Slogan: 'Every topos thinks that it is a category of sets'

Evidence for this: we can carry out set-theoretic constructions (constructively) in the internal logic of a topos.

in order to 'see' the non-set-theoretic aspects of a topos $\mathcal{E}$ (e.g. topological properties, being a topos of cohesive objects, etc.), we have to look 'outside' it, at its relationship with another topos $\mathcal{S}$ of 'abstract sets'.

However, toposes are really type theories, not set theories: we have only bounded quantifiers $(\forall x : A)\varphi$, $(\exists x : A)\varphi$ available, and can't 'quantify over the whole topos'.

So the first attempts to build models of set theory from (suitable) toposes (J. Cole, W. Mitchell, G. Osius, early 1970s) produced not models of ZF but models of 'weak Zermelo set theory' with only Δ_0 Separation (i.e. $\{x \in a \mid \varphi(x)\}$ exists if all quantifiers in φ are bounded) and no scheme of Replacement / Collection.

M. Fourman (1979) observed that if we start from a topos Set that was built from a model of ZF, and then construct Grothendieck toposes over it, we can give (not a model but) an interpretation of IZF (i.e. ZF with intuitionistic logic) in any such topos $\mathcal{E}$.

We can then use these interpretations to derive independence results in (I)ZF from topos-theoretic constructions.

Basic idea: construct the von Neumann hierarchy $(V_\alpha \mid \alpha \in On)$ inside $\mathcal{E}$.

We have a sequence of objects V_α and extensional relations $\epsilon_\alpha \rightarrowtail V_\alpha \times V_\alpha$ (extensional means that the 'twisted transpose' $V_\alpha \longrightarrow PV_\alpha$ of ϵ_α is monic), and for each $\alpha \leq \beta$ a mono $V_\alpha \rightarrowtail V_\beta$ such that

$$\begin{tikzcd} \epsilon_\alpha \arrow[r, tail] \arrow[d] & V_\alpha \times V_\alpha \arrow[d, tail] \\ \epsilon_\beta \arrow[r, tail] & V_\beta \times V_\beta \end{tikzcd}$$

is a pullback.

- $V_0 = \epsilon_0 = 0$.
- $V_{\alpha+1} = PV_\alpha$ and $V_\alpha \rightarrowtail V_{\alpha+1}$ is the twisted transpose of ϵ_α.

  $\epsilon_{\alpha+1}$ is the relation $\epsilon_{V_\alpha} \rightarrowtail V_\alpha \times PV_\alpha \rightarrowtail PV_\alpha \times PV_\alpha$.

  (so the squares
  $$\begin{tikzcd} \epsilon_\alpha \arrow[r] \arrow[d, tail] & \epsilon_{V_\alpha} \arrow[r] \arrow[d, tail] & \epsilon_{\alpha+1} \arrow[d, tail] \\ V_\alpha \times V_\alpha \arrow[r] & V_\alpha \times PV_\alpha \arrow[r] & PV_\alpha \times PV_\alpha \end{tikzcd}$$
  are pullbacks, and the twisted transpose of $\epsilon_{\alpha+1}$ is
  $$V_{\alpha+1} \xrightarrow{1} PV_\alpha \rightarrowtail PV_{\alpha+1})$$
- at nonzero limits λ, we define $V_\lambda = \varinjlim_{\alpha<\lambda} V_\alpha$ and $\epsilon_\lambda = \varinjlim_{\alpha<\lambda} \epsilon_\alpha$, and use the fact that filtered colimits commute with finite limits.

Given a formula φ in the language of set theory with free variables in the list $x_1, x_2, \dots x_n$, we define its interpretation in the <u>decorated context</u> $x_1^{\alpha_1}, x_2^{\alpha_2}, \dots x_n^{\alpha_n}$ (where the α_i are ordinals) as follows:

- atomic formulae $(x^\alpha = y^\alpha)$, $(x^\alpha \in y^\alpha)$ interpreted using $V_\alpha \xrightarrow{\Delta} V_\alpha \times V_\alpha$, $\epsilon_\alpha \rightarrowtail V_\alpha \times V_\alpha$. If decorations don't match, pull these subobjects back along appropriate $V_\alpha \times V_\beta \rightarrowtail V_\beta \times V_\beta$ (etc.)
- extend to quantifier-free formulae using usual Heyting algebra operations on subobjects.

Note: if $\alpha_i \leq \beta_i$ for all i, then we have a pullback square
$$\begin{tikzcd} [\![\vec{x}^{\vec{\alpha}} . \varphi]\!] \arrow[r, tail] \arrow[d, tail] & V_{\alpha_1} \times \cdots \times V_{\alpha_n} \arrow[d, tail] \\ [\![\vec{x}^{\vec{\beta}} . \varphi]\!] \arrow[r, tail] & V_{\beta_1} \times \cdots \times V_{\beta_n} \end{tikzcd}$$
for any quantifier-free φ.

Now suppose $\varphi = (\exists y)\,\psi$ where ψ is quantifier-free. For any $\gamma < \delta$, we have a commutative diagram

$$\begin{tikzcd}
[\![\vec{x}^{\vec{\alpha}}, y^{\gamma}.\ \psi]\!] \arrow[r, tail] \arrow[d, tail] & V_{\alpha_1} \times \cdots \times V_{\alpha_n} \times V_{\gamma} \arrow[r] \arrow[d, tail] & V_{\alpha_1} \times \cdots \times V_{\alpha_n} \\
[\![\vec{x}^{\vec{\alpha}}, y^{\delta}.\ \psi]\!] \arrow[r, tail] & V_{\alpha_1} \times \cdots \times V_{\alpha_n} \times V_{\delta} \arrow[ur] &
\end{tikzcd}$$

so the images $[\![\vec{x}^{\vec{\alpha}}.\ (\exists y^{\gamma})\,\psi]\!]$ form an increasing sequence of subobjects of $V_{\alpha_1} \times \cdots \times V_{\alpha_n}$; and since the squares are pullbacks, the subobjects $[\![\vec{x}^{\vec{\alpha}}.\ (\forall y^{\gamma})\,\psi]\!]$ form a decreasing sequence. Since $\mathcal{E}$ is well-powered, both sequences must stabilize: we define $[\![\vec{x}^{\vec{\alpha}}.\ (\exists y)\,\psi]\!]$ and $[\![\vec{x}^{\vec{\alpha}}.\ (\forall y)\,\psi]\!]$ to be these stable values.

Lemma The stable values still yield pullback squares

$$\begin{tikzcd}
[\![\vec{x}^{\vec{\alpha}}.\ \varphi]\!] \arrow[r, tail] \arrow[d, tail] & V_{\alpha_1} \times \cdots \times V_{\alpha_n} \arrow[d, tail] \\
[\![\vec{x}^{\vec{\beta}}.\ \varphi]\!] \arrow[r, tail] & V_{\beta_1} \times \cdots \times V_{\beta_n}
\end{tikzcd}$$

when $\alpha_i \leq \beta_i$ for all i. In fact, if the lower sequence stabilizes by the time the decoration of y reaches γ, so does the upper sequence.

Hence we can now iterate the argument, to interpret arbitrary first-order formulae-in-decorated-context. In particular, any sentence φ has an interpretation $[\![\varphi]\!]$ (in the empty context) which is simply a subobject of 1 in $\mathcal{E}$.

Theorem All the axioms of IZF set theory (Extensionality, Separation, Collection, Empty-set, Pair-set, Union, Power-set, $\in$-Induction, Infinity) are valid in $\mathcal{E}$.

If $\mathcal{E}$ is Boolean, we get an interpretation of (classical) ZF.

One problem: the Fourman interpretation doesn't 'see' the whole topos.

Given a geometric morphism $\mathcal{F} \xrightarrow{f} \mathcal{E}$, we can recursively define comparison maps $f^*(V_\alpha^{\mathcal{E}}) \longrightarrow V_\alpha^{\mathcal{F}}$ for all α; if f is open (e.g. if $\mathcal{E} = \mathbf{Set}$), then these comparisons are monic for all α, and if f is atomic they're isomorphisms for all α.

In particular, if $\mathcal{E} = \mathbf{Cont}(G)$ for a topological group G, then the Fourman interpretation in $\mathcal{E}$ satisfies the same sentences as that in $\mathbf{Set}$.

Definition By an <u>exponential variety</u> in a Grothendieck topos $\mathcal{E}$, we mean a full subcategory $\mathcal{F}$ closed under arbitrary limits, colimits and power-objects (equivalently, such that the inclusion $\mathcal{F} \longrightarrow \mathcal{E}$ is the inverse image of a connected geometric morphism).

In any $\mathcal{E}$, there's a smallest exponential variety $\mathcal{E}_{wf}$ (the well-founded part of $\mathcal{E}$).

Theorem Assuming Foundation holds in $\mathbf{Set}$, the full subcategory of $\mathcal{E}$ on objects A s.t. $A \rightarrowtail V_\alpha$ for some α is an exponential variety, and it coincides with $\mathcal{E}_{wf}$. (Hence the Fourman interpretation in $\mathcal{E}$ coincides with that in $\mathcal{E}_{wf}$.)

If $\mathcal{E}$ is localic over $\mathbf{Set}$ (equivalently, is expressible as $\mathbf{Sh}(P, J)$ where P is a partial order) then any hyperconnected morphism $\mathcal{E} \longrightarrow \mathcal{F}$ is an equivalence. Hence $\mathcal{E}_{wf} = \mathcal{E}$ in this case.

Remark: if $\mathcal{E}$ is localic over $\mathbf{Set}$ and Boolean, then it satisfies AC (in the form 'Every epimorphism splits') if $\mathbf{Set}$ does; so the Fourman interpretation is an interpretation of ZFC in this case.

Independence of the Continuum Hypothesis (Lawvere–Tierney)

Basic idea: suppose we have sets A, B s.t. $\mathbb{N} \rightarrowtail A \rightarrowtail B$ but there are no bijections $\mathbb{N} \to A$ or $A \to B$ (e.g. $A = P\mathbb{N}$, $B = PP\mathbb{N}$).

Suppose we build a new topos $\mathcal{E}$ by 'freely adjoining' an injection $B \rightarrowtail P\mathbb{N}$.

Then, provided we don't adjoin any new bijections $\mathbb{N} \to A$ or $A \to B$, the object A will become a counterexample to CH in $\mathcal{E}$ (and it necessarily lives in $\mathcal{E}_{wf}$, so the Fourman interpretation sees it).

We do this using classifying toposes. Consider the propositional theory $\mathbb{T}$ with primitive propositions $p(b,n)$, $q(b,n)$ $(b \in B, n \in \mathbb{N})$ (think 'b is/is not related to n' or 'n is/is not in the set $f(b)$') and axioms

$$\left.\begin{array}{l} ((p(b,n) \wedge q(b,n)) \vdash \bot) \\ (\top \vdash (p(b,n) \vee q(b,n))) \end{array}\right\} \text{ for all } b \in B, n \in \mathbb{N}$$

$$(\top \vdash \bigvee_{n \in \mathbb{N}} ((p(b,n) \wedge q(b',n)) \vee (q(b,n) \wedge p(b',n)))) \quad \text{for all } b \neq b'.$$

The theory $\mathbb{T}_0$ having only the first group of axioms is classified by $[\mathcal{P}^{op}, \mathrm{Set}]$ where $\mathcal{P}$ is the poset of finite consistent conjunctions of primitive propositions (equivalently, of partial functions $B \times \mathbb{N} \to \{\top, \bot\}$ with finite domain, ordered by $f \leq g$ iff f extends g). The other axioms generate a Grothendieck topology J on $\mathcal{P}$, such that $Sh(\mathcal{P}, J)$ is the classifying topos for $\mathbb{T}$.

Key lemma: all J-covering sieves are (stably) nonempty. Hence J is contained in the topology (K, say) corresponding to the Booleanization $sh_{\neg\neg}[\mathcal{P}^{op}, \mathrm{Set}]$, and the latter also contains a model of $\mathbb{T}$. But the latter is just an injection $p^*B \rightarrowtail P\mathbb{N}$ in $Sh(\mathcal{P}, K)$.

We then need to show that there are no epimorphisms $\mathbb{N} \twoheadrightarrow p^*A$ or $p^*A \twoheadrightarrow p^*B$ in $Sh(\mathcal{P}, K)$: this is a combinatorial argument based on the fact that any family of pairwise-disjoint elements of $\mathcal{P}$ is countable.

Applying the Fourman interpretation in $Sh(\mathcal{P}, K)$, we may thus deduce

Theorem CH is not deducible from the axioms of ZFC.

Independence of Suslin's Hypothesis (M. Bunge)

This is done by a similar method, relying on the result of Miller that $\neg SH$ is equivalent to the existence of a Suslin tree, i.e. an uncountable tree in which all chains and antichains are countable.

Idea: pick an uncountable well-ordered set W and consider a propositional theory $\mathbb{T}$ whose models are Suslin tree structures on W s.t. the tree-ordering $\preccurlyeq$ is contained in the well-ordering $\leq$. Primitive propositions $p(x,y), q(x,y)$ for all $x<y$ in W (think '$x \prec y$' and '$x \not\prec y$') with axioms $(T \vdash p(0,x))$ for all $x \neq 0$,

$((p(x,y) \wedge q(x,y)) \vdash \bot)$ and $(T \vdash (p(x,y) \vee q(x,y)))$ for all $x<y$,

$((p(x,y) \wedge p(y,z)) \vdash p(x,z))$ and $((p(x,z) \wedge p(y,z)) \vdash p(x,y))$ for all $x<y<z$, and

$(T \vdash \bigvee_{x,y \in S} p(x,y))$ and $(T \vdash \bigvee_{x,y \in S} q(x,y))$ for all uncountable $S \subseteq W$.

Once again, we can represent the classifying topos of $\mathbb{T}$ as $Sh(\mathcal{P}, J)$ where $\mathcal{P}$ is a suitable poset of finite conjunctions of primitive propositions, and J is contained in the topology K corresponding to the Booleanization of $[\mathcal{P}^{op}, Set]$. So there's a model of $\mathbb{T}$ in $Sh(\mathcal{P}, K)$.

The same combinatorial argument as before shows that p^*W is still uncountable in $Sh(\mathcal{P}, K)$; we also have to show that the $\mathbb{T}$-model structure on it really is a Suslin tree.

Independence of the Axiom of Choice (P. Freyd)

Assuming AC holds in Set, we can't negate it in a localic topos $Sh(\mathcal{P}, K)$.

Nor can we use a topos of the form $Cont(G)$ (though these do negate the topos-theoretic formulation of AC).

Freyd tried a more direct approach: we want a Boolean Grothendieck topos $\mathcal{B}$ containing a countable sequence of objects $A_n \rightarrowtail PN$ such that $\prod_{n=0}^{\infty} A_n \cong 0$ (and $A_n \twoheadrightarrow 1$).

We might as well assume:

- $\mathcal{B} = Sh(\mathcal{C}, K)$ where K consists of all stably nonempty sieves;
- each A_n is the associated K-sheaf of a representable $yU_n : \mathcal{C}^{op} \to Set$;
- the U_n are all the objects of $\mathcal{C}$. (This will ensure $\mathcal{B}$ is well-founded.)

In fact we assume

(1) $\mathcal{C}$ is countable, $ob\,\mathcal{C} = \mathbb{N}$ and $\mathcal{C}(m,n) \neq \emptyset$ iff $m \geq n$.

This ensures that $\prod_{n=0}^{\infty} y(n) \cong 0$ in $[\mathcal{C}^{op}, Set]$, though it's not enough on its own to ensure $\prod A_n \cong 0$. It also ensures that $A_n \twoheadrightarrow 1$ in $\mathcal{B}$.

(2) If
$$\begin{tikzcd} p \arrow[r, "k"] \arrow[d, "h"] & n \arrow[d, "f"] \\ n \arrow[r, "g"] & m \end{tikzcd}$$
commutes in $\mathcal{C}$, then $f = g$.

This ensures that each morphism of $\mathcal{C}$ is epic; hence representable functors are π-separated, i.e. $y(n) \longrightarrow A_n$ is monic for each n. It also ensures, along with countability of $\mathcal{C}$, that each $y(n)$ embeds in $\Omega_\pi^{\mathbb{N}}$, so A_n is the π-closure of $y(n) \rightarrowtail \Omega_\pi^{\mathbb{N}}$.

(3) Given $n \xrightarrow{f} m$ and $n \xrightarrow{g} m+1$ in $\mathcal{C}$, there exist $n+1 \underset{k}{\overset{h}{\rightrightarrows}} n$ such that $fh = fk$ but $gh \neq gk$.

This ensures that there is no partial map $y(m) \longrightarrow y(m+1)$ with nonempty domain, and hence that there is no map $A_m \longrightarrow A_{m+1}$ in $\mathcal{B}$. So $\prod_{n=0}^{\infty} A_n \cong 0$, as required.

Can we actually find a category $\mathcal{C}$ satisfying these conditions?

Key observation: if $\mathcal{C}$ satisfies the conditions, then there are at least $(m+1)^{n-m}$ morphisms $n \longrightarrow m$ for any $n \geq m$.

Example 1: take morphisms $n \longrightarrow m$ to be $(n-m)$-tuples $(i_1, i_2, \dots, i_{n-m})$ with $i_j \in \{0, 1, 2, \dots m\}$. The composite
$$p \xrightarrow{(j_1, \dots, j_{p-n})} n \xrightarrow{(i_1, \dots, i_{n-m})} m$$
is $(i_1, \dots, i_{n-m}, \bar{j}_1, \dots, \bar{j}_{p-n})$ where $\bar{j} = \min\{j, m\}$.

Condition **(2)** is clear since we can recover f from fk provided we know its length. For **(3)** we take h and k to be the 1-tuples (m) and $(m+1)$.

Example 2: take morphisms $n \to m$ to be functions $f : \{0,1,2,\dots,n\} \longrightarrow \{0,1,2,\dots,m\}$ which are split by the inclusion, i.e. satisfy $f(j) = j$ for all $j \leq m$. Condition **(2)** holds since h and k have a common splitting in Set. For **(3)**, given f and g, we define h and k by $h(n+1) = m+1$ and $k(n+1) = f(m+1)$.

Although they appear similar, the two examples are very different.

In Example 1 consider the functor $B_n : \mathcal{C}^{op} \longrightarrow \mathrm{Set}$ defined by

$$B_n(m) = \emptyset \quad \text{if } m < n$$
$$= \{0,1\} \text{ if } m \geq n$$

if $f = (i_1, \dots, i_{p-m}) : p \to m$ with $m \geq n$, then

$$B_n(f) = \left\{ \begin{matrix} \text{identity} \\ \text{twist map} \end{matrix} \right\} \text{ if } \#\{j : i_j < n\} \text{ is } \left\{ \begin{matrix} \text{even} \\ \text{odd} \end{matrix} \right\}.$$

Let $\overline{B}_n$ denote the associated K-sheaf of B_n; then each $\overline{B}_n$ is finite, but $\prod_{n=0}^{\infty} \overline{B}_n \cong 0$.

On the other hand, in Example 2, the lexicographic ordering on morphisms $n \to m$ (is total and) satisfies $f_1 < f_2 \Rightarrow f_1 g < f_2 g$. Using this (plus AC in Set), can show that every object of $Sh(\mathcal{C}, K)$ admits a total ordering. So the Fourman interpretation in this topos shows that AC is independent of the Ordering Principle.

What geometric theories do these two toposes classify?

Remark: if $\mathcal{C}$ satisfies Freyd's axioms and attains the lower bound on size of hom-sets, then $Sh(\mathcal{C}, K)$ has no points (i.e. the theory has no models in Set).