation of a homeomorphism, linked to the use of computers. At the time it has no applications, but we will see how Alpern and Prasad used it in the study of homeomorphisms.

***Theorem* (Lax).** — *Let $h \in \mathrm{Homeo}(X, \mathrm{Vol})$, and $\varepsilon > 0$. Then there exists a dyadic cyclic*[(1)] *permutation σ such that $||h - \sigma|| < \varepsilon$.*

***Lemma* (Marriage lemma).** — *Let E, F be two (finite) sets, and $\approx$ be a relation between elements of E and F. Under the following condition:*

$$\forall E' \subset E, \#E' \leq \#\{f \in F, \exists e \in E', e \approx f\}$$

there exists a one-to-one map Φ from E to F, such that, for every $e \in E$, $e \approx \Phi(e)$.

Proof. — Consider a dyadic subdivision. For every cubes C, C' of the subdivision, write $C \approx C'$ if the image of C meets C'. Note that because of the volume preserving hypothesis, for any family $C_1, \dots, C_k$ of cubes, the image of their union meets at least k cubes. According to the marriage lemma, there exists a (non necessarily cyclic) permutation σ of the family of dyadic cubes such that, for every cube C, $h(C)$ meets $\sigma(C)$. Then the norm $||h - \sigma||$ is less than the supremum of the diameters of the images of the cubes under h, plus the diameter of a cube. By uniform continuity of f, this is certainly less than ε if the order of the subdivision is large enough.

It remains to prove that any permutation is uniformly approximated by a cyclic permutation. This is a purely combinatorial statement, which essentially amounts to the following fact (where n is any positive integer). *For any permutation σ of the set $\{1, \dots, n\}$, there exists a permutation τ satisfying $\mid \tau(k) - k \mid \leq 2$ for every*

[(1)]Remember that a permutation is cyclic if it has a single orbit.

k, and such that $\sigma' = \tau\sigma$ is a cyclic permutation. Given the fact, we can order our dyadic cubes such that any two consecutive cubes are adjacent; then we use the numbering to see σ as a permutation on the set of numbers, and get a new dyadic permutation σ' with $||\sigma' - \sigma||$ less than two times the diameter of the cubes.

Let us prove the fact. Remember that the set $\{1, ..., n\}$ is partitionned by the orbits (the *cycles*) of σ. Furthermore, the composition by a transposition has the following effect on cycles. When $k, k+1$ do not belong to the same cycle of σ, then the cycles of the permutation $(kk+1) \circ \sigma$ are the cycles of σ, except that the cycles of k and $k+1$ have been merged into a single cycle.

For simplicity we assume n is even. If 1 and 2 belongs to the same cycle, then we define r_1 as the identity, otherwise r_1 is the permutation (12). We now consider the cycles of the permutation $r_1\sigma$; if 3 and 4 belongs to the same cycle, r_2 is the identity, otherwise it permutes 3 and 4. We proceed this way to produce a permutation $\tau_1 = r_{n/2} \dots r_1$ such that for every k, the integers $2k-1$ and $2k$ belongs to the same cycle of $\tau_1\sigma$. A similar process produces a permutation τ_2 such that for every k, the integers k and $k+1$ belongs to the same cycle of $\tau_2\tau_1\sigma$. Thus this permutation has only one cycle □

Proof of the marriage lemma. — We use the following classical vocabulary. E is the set of "girls", F is the set of "boys", if $e \approx f$ we say that e "knows" f, and if $e = \Phi(f)$ we say that e is married with f. We prove the lemma by induction on the number n of girls. Let E be a set of $n+1$ girls as in the lemma. First assume the following stronger hypothesis is satisfied: for every subset of k girls, $0 < k \leq n$, the number of boys known by some girl of the subset is greater than k. Then the solution is easy: select any girl, marry her to any boy she happens to know, check that the remaining sets of girls and boys satisfy the induction hypothesis, so they can also be married. In the opposite case, there exists a subset of k girls, $0 < k \leq n$, knowing exactly k boys altogether. Use the induction hypothesis to marry all of them, then check that the unmarried boys and girls again satisfy the induction hypothesis, which complete the ceremony. □

2.2. Genericity of topological transitivity

We make a small digression to show the genericity of transitivity as an easy consequence of Lax's theorem (together with a fundamental extension lemma).

A homeomorphism h is called *topologically transitive* if for every open sets U, V, some iterate of U meets V (in a nice space(2) like a manifold, this is equivalent to the existence of a dense orbit). The Anosov automorphisms are topologically transitive. A countable intersection of open sets is called a G_δ-subset. A property is called *generic* if it is shared by all the elements of a dense G_δ subset. The Baire category theorem applies in the complete metric space $\mathrm{Homeo}(\mathbb{T}^n, \mathrm{Vol})$, so that any countable intersection of dense open sets is dense (and in particular, the intersection of two G_δ dense subsets is again a G_δ dense subset). A corollary of Lax's theorem is the genericity of transitive homeomorphisms in $\mathrm{Homeo}(\mathbb{T}^n, \mathrm{Vol})$.

Proposition. — *The set of topologically transitive homeomorphisms is a G_δ dense subset of the space* $\mathrm{Homeo}(\mathbb{T}^2, \mathrm{Vol})$ *with respect to the uniform topology.*(3)

(2) Complete separable with no isolated points.

(3) Definition, completeness.

Proof. — Let $\tau_1, \ldots$ be an enumeration of all the (open) dyadic cubes of all orders. Define $T_{i,j}$ as the set of homeomorphisms for which there exists some $k > 0$ such that $h^k(\tau_i)$ meets τ_j. This is an open set. The intersection of all these sets is precisely the set of transitive homeomorphisms. Thus it remains to see that each set $T_{i,j}$ is dense.

Fix i, j and $\varepsilon > 0$. Let $h \in \mathrm{Homeo}(\mathbb{T}^n, \mathrm{Vol})$, and let σ be a cycle dyadic permutation (of sufficiently large order, larger than the order of τ_i and τ_j) such that $||h - \sigma|| < \varepsilon$. Let $p_1, \ldots,$ be the centers of the dyadic cubes, enumerated in the order of the action of σ. Note that for any k, $||h(p_k) - p_{k+1}|| < \varepsilon$. We now make use of the lemma asserting that the map sending each point $h(p_k)$ on p_{k+1} may be extended to a volume-preserving homeomorphism φ such that $||\varphi|| < \varepsilon$ (see below, *extension of finite maps*). Let $h' = \varphi h$. Then $||h' - h|| < \varepsilon$ and h' belongs to $T_{i,j}$. □

The same proof works in the cube, where the mere existence of a single topologically transitive homeomorphism is not at all obvious, althoug there is a construction using Anosov torus maps.

Exercise (may be difficult, or even wrong?...) Prove that a generic $h \in \mathrm{Homeo}(\mathbb{T}^n, \mathrm{Vol})$ is not *topologically mixing* (h is called topologically mixing if for every open sets U, V, every iterates of U, but a finite number, meets V.

2.3. A fixed point theorem

Consider an isotopy (h_t) in $\mathrm{Homeo}_0(\mathbb{T}^n, \mathrm{Vol})$, with $h_0 = \mathrm{Id}$, and the trajectory $(h_t(x))$ of some point $x \in \mathbb{T}^n$. We can lift this trajectory to $\mathbb{R}^n$, that is, consider a curve $\gamma : [0,1] \to \mathbb{R}^n$ such that $\pi \circ \gamma(t) = h_t(x)$ for every t. Note that this curve joins a lift of x to a lift of $h(x)$. Denote by $D(x)$ the displacement vector $\overrightarrow{\gamma(0)\gamma(1)}$ of x (this does not depend on the choice of γ). The mean rotation vector of the isotopy is

$$\rho(h_t) := \int_{\mathbb{T}^n} D(x) d\mathrm{Vol}(x).$$

This vector is also named the mass flow, when interpreted as the quantity of matter that flows across the hypersurfaces of the torus. When considered modulo $\mathbb{Z}^n$, it does only depend on $h_1 = h$, not on the isotopy.

Let $h \in \mathrm{Homeo}_0(\mathbb{T}^n, \mathrm{Vol})$. A homeomorphisms of $\mathbb{R}^n$ is called a *lift* of h if $\pi H = h\pi$, that is if any lift of any point x of the torus is sent by H on a lift of $h(x)$. Any isotopy $(h_t$ may be lifted to an isotopy (H_t), with $H_0 = \mathrm{Id}$ and each H_t a lift of h_t (this is a general result of the theory of covering maps, and of course it has nothing to do with the volume). The mean rotation vector of the isotopy (h_t) is then equal to the mean translation vector of $H = H_1$,

$$\rho(h_t) := \int_{\mathbb{T}^n} (D(x)) \, \mathrm{dVol}(x)$$

where $D(x)$ is defined as $H(\tilde{x}) - \tilde{x}$ for any lift $\tilde{x}$ of x. Assume in addition that h belongs to $\mathrm{Homeo}_{0,0}(\mathbb{T}^2, \mathrm{Vol})$, *i.e.* $\rho(h) = 0 \pmod{\mathbb{Z}^n}$. Then there is a unique lift H with mean vanishing translation vector. We can give the following more precise version of Franks' theorem.

Theorem. — *Let us assume $n = 2$. In this situation, H has at least one fixed point.*

P. Le Calvez has proved that is this situation, H has infinitely many periodic points. Also note that the techniques used to prove the genericity of transitivity may also be used to prove that generically in $\mathrm{Homeo}_{0_0}(\mathbb{T}^n, \mathrm{Vol})$, the favorite lift is transitive in $\mathbb{R}^n$.

Let us recall the Brouwer plane translation theorem.

Theorem **(Brouwer plane translation theorem, baby version)**
Let H be a homeomorphism of the plane which is isotopic to the identity. If H has a periodic point then H has a fixed point.

This is a deep theorem that we will not prove here. The case of a periodic point of period two has an easy proof (by Morton Brown). The idea of the other cases is to construct, starting with a (would be) fixed point free homeomorphism with a periodic point, another fixed point free homeomorphism with a periodic point of period 2.

We will deduce Franks' theorem from the plane translation theorem.

Proof. — Let H be as in the theorem. We argue by contradiction, assuming that H has no fixed point. Let $0 < \varepsilon < \inf_{x \in \mathbb{R}^2} d(x, H(x))$. Note that any plane homeomorphism H' satisfying $||H' - H|| < \varepsilon$ has no fixed point. We assume $\varepsilon < 1/2$. We apply Lax's theorem to find a first dyadic permutation σ with $||h - \sigma|| < \varepsilon/3$. Then there exists a unique lift Σ such that $||H - \Sigma|| < \varepsilon/3$ (this is because the ball with radius less than $1/2$ in $\mathbb{R}^n$ projects one-to-one on the torus). Define the translation vector $\rho(\Sigma)$ by the same formula that we used for homeomorphisms. Then we have $|\rho(\Sigma)| = |\rho(\Sigma) - \rho(H)| < \varepsilon/3$. On the other hand the coordinates of $\rho(\Sigma)$ are (small) dyadic numbers (the displacement of each x is a multiple of the side of the dyadic cubes, the displacements are summed and the sum is divide by the number of cubes, which is a power of 2). The translation of $\mathbb{R}^2$, $x \mapsto x - \rho(\Sigma)$, induces a dyadic permutation σ' of the torus (maybe of a greater order), and its translation vector is the opposite of $\rho(\Sigma)$. Then $\tau = \sigma' \circ \sigma$ has a lift with translation number 0, and $||\tau - h|| < \varepsilon$.

We can now modify τ, using the same process as in the proof of Lax's theorem, in order to get a cyclic permutation υ, close to τ, and thus also close to h. It is easy to check that the rotation vector of Υ (the unique lift of υ that is close to H) is still equal to 0.

Now for any point X in the plane we have $\Upsilon^M(X) = X$, where M is the order of υ (the number of cubes). Indeed, since $\upsilon^M = \mathrm{Id}$, we get $\Upsilon^M(X) = X + \vec{v}$ where $\vec{v}$ is an integer vector; since υ is cyclic, this implies $\Upsilon^M(Y) = Y + \vec{v}$ for every Y, so that $\vec{v}$ is the translation vector of Υ, which is zero. Since $||H - \Upsilon|| < \varepsilon$ we may modify H into a plane homeomorphism H' with $||H' - H|| < \varepsilon$, having a periodic orbit (using the extension of finite maps, as in the proof of generic transitivity). By the plane translation theorem, H' must have a fixed point. This contradicts the definition of ε at the beginning of the proof. □

2.4. Extension of finite maps

The following proposition says that the group $\mathrm{Homeo}(\mathbb{T}^n, \mathrm{Vol})$ acts transitively on k-uplets of points.

Proposition. — *Let $x_1, \ldots, x_k; y_1, \ldots, y_k$ be two sequences of k pairwise distinct points of X. Then there exists $h \in \mathrm{Homeo}(\mathbb{T}^n, \mathrm{Vol})$ such that for every $i = 1, ..., k$, $h(x_i) = y_i$.*

Furthermore, the map h may be chosen so that

1. *it is equal to a torus translation in a neighborhood of each point x_i;*
2. *if no pair of point (x_i, y_i) is more than a distance δ apart, then $d(h, \mathrm{Id}) < \delta$.*

This proposition is used to get a "closing lemma" (approximate a homeomorphism by another one having a periodic orbit). It will be also more fundamentally used to get a Lusin-like theorem (in particular, this instance will require the first property).

We need a lemma, which is some kind of infinitesimal version of the transitivity.

Lemma. — *Let $\varepsilon > 0$. There exists a continuous map, defined on the tangent bundle of the torus, with values in the space of C^∞ vector fields on the torus,*

$$\begin{array}{ccc} T\mathbb{T}^n & \to & \Xi(\mathbb{T}^n) \\ (x, \vec{v}) & \mapsto & X_{x,\vec{v}} \end{array}$$

such that

1. *the vector field $X_{x,\vec{v}}$, at point x, is equal to $\vec{v}$, and it is constant on some neighbourhood of x;*
2. *this vector field vanishes outside the ball of radius ε around x;*
3. $||X_{x,\vec{v}}|| \leq ||\vec{v}||$.

Proof. — We deal only with dimension 2, leaving the other dimensions to the reader. We first define the vector field $X_{x,\vec{v}}$ for $x = (0, 0)$ and $\vec{v} = (\ell, 0)$. Let H_1 be any hamiltonian function satisfying :

1. $\frac{\partial H}{\partial x}(x, y) = 0, \quad \frac{\partial H}{\partial y}(x, y) = 1$ for any (x, y) in some neighbourhood of $(0, 0)$;
2. H is supported inside the ball of radius ε around $(0, 0)$;
3. $||\vec{\mathrm{grad}} H|| \leq 1$ everywhere.

Then define $H_\ell = \ell H_1$, and let $X_{(0,0,(\ell,0)}$ be the hamiltonian vector field associated to the hamiltonian function H_ℓ. Now we extend the map, first for $x = (0, 0)$ and any $\vec{v}$ by rotating $X_{0,(\ell,0)}$: more precisely, let $X_{0,\vec{v}}$ be the image of $X_{0,(\ell,0)}$, where $\ell = ||\vec{v}||$, under the unique (local) rotation with center 0 that sends the vector $(\ell, 0)$ on $\vec{v}$. Finally we extend the map to any couple $(x, \vec{v})$ by defining $X_{x,\vec{v}}$ be the image of $X_{0,\vec{v}}$ under the unique torus translation that sends the point $(0, 0)$ on x. □

***Lemma* (Extension of isotopies)**. — *Let $\gamma_1, \ldots, \gamma_k$ be k smooth curves in $\mathbb{T}^n$, such that for every t, for every $i \neq j$, $\gamma_i(t) \neq \gamma_j(t)$ ("no collision").*

Then there exists a smooth, time-dependent vector field $(X_t)_{t\in[0,1]}$ on $\mathbb{T}^n$, integrating into a smooth isotopy (Φ_t) such that

1. *for every $i = 1, \ldots, k$, for every $t \in [0, 1]$, $\Phi_t(\gamma_i(0)) = \gamma_i(t)$, in other words γ_i is a trajectory of the vector field X_t; furthermore, in the neighborhood of each $\gamma_i(0)$, the map Φ_1 coincides with a translation of the torus;*
2. *for every t, $||X_t|| \leq \max_i ||\gamma_i'(t)||$.*

Proof. — Let ε be less than half the infimum of all the distances $d(\gamma_i(t), \gamma_j(t)), i \neq j, t \in [0, 1]$. We use the previous lemma and set, for every t,

$$X_t = \sum_{i=1}^{k} X_{\gamma_i(t),\gamma_i'(t)}.$$

Note that, due to the choice of ε, at each time t all the vector fields in the sum have disjoint support. Thus we get the first condition, and also

$$||X_t|| \leq \max_i ||X_{\gamma_i(t),\gamma_i'(t)}|| \leq \max_i ||\gamma_i'(t)||.$$

□

Proof of the proposition. — For every i, let γ_i be the curve running at constant speed on a segment joining x_i to y_i, of length less than δ. We can avoid collisions between these curves by changing the speed a little bit if necessary, keeping the speed less than δ. Then we apply the previous lemma. $\square$

Exercise. Write a similar proof in the C^0 category. Solution : see the book by Alpern and Prasad, proof of theorem 2.4.

CHAPTER 3

GENERIC ERGODIC PROPERTIES

3.1. Generic properties in Homeo *versus* Auto

The aim of this chapter is to prove the following.

Theorem. — *A dynamical property which is generic in* $\mathrm{Auto}(\mathbb{T}^n, \mathrm{Vol})$ *(with respect to the weak topology) is also generic in* $\mathrm{Homeo}(\mathbb{T}^n, \mathrm{Vol})$ *(with respect to the topology of uniform convergence). More precisely, let* P *be a dense* G_δ *subset of* $\mathrm{Auto}(\mathbb{T}^n, \mathrm{Vol})$ *which is invariant under conjugacies by automorphisms. Then* $P \cap \mathrm{Homeo}(\mathbb{T}^n, \mathrm{Vol})$ *is a dense* G_δ *subset of* $\mathrm{Homeo}(\mathbb{T}^n, \mathrm{Vol})$.

Remember that $\mathrm{Auto}(\mathbb{T}^n, \mathrm{Vol})$ is the set of bijections of $\mathbb{T}^n$ that are bi-measurable and preserves the volume, up to equality on a full volume subset. It is equipped with the *weak topology*, for which a sequence (h_k) converges to h if and only if, for every measurable set A, the sequence $(\mathrm{Vol}(h_k(A)\Delta h(A))$ converges to 0 (where Δ denotes the *symmetric difference*, points that are in one of the sets but not in both). It is also the topology induced by the *weak metric*

$$d_{\mathrm{weak}}(f, g) = \inf\{\lambda | \mathrm{Vol}\left(\{x | d(f(x), g(x)) > \lambda\}\right) < \lambda\}.$$

In other words, (h_k) converges to h if and only if, for every $\varepsilon > 0$, for every n large enough, the set $\{x | d(f(x), g(x)) > \varepsilon\}$ has volume less than ε.[(1)] With this topology the group $\mathrm{Auto}(\mathbb{T}^n, \mathrm{Vol})$ is a topological group. It is also a complete metric space. Note that this topology is weaker than the topology of the (essential) sup metric used in the previous chapter: two maps f and g are uniformly close if $f(x)$ and $g(x)$ are close except on a set of measure 0, whereas f and g are weakly close if $f(x)$ and $g(x)$ are close except on a set of small measure.

We essentially follow the proof in the book by Alpern and Prasad, with a small simplification, namely the use of a tower with only two columns.

(1) It may seem odd to make use of the distance on the manifold in the description of the purely measure-theoretic notion of weak convergence, but this will turn out to be useful! For the equivalence of the two topologies: if $d_{\mathrm{weak}}(h_k, h)$ tends to 0, then $(\mathrm{Vol}(h_k(c)\Delta h(c))$ converges to 0 for every cube. Then the cubes generate the σ-algebra of measurable sets, ant the family of sets A such that $(\mathrm{Vol}(h_k(A)\Delta h(A))$ converges to 0 is a σ-algebra. For the other direction, for a cube c,if $(\mathrm{Vol}(h_k(c)\Delta h(c))$ tends to zero then for n large enough a proportion bigger than $1-\varepsilon$ of the points of $h_k(c)$ belongs to $h(c)$, and if c is small enough, then $h(c)$ has diameter less than ε, thus all these points satisfies $d(h_k(x), h(x)) < \varepsilon$.

3.2. Density of conjugacy classes of aperiodic automorphisms

Proposition. — *Let $f \in \mathrm{Auto}(\mathbb{T}^n, \mu)$. Assume f is aperiodic (*i. e. *the set of periodic points has measure zero). Then the conjugacy class of f in $\mathrm{Auto}(\mathbb{T}^n, \mu)$ is dense in $\mathrm{Homeo}(\mathbb{T}^n, \mu)$ for the uniform topology (given by the sup metric).*

The following is a special case of Alpern's multiple Rokhlin tower theorem (and the proof below is due to Eigen and Prasad).

Lemma. — *Let $f \in \mathrm{Auto}(\mathbb{T}^n, \mathrm{Vol})$. Assume f is aperiodic. Let p, q be two mutually prime integers. Then there exists two sets t_1, t_2 such that the sets*

$$t_1, f(t_1), \dots, f^{p-1}(t_1)$$
$$t_2, f(t_2), \dots, f^{q-1}(t_2)$$

constitutes a partition of $\mathbb{T}^n$. Furthermore, t_1 and t_2 have the same measure.

In other words, the lemma asserts the existence of a *full tower with two columns, of heights p and q*. A *tower* is a (measurable) set t with positive measure, called the *basis* of the tower, equipped with a (measurable) partition, such that the return-time function $\tau_t(x) = \min\{h > 0 \mid S^h(x) \in t\}$ is constant on every set of the partition. Given an element a of this partition, and $h = \tau_t(a)$, the *column over a* is the sequence of sets $(a, S(a), \dots S^{h-1}(a))$; the *height* of the column is the number h; a is called the basis of the column. The tower is *full* if the iterates of t cover $\mathbb{T}^n$. Thus the lemma asserts the existence of a set $t = t_1 \sqcup t_2$ such that the return-time function takes is equal to p on t_1 and to q on t_2, both sets having volume $1/(p+q)$.

Remark. — The arithmetic condition is unavoidable, for example any tower with two columns for an irrational circle rotation R has columns of mutually prime heights. Indeed, we can reorganize a tower t with two columns of height kp and kq into a tower t' with a single column of height k (taking as a basis one level of t out of k). Thus t' is a periodic set of period k, which implies that R^k is not ergodic.

Proof of the lemma. — Let $m = pq$. We begin by applying Rokhlin's lemma.

Lemma. — *There exists a set F such that $\cup_{n\in\mathbb{Z}} f^n(F) = \mathbb{T}^n$, and whose positive iterates $F, f(F), \dots, f^{m-1}(F)$ are disjoint.*

Proof of Rokhlin's lemma. — The proof is easy if we assume f is ergodic : choose some set F' with measure less than $\frac{1}{m}$; then the set $F = F' \setminus (f^{-1}(F') \cup \dots \cup f^{-(m-1)}(F'))$ has positive measure, and is disjoint from its $m-1$ first iterates.

Here is a proof when f is not assumed to be ergodic. First note that the property "$\cup_{n\in\mathbb{Z}} f^n(F) = \mathbb{T}^n$" may be replaced by the weaker property "$\cup_{n\in\mathbb{Z}} f^n(F)$ has measure greater than $1/2$" (then we will get the original property by applying repeatedly the lemma, infinitely many times if necessary). Let $\varepsilon > 0$. Fix some positive integer N, and consider the dyadic subdivision of order N. Let G_N be the set of "good" points x whose iterates $x, f(x), \dots, f^{m-1}(x)$ belongs to distinct cubes of the subdivision. (All along this paragraph, we exclude the points which have some positive iterates in the boundary of some cube of the subdivision; this is a set of measure 0). Since f is aperiodic[(2)], almost every point belongs to G_N for some N, thus for N big enough the set G_N has measure greater than $1 - \varepsilon/m$. Note that for every dyadic

[(2)]Note that we only make use of the fact that the set of points of period less than m has measure 0.

cube C of order N, the set $G_N \cap C$ is disjoint from its m first iterates [(3)] Then we apply the following process, where the dyadic cubes are denoted by $C_1, \dots, C_d$.

$F_0 := \emptyset$
for $i = 0$ to d
$F_i := F_{i-1} \cup ((C_i \cap G) \setminus (f^{-m-1}(F_{i-1}) \cup \cdots \cup f^{m-1}(F_{i-1})))$.

We set $F := F_d$. Note that the family (F_i) is increasing, and that F is disjoint from its m first iterates. Then it can be checked that the set $\cup_{n\in\mathbb{Z}} f^n(F)$ cover the set $G \cap \cdots \cap f^{m-1}(G)$, whose measure is greater than $1 - \varepsilon$. □

Note that the sets $\cup_{n\in\mathbb{Z}} f^n(F)$ and $\cup_{n\in\mathbb{N}} f^n(F)$ coincides up to measure 0. Consider the Kakutani tower over F, that is, partion F according to the return-time function. Every column has height greater than or equal to m. Now any integer $k \geq pq$ may be written $\alpha p + \beta q$ with $\alpha, \beta \geq 0$. Thus we may partitionned each column into subcolumns of height p or q. Let t_1 be the union of the first levels of all the columns of height p, and t_2 be the union of the first levels of all the columns of height q. The tower over $t_1 \sqcup t_2$ suits our needs, except that the basis do not necessarily have the same measure.

Here are the modifications in order to get this additional property. Let α be the smallest of $\frac{p}{p+q}$ and $\frac{q}{p+q}$, and let N be alagre integer (more precisely, we will need $2/(N-1) < \alpha$, see below). We begin with a much higher tower, of height greater than Npq (instead of pq). For every height $h \geq Npq$, we write the euclidean division $h = Apq + B = (A-1)pq + (B + pq)$ (avec $0 \leq B < pq$). At the top of the column we consider the subcolumn of height $B + pq$ and we subdivide it as before into pieces of length p and q. For the time being the remaining part is not attributed. Since $B + pq < 2pq$, the volume ratio of the part that is attributed and the total volume of the column is less than

$$\frac{2pq}{Npq} < \frac{2}{N} < \alpha.$$

Now the remaining part can be subdivided into columns of height pq. Gathering our pieces, we get an intermediary tower with three columns, of respective heights p, q and pq. Note that the first two bases have volume less than $1/(p+q)$. The last column might be subdivided into q columns of height p that will increase the volume of the first column, and then its basis would be too big. Similarly the volume could be entirely attributed to the second column, which would then be too big. Thus there exists a good ratio according to which we may divide vertically our last column into two columns of height pq, the first is divided into q columns of height p added to the first column, the second into p columns of height q added to the second column, so that the basis have exactly the wanted volume $\frac{1}{p+q}$.

□

Proof of the proposition. — Let $h \in \mathrm{Homeo}(\mathbb{T}^n, \mu)$, and $\varepsilon > 0$. We consider a dyadic subdivision of $\mathbb{T}^n$ into cubes; let ε denotes the diameter of the cubes, which may be chosen arbitrarily small. We apply Lax's theorem to find an automorphism $T \in \mathrm{Auto}(\mathbb{T}^n, \mu)$ which is a cyclic permutation of the dyadic cubes, and such that $||T - h|| \leq \varphi(\varepsilon)$.

Fact: there exist two adjacent dyadic cubes $C_1, C_2 = T^n(C_1)$ such that the transition time n is odd.

Let us prove the fact, by contradiction. Let C_1 be any cube, and $C_1' = T(C_1)$. The integers n such that $C_1' = T^n(C_1)$ are odd (since the order of the permutation

(3) For an easy proof using transfinite induction from this point, see Halmos 56.

T is even). Consider a sequence of successively adjacent cubes from C_1 to C_1'. If all the transition times from one cube to the following one are even, then we find an even transition time from C_1 to C_1', which is a contradiction.

We go back to the proof of the proposition. Let σ be the automorphism which permutes two adjacent dyadic cubes with odd transition time (and is the identity everywhere else). Let $T' = \sigma \circ T$. We get $||T' - h|| \leq ||T - h|| + ||T' - T|| \leq \varphi(\varepsilon) + \varepsilon$. Furthermore, the permutation T' decomposes into two disjoint cycles of odd order p and q. Since $p+q$ is the number of cubes, thus a power of 2, p and q are mutually prime. the two adjacent cubes C_1, C_2 belongs to different cycles, to fix idea we assume the order of C_1 is p and the order of P_2 is q.

We now apply the above lemma, and get a tower for the automorphism f, with basis $t_1 \cup t_2$, where t_1 is the basis of a column of height p and t_2 is the basis of a column of height q. Furthermore, the sets t_1 and t_2 have the same mass, namely $\frac{1}{p+q}$, which is also the mass of the dyadic cubes. Let Φ be any automorphism[4] of $(\mathbb{T}^n, \mu)$ sending $f^k(t_1)$ on $T'^k(C_1)$ and $f^\ell(t_2)$ on $T'^\ell(C_2)$, for each $0 \leq k \leq p$ and $0 \leq \ell \leq q$. Let $f' = \Phi f \Phi^{-1}$.

Note that almost all the dyadic cubes of the decomposition has the same image under f' and h. The only exception are the two cubes $T'^{p-1}(C_1)$ and $T'^{q-}(C_2)$: T' send them respectively on C_1 and C_2, whereas f' send both of them into $C_1 \cup C_2$. Since the two cubes are adjacent, we get that for every x, the points $f'(x)$ and $T'(x)$ either belong to the same cube or to adjacent cubes. Thus $||f' - T'|| \leq 2\varepsilon$. Finally $||T' - h|| \leq \varphi(\varepsilon) + 3\varepsilon$.

□

Remark. — . We have implicitly and repeatedly used the following "obvious" result.

Lemma. — *For any measurable sets E, F in $\mathbb{T}^n$ having the same measure, there exists an automorphism Φ such that $\Phi(E) = F$ (mod 0).*

Exercise Write a proof. Solution : use the existence of an ergodic automorphism, and a transfinite induction, see Halmos p74. Does anybody know a proof without transfinite induction??

3.3. Lusin-like theorem: density of homeomorphisms in automorphisms

B_ε denotes the set of g such that $||g|| < \varepsilon$.

Proposition. — *The space* $\mathrm{Homeo}(\mathbb{T}^n, \mathrm{Vol})$ *is dense in the space* $\mathrm{Auto}(\mathbb{T}^n, \mathrm{Vol})$ *for the weak topology. More precisely, for every $\varepsilon > 0$, the space* $B_\varepsilon \cap \mathrm{Homeo}(\mathbb{T}^n, \mathrm{Vol})$ *is dense in the space B_ε.*

Note that, as a consequence, for any $f \in \mathrm{Homeo}$, the space $B_\varepsilon(f) \cap \mathrm{Homeo}(\mathbb{T}^n, \mathrm{Vol})$ is dense in the space $B_\varepsilon(f) = B_\varepsilon \circ f$.

Proof. — Three steps :

– Approximation of automorphisms by permutations,

– (purely combinatorics) Approximation by uniformly small permutations,

– Approximation of (uniformly small) permutations by (uniformly small) homeomorphisms.

We combine the first two steps into a single lemma.

[4] See the remark below.

Lemma. — *Let* $\Phi \in \text{Auto}$. *Then there exists a dyadic permutation* $\sigma \in \text{Auto}$ *arbitrarily close to* Φ *in the weak topology. Furthermore, if* $||\Phi|| < \varepsilon$ *then* σ *may be chosen such that* $||\sigma|| < \varepsilon$.

Proof. — We fix $\delta > 0$ (which says how close we want σ to be to Φ). We consider a dyadic subdivision such that the diameter of the cubes is smaller than δ. For each cube C_i, by regularity of the Lebesgue measure, there exists a compact set K_i included in the inverse image $\Phi^{-1}(C_i)$, whose volume is more than a proportion $1-\delta$ of the volume of C_i. The compact sets K_i's are pairwise disjoint, thus we may find another dyadic subdivision, by cubes so small that no cube meets two different K_i's. Let O_i be the union of the new cubes that meet K_i. If the order of the subdivision is sufficiently large, then for every i the volume of O_i is less than the volume of C_i. We can now find a permutation σ of the new dyadic subdivision that sends each O_i into a subset of C_i (to start with we impose no condition on the image of the cubes that are not in the O_i's). Then for each point x in some K_i, the images $\sigma(x)$ and $\Phi(x)$ belongs to the same cube C_i and so are less than δ apart, and the union of the K_i's cover in volume more than $1 - \delta$ of the torus: in other words we have $d_{\text{weak}}(\sigma, \Phi) < \delta$.

Idea for uniform smallness. — It remains to get σ uniformly small when Φ is supposed uniformly small, so from now on we assume $||\Phi|| < \varepsilon$. We can have chosen δ so small that we still have $||\Phi|| < \varepsilon - 3\delta$, and then a point x in K_i still satisfies $d_{\mathbb{T}^n}(x, \sigma(x)) < \varepsilon - 2\delta$. Observe that because σ is a dyadic permutation, the set of points satisfying this inequality is a union of cubes of the dyadic subdivision permuted by σ, so it includes all the points of the O_i's. We call the remaining cubes the "bad cubes". We will now slightly modify σ into a permutation σ' meeting the condition $||\sigma'|| < \varepsilon$; for this the idea is to decide that σ' will fix all the bad cubes, and try to rearrange the images of the good cubes to fit this decision. Of course the new image must not be too far from the old image if we want to keep the properties $d_{\text{weak}}(\sigma, \Phi) < \delta$ and get $||\sigma'|| < \varepsilon$.

Number of bad cubes. — Until now we have considered two dyadic subdivisions. Let us denote by M be the number of cubes in the first subdivision, and by NM be the number of cubes in the second one. In the previous construction the sets K_i's are chosen after the first dyadic subdivision, so we can require that their union covers a proportion of the torus larger than $1-\frac{1}{M}$. Thus the proportion of bad cubes (in the second subdivision) is less than $\frac{1}{M}$, that is, the number of bad cubes is less than N, which is precisely the number of cubes of the second subdivision contained in any cube of the first.

Numbering. — We now need to number the (big) cubes of the first subdivision, and we do so in order that two consecutive cubes are adjacent. Then we also number the (small) cubes in the second subdivision: we attribute the numbers 1 to N to the N small cubes contained in the first big cube, then the N small cubes in the second big cube are labelled from $N + 1$ to $2N$, and so on. As a consequence, note that *two small cubes whose numbering differs by less than* N *are either included in the same big cube or in two adjacent big cubes*, and in any case their points are a distance less than 2δ apart in the torus.

Modification of σ. — We will now construct σ' as the composition $\tau \circ \sigma$. As we said we impose that σ' fixes the bad cubes, that is, τ sends each σ-image of a bad cube C back to itself. It remains to define $\tau : \mathcal{F} \to \mathcal{F}'$, where $\mathcal{F}$ is the family of the σ-images of the good cubes, and $\mathcal{F}'$ is the family of the good cubes (with the condition that it does not moves the cubes too much). For this we just define τ as the unique order-preserving bijection between $\mathcal{F}$ and $\mathcal{F}'$, where the order is given by the numbering of the small cubes. To see that the cubes are not moved too much by τ, observe that the families $\mathcal{F}$ and $\mathcal{F}'$ contains all the cubes but the (at most N) bad cubes, so that, seeing τ as a permutation of the numbering, we have $|\tau(i) - i| \leq N$ for every i. Which means that every small cube in $\mathcal{F}$ is moved by τ a distance less than 2δ. Thus, according to the above estimate on σ, we get

$$||\sigma'|| \leq \sup\{d(\sigma(x), x) | x \text{ in a good cube}\} + 2\delta < \varepsilon.$$

Finally we note that the good cubes fill a volume $1-\delta$ and a point x in a good cube satisfies $d(\sigma'(x), \Phi(x)) < 3\delta$, so that $d_{\text{weak}}(\sigma', \Phi) < 3\delta$. □

The last lemma says that (small) dyadic permutations may be approximated by (small) homeomorphisms.

Lemma. — *Let $\sigma \in$ Auto be a dyadic permutation. Then there exists $h \in$ Homeo arbitrarily close to σ for the weak topology. Moreover, if $||\sigma|| < \varepsilon$, then h may be chosen such that $||h|| < \varepsilon$.*

Proof. — We fix $\delta > 0$ (which says how close we want h to be to σ). Let $x_1, \ldots, x_k$ be the centers of the dyadic cubes C_i permuted by σ, and $y_1, \ldots, y_k$ be their images under σ. According to the proposition allowing the extension of finite maps, there exists $g \in$ Homeo such that $g(x_i) = y_i = \sigma(x_i)$ for every i. Furthermore, g coincides with σ in some neighborhood of each x_i (see point 1 of the proposition), say in a small cube c_i included in the dyadic cube C_i, with the same center. We assume all the c_i's have the same size. Now choose a homeomorphism $\Psi : C_1 \to C_1$ such that

- on c_1, Ψ is a homothetic transformation, and $\Psi(c_1)$ is a cube that fills a proportion $1-\delta$ of C_1,
- Ψ has constant jacobian on $C_1 \setminus c_1$,
- Ψ fixes every point of the boundary of C_1.

Extend Ψ to a homeomorphism of the whole torus, such that on each cube C_i, Ψ is the conjugate of $\Psi_{|C_1}$ by the translation that takes C_1 to C_i. Consider the map $h = \Psi g \Psi^{-1}$. Since Ψ has constant jacobian outside the union of the c_i's, h preserves the volume, and thus belongs to Homeo$(\mathbb{T}^n, \text{Vol})$. Since the conjugate of a translation under a homothetic transformation is a translation, h coincides with σ on each cube $\Psi(c_i)$. The union of these cubes has measure greater than $1-\delta$, thus for the weak distance we certainly have $d(h, \sigma) < \delta$.

It remains to check that the construction can be done so that $||h|| < \varepsilon$. If $||\sigma|| < \varepsilon' < \varepsilon$, then the second point of the extension proposition says that we can choose g such that $||g|| < \varepsilon'$. Furthermore $||\Psi||$ and $||\Psi^{-1}||$ are less than the diameter of the dyadic cubes, which may be assumed to be arbitrarily small (less than $(\varepsilon - \varepsilon')/2$), up to considering a finer dyadic subdivision right from the start. Then we get that $||h|| < ||g|| + ||\Psi|| + ||\Psi^{-1}||$ is less than ε. □

Of course, the combination of both lemmas gives the proposition. □

3.4. Proof of the genericity theorem

Let P be a dense G_δ subset in Auto, as in the statement of the theorem. We first note that P contains an aperiodic automorphism. Indeed:

Lemma. — *The set of aperiodic automorphisms is a G_δ-dense subset of* Auto$(\mathbb{T}^n, \text{Vol})$.

Proof. — Let A_n be the set of automorphisms Φ such that the set of fixed points of Φ^n has measure less than $1/n$. Clearly, the set of aperiodic automorphisms is the intersection of the A_n's.

Each A_n is open. Indeed, given $\Phi \in A_n$, the Rokhlin lemma provides a set E such that $E \cap \Phi^n(E) = \emptyset$ and $E \cup \Phi^n(E)$ has measure greater than $1 - 1/n$ (and the measure of E is half of this). If Ψ is close to Φ in the weak topology, then the set $E' = E \setminus \Psi^n(E)$ has still measure greater than $(1 - 1/n)/2$, and is disjoint from its image under Ψ^n, which proves that $\Psi \in A_n$.

Each A_n is dense: we prove that aperiodic automorphisms are dense. Let Ψ be any automorphism. Let E_1 be the fixed point set of Ψ. Subdivide E into small subsets, and replace Ψ on each subset by a fixed point free automorphism. Thus we get Φ, close to Ψ in the weak topology (or even such that $||\Phi - \Psi||$ is small), with no periodic point. We apply Rokhlin's lemma for the set of periodic points of period 2, getting a set E_2 such that $E_2 \cap \Psi(E_2) = \emptyset$, and any period-2 point is in $E_2 \cap \Psi(E_2) = \emptyset$. Again on E_2 we compose Ψ by a small automorphism with no periodic point. Proceeding this way successively with all the periods, we get an aperiodic automorphism close to Ψ. □

By density of conjugacy classes of aperiodic automorphisms, P is uniformly dense in Homeo. Now write $P = \cap P_k$ with each P_k an open and dense subset of Auto. If a sequence of homeomorphisms converges uniformly then it certainly converges for the weak topology, thus the set $P_k \cap$ Homeo is open in Homeo for the uniform topology. We want to prove that $P_k \cap$ Homeo is dense in Homeo. Let $\varepsilon > 0$ and $f \in$ Homeo. Since P_k is dense for the uniform topology, $B_\varepsilon(f) \cap P_k$ contains some automorphism g. We apply Lusin-like theorem: the set $B_\varepsilon(f) \cap$ Homeo is (weakly) dense in $B_\varepsilon(f)$, thus we can approximate weakly g by a homeomorphism g' in $B_\varepsilon(f) \cap P_k$, as wanted. This completes the proof.

3.5. Example one: strong mixing

Now we can reap what we sow: we prove that some ergodic property is generic in Auto, and we conclude that it is also generic in Homeo.

History: Halmos, 1944, *In general a measure-preserving transformation is mixing*; Rokhlin, 1948, *In general a measure-preserving transformation is not mixing.* Concerning homeomorphisms, Oxtoby-Ulam proved the genericity of ergodicity, then Katok Steppin proved weak-mixing, then Alpern proved the general theorem that provides the link between Halmos and Katok-Steppin theorems.

An automorphism $T \in \mathrm{Auto}(\mathbb{T}^n, \mathrm{Vol})$ is said to be *strongly mixing* if for every borelian sets A, B, the volume of $T^{-n}(A) \cap B$ converges to the product $\mathrm{Vol}(A)\mathrm{Vol}(B)$.

Proposition. — *Generically in* Auto *(and in* Homeo*), the elements are not strongly mixing.*

Proof. — We follow Halmos explaining Rokhlin proof. We have seen that dyadic permutation are dense in $\mathrm{Auto}(\mathbb{T}^n, \mathrm{Vol})$ for the weak topology. Using the technique of the proof of Lax's theorem, we even get the density of cyclic dyadic permutations of arbitrarily high order: for every n, the set

$$\cup_{k \geq n} P_k$$

is dense, where P_k is the set of cyclic dyadic permutation T such that $T^k = \mathrm{Id}$.

Let A be a dyadic cube, say $A = [0, 1/2]^n$. Let M_k be the set of automorphisms T such that the measure of $T^k(A) \cap A$ belongs to $[1/2^{2n} \pm 1/2^{4n}]$. It is a closed set for the weak topology. Furthermore the set

$$\cup_{n\geq 0} \cap_{k\geq n} M_k$$

contains all the strongly mixing automorphisms.

Now clearly P_k is disjoint from M_k. Thus $\cup_{k\geq n} P_k$ is disjoint from $\cap_{k\geq n} M_k$. The first set is dense, thus the second one has empty interior (is nowhere dense). Finally the strong mixing automorphisms are included in a countable union of nowhere dense closed sets. □

3.6. Example two: weak mixing

Here we follow Halmos. There are several definitions of weak mixing. Equivalently:

1. for every borelian sets A, B, the sequence $|\mathrm{Vol}(T^{-n}(A) \cap B - \mathrm{Vol}(A)\mathrm{Vol}(B)|$ tends to 0 in the sense of Cesaro, or equivalently outside a subset of the integers of density 0;
2. the cartesian square $T \times T$ is ergodic;
3. the only proper vectors for the unitary L^2 operator $f \mapsto f \circ T$ are the constant functions, in other words the only eigenvalue of this operator is 1, and it is simple (this is called *continuous spectrum*).

Proposition. — *Generically in* Auto *(and in* Homeo*), the elements are not strongly mixing.*

Proof. — We use the existence of at least one weakly mixing automorphism (Anosov linear automorphisms are strongly mixing). Since such a transformation is clearly aperiodic, the set of weakly mixing elements is dense. Thus it is enough to prove that it is a G_δ set.

Let (f_i) be a countable dense family in $L^2(\mathbb{T}^n)$. Then an element $T \in \mathrm{Auto}(\mathbb{T}^n, \mathrm{Vol})$ is weakly-mixing if and only if $\liminf |\int f_i \circ T^n f_j - \int f_i \int f_j| = 0$ for every i and j, that is,

$$\forall i, j \forall k \exists n \left| \int f_i \circ T^n f_j - \int f_i \int f_j \right| < \frac{1}{k}.$$

This gives an expression of the set of weakly mixing elements as a G_δ subset of $\mathrm{Auto}(\mathbb{T}^n, \mathrm{Vol})$.

How to prove the above equivalence? The characterization in terms of convergence outside a set of density 0 gives the direct implication. Now if T is not weakly mixing, then there exists an eigenfunction $f \in L^2$ and a constant c such that $f \circ T = cf$. By density of our family there exists f_i very close to f. Then one can see that T does not satisfy the property

$$\exists n \left| \int f_i \circ T^n f_i - \int f_i \int f_i \right| < \frac{1}{2}.$$

Thus T does not satisfy the above criterion. □

CHAPTER 4

REALIZATION OF ERGODIC SYSTEMS ON THE TWO-TORUS

This chapter is extracted from my paper with François Béguin and Sylvain Crovisier, *Realisation of measured dynamics as uniquely ergodic minimal homeomorphisms on manifolds* which you can find on ArXiv or Hal.

4.1. General outline

A natural problem in dynamical systems is to determine which measured dynamics admit topological or smooth realisations. Results in this direction include:

– constructions of smooth diffeomorphisms on manifolds satisfying some specific ergodic properties,

– general results about topological realisations on Cantor sets (Jewett-Krieger theorem and its generalisations).

In this paper, we tackle the following question: *given a manifold $\mathcal{M}$, which measured dynamical systems can be realised as uniquely ergodic minimal homeomorphisms on $\mathcal{M}$?* Our main result asserts that this class of dynamical systems is stable under extension.

Theorem. — *Let $\mathcal{M}$ be a compact topological manifold of dimension at least two. Assume we are given*

– *a uniquely ergodic minimal homeomorphism F on $\mathcal{M}$, with invariant measure m;*

– *an invertible ergodic dynamical system (Y, ν, S) on a standard Borel space, which is an extension of $(\mathcal{M}, m, F)$.*

Then there exists a uniquely ergodic minimal homeomorphism G on $\mathcal{M}$, with invariant measure v, such that the measured dynamical system $(\mathcal{M}, v, G)$ is isomorphic to (Y, ν, S).

Let us recall the classical definitions. A *measured dynamical system* is given by (X, μ, R) where (X, μ) is a probability space and $R : X \to X$ is a bi-measurable bijective map for which μ is an invariant measure. Given two such systems (X, μ, R) and (Y, ν, S), the second is an *extension* of the first through a measurable map $\Phi : Y_0 \to X_0$ if X_0, Y_0 are full measure subsets of X, Y, the map Φ sends the measure ν to the measure μ, and the conjugacy relation $\Phi S = R\Phi$ holds on Y_0. If, in addition, the map Φ is bijective and bi-measurable, then the systems are *isomorphic*. A measurable space $(X, \mathcal{A})$ is called a *standard Borel space* if $\mathcal{A}$ is the Borel σ-algebra of some topology on X for which X is a Polish space (*i.e.* a metrizable complete separable space). Throughout the text all the topological spaces will be implicitly equipped with their Borel σ-algebra, in particular all the measures on topological spaces are Borel measures. Note that if some measured dynamical system (Y, ν, S) satisfies the conclusion of theorem 4.1, then it is obviously isomorphic to a dynamical system on a standard Borel space, thus it is reasonable to restrict ourselves to such systems. We say that a homeomorphism F on a topological space $\mathcal{M}$, with an invariant measure m, is a *realisation* of a measured dynamical system (X, μ, R) if the measured dynamical system $(\mathcal{M}, m, F)$ is isomorphic to (X, μ, R).

Independently of theorem 4.1 we will prove the following result.

Theorem. — *Any irrational rotation of the circle admits a uniquely ergodic minimal realisation on the two-torus.*

The proof of this result relies on a rather classical construction. The realisation is a skew-product $F(x, y) = (x + \alpha, A(x).y)$ where $A : \mathbb{S}^1 \to \mathrm{SL}(2, \mathbb{R})$ is a continuous map. It will be obtained as the limit of homeomorphisms conjugated to the "trivial" homeomorphism $(x, y) \mapsto (x + \alpha, y)$.

Theorems 4.1 and 4.1 can be associated to provide a partial answer to the realisation problem on the two-torus. Remember that a measured dynamical system (X, μ, R) is an extension of some irrational circle rotation if and only if the spectrum of the associated operator on $L^2(X, \mu)$ has an irrational eigenvalue. Therefore, as an immediate consequence of theorem 4.1 and 4.1, we get the following corollary.

corollary. — *If an ergodic measured dynamical system on a standard Borel space has an irrational eigenvalue in its spectrum, then it admits a uniquely ergodic minimal realisation on the two-torus.*

This result is known not to be optimal: indeed there exist uniquely ergodic minimal homeomorphisms of the two-torus that are weakly mixing.Actually, it might turn out that *every* aperiodic ergodic system admits a uniquely ergodic minimal realisation on the two-torus; at least no obstruction is known to the authors. For instance, we are not able to answer the following test questions:

– does an adding machine admit a uniquely ergodic minimal realisation on the two-torus?
– what about a Bernoulli shift?

Our original motivation for studying realisation problems was to generalise Denjoy counter-examples in higher dimensions, that is, to construct (interesting) examples of homeomorphisms of the n-torus that are topologically semi-conjugate to an irrational rotation. The proof of the first theorem actually provides a topological semi-conjugacy between the maps G and F.Thus another corollary of our results is: *any ergodic system which is an extension of an irrational rotation R of the two-torus can be realised as a uniquely ergodic minimal homeomorphism which is topologically semi-conjugate to R.* For more comments on realisation problems and generalisations of Denjoy counter-examples, we refer to the introduction of our previous work (Construction of curious minimal uniquely ergodic homeomorphisms on manifolds: the Denjoy-Rees tech- nique. *Ann. Sci. École Norm. Sup.* **40** (2007), 251–308). The reader interested in the smooth version can consult the paper by Bassam Fayad and Anatole Katok, (Constructions in elliptic dynamics. *Ergod. Th. Dyn. Sys.* **24** (2004), 1477–1520) and especially the last section.

4.2. Realization of circle rotations on the torus

In this section, we prove the easy part of the announced results, namely the existence of uniquely ergodic minimal realisations of circle rotations on the two-torus. The construction has some additional properties that requires the following definition. If $\varphi : \mathbb{S}^1 \to \mathbb{S}^1$ is a measurable function, then the set $\{(x, \varphi(x)), x \in \mathbb{S}^1\}$ is called a *measurable graph.* If φ is continuous then this set is called a *continuous graph.* Remember that the group $\mathrm{SL}(2,\mathbb{R})$ acts projectively on the circle. We will prove the following statement.

Theorem. — *For every $\alpha \in \mathbb{R}\setminus\mathbb{Q}$, there exists a continuous map $A : \mathbb{S}^1 \to \mathrm{SL}(2,\mathbb{R})$ homotopic to a constant such that the skew-product homeomorphism $f : \mathbb{T}^2 \to \mathbb{T}^2$ defined by $f(x,y) = (x+\alpha, A(x).y)$ has the following properties:*
- *f is minimal;*
- *f is uniquely ergodic and the invariant measure μ is supported on a measurable graph.*

If f is a map given by theorem 4.2 then the first coordinate projection $\pi_1 : \mathbb{T}^2 \to \mathbb{S}^1$ induces an isomorphism between $(\mathbb{T}^2, \mu, f)$ and $(\mathbb{S}^1, \mathrm{Leb}, R_\alpha)$, where R_α is the rotation $x \mapsto x + \alpha$. Therefore, the realization of circle irrational rotations on the torus will follow from theorem 4.2. The core of the proof of theorem 4.2 is the following technical lemma.

Lemma. — *For every $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, and every $\varepsilon > 0$, there exists a homeomorphism $g : \mathbb{T}^2 \to \mathbb{T}^2$ with the following properties.*
1. *There exists a continuous map $m : \mathbb{S}^1 \to \mathrm{SL}(2,\mathbb{R})$ homotopic to a constant, such that*
$$g = M^{-1} \circ (R_\alpha \times \mathrm{Id}) \circ M$$
where $M(x,y) = (x, m(x).y)$. In particular, the homeomorphism g is a skew-product over the circle rotation R_α, and is conjugated to $R_\alpha \times Id$.
2. *The homeomorphism g is ε-close to $R_\alpha \times \mathrm{Id}$ in the C^0-topology.*
3. *Every g-invariant continuous graph C is ε-dense in $\mathbb{T}^2$.*
4. *There exists a horizontal open strip $\Gamma = \{(x,y) \in \mathbb{T}^2 \mid y \in V_x\}$ of width ε (by such we mean that V_x is an interval of length ε depending continuously on x) such that, for every g-invariant continuous graph $C = \{(x, \varphi(x)) \mid x \in \mathbb{S}^1\}$, one has*
$$Leb\,(\pi_1(C \cap \Gamma)) = Leb\left(\{x \in \mathbb{S}^1 \mid \varphi(x) \in V_x\}\right) \geq 1 - \varepsilon.$$

Proof of the lemma. — The following fact, which is a refinement of the classical Rokhlin lemma, will be applied to the circle rotation R_α.

Fact. — *Let us consider a compact manifold X, a homeomorphism h of X and a h-invariant probability measure μ which has no atom. Let N, k be any positive integers and ε be any positive real number. Then there exist some compact sets $D_1, \dots, D_k \subset X$ such that:*
1. *the sets $h^i(D_j)$ for $0 \leq i \leq N$ and $1 \leq j \leq k$ are pairwise disjoint;*
2. *the union of the $h^i(D_j)$ for $0 \leq i \leq N$ and $1 \leq j \leq k$ has μ-measure larger than $1 - \varepsilon$;*
3. *for any $j_0 \in \{1, \dots, k\}$, the μ-measure of the union $\bigcup_{i=0}^N h^i(D_{j_0})$ is smaller than k^{-1}.*

Proof. — Rokhlin lemma ensures the existence of a measurable set $A \subset X$ such that the iterates $h^i(A)$ for $0 \le i \le N$ are pairwise disjoint and their union has μ-measure larger than $1-\varepsilon$. Then, using the fact that the measure μ is regular, we can find a compact set $B \subset A$ such that these properties are is still satisfied when we replace A by B. One can assume that B is the union of disjoint compact sets with arbitrarily small measure: this allows to decompose B as a disjoint union $D_1 \cup \cdots \cup D_k$ of compact sets whose μ-measure is close to $\mu(B)/k$. These sets satisfy the conclusion of the fact. □

Let us come to the proof of the lemma. We see the torus $\mathbb{T}^2$ as the product of two copies of $\mathbb{S}^1$. For sake of clarity, we shall distinguish between these two copies, denoting them respectively by $\mathbb{S}^1_h$ and $\mathbb{S}^1_v$ (where "h" and "v" stand for "'horizontal" and "vertical"). The rotation R_α acts on $\mathbb{S}^1_h$, whereas the elements of $\mathrm{SL}(2,\mathbb{R})$ defined below act on $\mathbb{S}^1_v$.

Construction of the homeomorphism g.— According to the first item of the lemma we have to construct a continuous map $m : \mathbb{S}^1_h \to \mathrm{SL}(2,\mathbb{R})$ homotopic to a constant. We proceed step by step.

- We first choose a finite collection of intervals $A_1, \dots, A_k \subset \mathbb{S}^1_v$ such that
  - for every j, the length of the interval A_j is less than $\varepsilon/2$;
  - the union $A_1 \cup \cdots \cup A_k$ covers the whole circle $\mathbb{S}^1_v$.

  Note that in particular one has $k\varepsilon/2 \ge 1$.
- We choose some pairwise disjoint intervals $R_1, \dots, R_k \subset \mathbb{S}^1_v$ such that, for each $j \in \{1, \dots, k\}$, the interval R_j is disjoint from the interval A_j.
- For each $j \in \{1, \dots, k\}$, we choose a hyperbolic map $S_j \in \mathrm{SL}(2,\mathbb{R})$ whose attractive fixed point is included in the interior of A_j and whose repulsive fixed point is included in the interior of R_j, and which is ε-close to the identity of $\mathbb{S}^1_v$ (for the C^0 topology). Note that S_j maps A_j into itself. Then we choose an integer ℓ_j such that $S_j^{\ell_j}$ maps $\mathbb{S}^1_v \setminus R_j$ into A_j. We set $\ell := \max_{j \in \{1 \dots k\}} \ell_j$. So, for every j, the homeomorphism S_j^ℓ maps $\mathbb{S}^1_v \setminus R_j$ into A_j.
- We choose an integer $N > 8\ell/\varepsilon$ large enough so that for every $x \in \mathbb{S}^1_h$, the orbit segment $x, R_\alpha(x), \dots, R_\alpha^{N-2\ell}(x)$ is $(\varepsilon/2)$-dense in $\mathbb{S}^1_h$.
- We choose by the above fact a finite number of compact sets $D_1, \dots, D_k \subset \mathbb{S}^1_h$ such that:
  - the sets $R_\alpha^i(D_j)$ for $j = 1, \dots, k$ and $i = 0, \dots, N$ are pairwise disjoint;
  - the Lebesgue measure of the union $\bigcup_{j=1}^k \bigcup_{i=0}^N R_\alpha^i(D_j)$ is larger than $1 - \varepsilon/4$;
  - for each $j_0 \in \{1, \dots, k\}$, the measure of $\bigcup_{i=0}^N R_\alpha^i(D_{j_0})$ is smaller than $k^{-1} \le \varepsilon/2$.
- For $j \in \{1, \dots, k\}$ and $i \in \{0, \dots, N\}$ we define the map m on $R_\alpha^i(D_j)$ by

$$m := \begin{cases} S_j^i & \text{if } 0 \le i \le \ell \\ S_j^\ell & \text{if } \ell \le i \le N-\ell \\ S_j^{N-i} & \text{if } N-\ell \le i \le N. \end{cases}$$

  Note that $m(x+\alpha)^{-1}m(x)$ is ε-close to the identity for every $x \in \bigcup_{j=1}^k \bigcup_{i=0}^{N-1} R_\alpha^i(D_j)$. Now, we extend continuously the map m on $\mathbb{S}^1_h$ with the constraints that m is homotopic to a constant and $m(x+\alpha)^{-1}m(x)$ is ε-close to the identity for every $x \in \mathbb{S}^1_h$. This can be done as follow. For each $j \in \{1, \dots, k\}$, we choose a small neighbourhood U_j of D_j and a continuous map $\varphi \colon U_j \to SL(2,\mathbb{R})$, ε-close to the identity of $\mathbb{S}^1_v$, equal to S_j on D_j, and that coincides with the identity on the boundary of U_j. Then we set for $x \in R_\alpha^i(U_j)$,

$$m(x) := \begin{cases} (\varphi(R_\alpha^{-i}(x)))^i & \text{if } 0 \le i \le \ell \\ (\varphi(R_\alpha^{-i}(x)))^\ell & \text{if } \ell \le i \le N-\ell \\ (\varphi(R_\alpha^{-i}(x)))^{N-i} & \text{if } N-\ell \le i \le N. \end{cases}$$

If the neighbourhoods U_j are chosen small enough then the sets $R_\alpha^i(U_j)$ for $0 \le i \le N, 1 \le j \le k$ are pairwise disjoint and the above formulae do make sense. For the points x that do not belong to one of these sets, $m(x)$ is defined to be the identity map of $\mathbb{S}_v^1$.

– We choose an open interval $V_x \subset \mathbb{S}_v^1$ of length ε which depends continuously on $x \in \mathbb{S}_h^1$ and such that for every $1 \le j \le k$ it contains A_j whenever x belongs to $R_\alpha^i(D_j)$ for some $0 \le i \le N$. Then we consider the horizontal strip $\Gamma := \{(x,y) \mid y \in V_x\}$.

Properties of the homeomorphism g.— Let us check that the maps g and M associated to m as in the statement of the lemma satisfy the required properties:

1. The first item of the lemma is a consequence of the definition of g.
2. Since the map $x \mapsto m(x+\alpha)^{-1}m(x)$ is ε-close to the identity map of the circle $\mathbb{S}_v^1$, the homeomorphism $g : (x,y) \mapsto (x+\alpha\ ,\ m(x+\alpha)^{-1}m(x).y)$ is ε-close to $R_\alpha \times \mathrm{Id}$.
3. Let C be a g-invariant continuous graph. Then $M(C)$ is a $(R_\alpha \times \mathrm{Id})$-invariant continuous graph. Hence there exists a point $y \in \mathbb{S}_v^1$ such that $C = M(\mathbb{S}_h^1 \times \{y\})$. Let $j \in \{1,\dots,k\}$ be an integer such that $y \notin R_j$. We claim that C is $\varepsilon/2$-dense in $\mathbb{S}_h^1 \times A_j$. Indeed consider a point x in D_j. On the one hand, for every $i \in \mathbb{Z}$, the point $M(R_\alpha^i(x),y)$ belongs to the graph C. On the other hand, for $i \in \{\ell,\dots,N-\ell\}$, by our choice of x we have
$$M\left(R_\alpha^i(x),y\right) = \left(R_\alpha^i(x), m\left(R_\alpha^i(x)\right).y\right) = \left(R_\alpha^i(x), S_j^\ell.y\right)$$
and this point belongs to $\{R_\alpha^i(x)\} \times A_j$ by our choice of S_j and since $y \notin R_j$. Now remember that the length of the interval A_j is less than $\varepsilon/2$ and that the integer N was chosen in such a way that the sequence $R_\alpha^\ell(x),\dots,R_\alpha^{N-\ell}(x)$ is $\varepsilon/2$-dense in $\mathbb{S}^1$. This shows the claim.
   Since the intervals R_j are pairwise disjoint, there is at most one integer $j_0 \in \{1,\dots,k\}$ such that $y \in R_{j_0}$. By construction the union $\bigcup_{j\neq j_0} A_j$ is $\varepsilon/2$-dense in $\mathbb{S}_v^1$. Therefore the graph C is ε-dense in $\mathbb{T}^2 = \mathbb{S}_h^1 \times \mathbb{S}_v^1$.
4. Let us once again consider a g-invariant continuous graph $C = M(\mathbb{S}_h^1 \times \{y\})$. As above for every $i \in \{\ell,\dots,N-\ell\}$ and $j \in \{1,\dots,k\}$ such that $y \notin R_j$, the point $M\left(R_\alpha^i(x),y\right)$ belongs to the set $\{R_\alpha^i(x)\} \times A_j$ which is included in the strip Γ. There is at most one $j_0 \in \{1,\dots,k\}$ such that $y \in R_{j_0}$. Hence,
$$\pi_1(C \cap \Gamma) \supset \bigcup_{j\notin j_0} \bigcup_{i=\ell}^{N-\ell} R_\alpha^i(D_j).$$
   As a consequence, we get
$$\begin{aligned} \mathrm{Leb}\left(\pi_1(C\cap\Gamma)\right) &\ge \frac{N-2\ell}{N}\left(\mathrm{Leb}\left(\bigcup_{j=1}^{k}\bigcup_{i=1}^{N} R_\alpha^i(D_j)\right) - \mathrm{Leb}\left(\bigcup_{i=1}^{N} R_\alpha^i(D_{j_0})\right)\right) \\ &\ge (1-\varepsilon/4)(1-\varepsilon/4-\varepsilon/2) \ge 1-\varepsilon \end{aligned}$$
   (the second inequality follows from the definition of the intervals $D_1,\dots,D_k$ and the integer N).

This completes the proof of the lemma.

Proof of the theorem. — We will use the lemma to construct inductively a sequence of homeomorphisms $(M_k)_{k\geq 0}$. This will give rise to the sequences $(\Phi_k)_{k\geq 0}$ and $(f_k)_{k\geq 0}$ defined by

$$\Phi_k = M_0 \circ \cdots \circ M_k \text{ and } f_k = \Phi_k \circ (R_\alpha \times \mathrm{Id}) \circ \Phi_k^{-1}.$$

We set $M_0 = \mathrm{Id}$. Assuming that the sequence $M_0, \ldots, M_k$ has been constructed, we consider a small positive number ε_{k+1} (the conditions on ε_{k+1} will be detailed below) and we apply the lemma to $\varepsilon = \varepsilon_{k+1}$. The lemma provides the maps g_{k+1} and M_{k+1}, and we have

$$f_{k+1} = \Phi_k \circ M_{k+1} \circ (R_\alpha \times \mathrm{Id}) \circ M_{k+1}^{-1} \circ \Phi_k^{-1} = \Phi_k \circ g_{k+1} \circ \Phi_k^{-1}.$$

By the second item of the lemma the homeomorphisms g_{k+1} and $R_\alpha \times \mathrm{Id}$ are ε_{k+1}-close, so if ε_{k+1} has been chosen small enough, then f_{k+1} is 2^{-k}-close to $f_k = \Phi_k \circ (R_\alpha \times \mathrm{Id}) \circ \Phi_k^{-1}$. One can thus assume that the sequence (f_k) is a Cauchy sequence. It converges to a homeomorphism f of the two-torus, which will be a skew-product of the form required by the theorem (because of the first item of the lemma and since $SL(2,\mathbb{R})$ is closed in the space of circle homeomorphisms). It remains to check that, as soon as the sequence (ε_k) decreases sufficiently fast, the map f satisfies the conclusions of the theorem.

Let us address the minimality. By item 3 of the lemma, at step k every g_k-invariant continuous graph is ε_k-dense in $\mathbb{T}^2$. Consequently we can choose ε_k small enough so that every f_k-invariant graph is $1/k$-dense in $\mathbb{T}^2$. Since f_k is conjugate to $R_\alpha \times \mathrm{Id}$, every orbit is dense in an invariant circle, and there exists a positive integer N_k such that every piece of orbit $p, f_k(p), \ldots, f^{N_k}(p)$ is again $1/k$-dense in $\mathbb{T}^2$. This last property is open: the sequence $(\varepsilon_\ell)_{\ell>k}$ can be chosen so that this property is shared by the limit map f, namely, every piece of f-orbit of length N_k is $1/k$-dense. This entails the minimality of f.

We now turn to the ergodic properties. Let Γ be the horizontal strip given by item 4 of the lemma when applied at step k and let $\Gamma_k := \Phi_{k-1}(\Gamma)$. Then Γ_k is a horizontal strip: the intersection of Γ_k with any circle $\{x\} \times \mathbb{S}^1_v$ is an interval, whose length is less than $1/k$ if ε_k is small enough. Furthermore, for every f_k-invariant continuous graph C, the set $\pi_1(C \cap \Gamma_k)$ is an open subset of the circle whose Lebesgue measure is bigger than $1 - \varepsilon_k$. Then the unique ergodicity of the rotation R_α entails the following fact.

Fact. — *There exists a positive integer N_k with the following property: every f_k-orbit of length N_k spends more than a ratio $(1 - \varepsilon_k)$ of its time within the strip Γ_k, in other words for every point p,*

$$\mathrm{Card}(\{0 \leq n < N_k, f_k^n(p) \in \Gamma_k\}) > (1 - \varepsilon_k)N_k.$$

Since Γ_k is open, every point p has a neighbourhood V_p such that this inequality remains true when we replace p by any point $p' \in V_p$ and f_k by any map which is ε_p-close to f_k (for some positive ε_p). Thus, by compactness, the property expressed in the fact is open, and shared by the map f as soon as the sequence $(\varepsilon_\ell)_{\ell>k}$ tends to 0 fast enough. As a consequence, for any f-invariant measure μ one has

$$\mu(\Gamma_k) \geq 1 - \varepsilon_k.$$

For any positive integer k_0 we set

$$\mathcal{C}_{k_0} = \bigcap_{i\geq k_0} \Gamma_i \qquad \text{and } \mathcal{C} = \bigcup_{k_0 \geq 0} \mathcal{C}_{k_0}.$$

For any f-invariant measure μ, the measure of $\mathcal{C}_{k_0}$ is bounded from below by $1 - \sum_{k \geq k_0} \varepsilon_k$ and is positive if the sequence (ε_k) goes to 0 fast enough; hence $\mathcal{C}$ has measure 1. Remember also that Γ_k is a strip of thickness less than $1/k$. Thus the intersection of $\mathcal{C}$ with any vertical circle is empty or reduced to a point: $\mathcal{C}$ is a measurable graph over a set of full Lebesgue measure. The unique ergodicity of the rotation R_α implies that f is also uniquely ergodic, the only invariant measure being the measure μ defined by the formula

$$\mu(E) = \mathrm{Leb}(\pi_1(E \cap \mathcal{C})).$$

This completes the proof of the theorem.